Discouraging "Proof By Example"

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Abstract

One of the difficult and frustrating aspects of teaching mathematics is getting students to think logically and develop an ability to prove theorems. Too frequently, students resort to what might be called "proof by example." A classic illustration of "proof by example" comes from years of teaching "mathematics for economists." The exercise is to prove a general principle about sets; for example: "show that for arbitrary sets A and B, the sets $A \setminus B$, $B \setminus A$, and $A \bigcap B$ are pairwise disjoint." Too frequently the student "proof" begins with: "Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$. Then …" Other examples, in other areas of mathematics, no doubt abound. This note presents an obviously preposterous "theorem" and "proves" it with many examples. This "theorem," once "proved" in class, successfully discourages future proofs "by example."

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Discouraging "Proof By Example"

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One of the difficult and frustrating aspects of teaching mathematics is getting students to think logically and develop an ability to prove theorems and general principles. Too frequently, students resort to what might be called "proof by example." A classic illustration of "proof by example" comes from years of teaching "mathematics for economists." The exercise is to prove a general principle about sets; for example: "show that for arbitrary sets A and B, the sets $A \setminus B$, $B \setminus A$, and $A \bigcap B$ are pairwise disjoint." Too frequently the student "proof" begins with: "Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$. Then ..." Other examples, in other areas of mathematics, no doubt abound. To combat "proof by example," I present the students with an obviously preposterous "theorem," then "prove" it with *many* examples. This "theorem" and "proof," once presented in class, successfully discourage future proofs "by example."

The "Theorem"

Notational convention usually leads us to interpret "*xyz*" as the product of the three real numbers, x, y, and z. In this note, upper case letters will be treated as single-digit numbers and will take a value from the elements of the set $\{0, 1, \ldots, 9\}$, so that "XYZ" will have the numerical value of 100X + 10Y + Z, rather than the product of X, Y, and Z. Having established this notational convention, we now state the "theorem."

> **"Theorem."** *If the numerator and denominator of a fraction have a digit in common, then the fraction can be reduced by simply striking the common digit from the numerator and denominator.*

Of course the "Theorem" is quite valid in one very restrictive case: when the numerator and denominator both end in zero. In such a case the trailing zeros can be struck or deleted without changing the value of the fraction; e.g. $\frac{12\not{0}}{34\not{0}} = \frac{12}{34}$. However, the "Theorem" claims a more general applicability. For example, suppose the ratio of three-digit numbers, $\frac{\text{XYZ}}{\text{TUV}}$, is formed using X = 2, Y = 3, Z = 4, T = 9, U = Y = 3, and V = 6. Then the fraction $\frac{\text{XYZ}}{\text{TUV}} = \frac{\text{XYZ}}{\text{TYV}}$ is $\frac{234}{936}$. According to the "Theorem," we can simply strike the 3's that occur in the numerator and denominator, leaving a ratio of two-digit numbers that is equal to the original ratio of three-digit numbers:

$$\frac{2\not{3}4}{9\not{3}6} = \frac{24}{96}.$$

Of course this "Theorem" is absurd. However, consideration of the ratio above indicates that there *are* some fractions for which the claim of the "Theorem" holds because $\frac{234}{936}$ *is in fact equal to* $\frac{24}{96} = \frac{1}{4}$! It turns out that there are many such examples, so I offer students a proof to the "Theorem," "by example."

> **"Proof."** *The "proof by example" amounts to giving example after example that supports the "Theorem." Among ratios of three-digit numbers there are hundreds of examples that support the "Theorem," but refer to Table 1, which identifies labelled "lists" in the Appendix of the most "elegant" examples. See also (1) and (2) on the next page.* ■

For example, we see from Table 1 that in "expression" (9) of the Appendix we can find 36 ratios of three-digit numbers supporting the "Theorem" that have re-used or repeated digits in the numerator *and* the denominator (i.e. X = Z and T = V); in expression (14) we can find another 15 examples that have neither repeated digits nor leading zeros. As indicated in the last two lines of the Table, one can present 149 examples that have no repeated digits save for the two that "cancel." Before one gets to the end of these 149, however, the point will have been made to the students and one can proclaim "**Q.E.D.**" (if only with tongue in cheek)! This "proof" produces many smiles

Property of example	**"Equation" number: number of examples**			
	X = U	**X = V**	**Y = U**	**Y = V**
Repeated digits only *within*				
numerator *and* denominator	(6): 4		(9): 36	(13): 4
numerator *or* denominator			(10): 17	
No repeats: no leading zeros	(7): 22		(11): 95	(14): 15
No repeats: leading zeros		(8): 3	(12): 10	(15): 4

Table 1: Catalogue of examples of ratios $\frac{XYZ}{TUV}$ that support the "Theorem"

and amused looks, but it also has the desired effect of virtually eliminating “proofs by example” for the remainder of the semester.

Other Cases

While the examples presented in the Appendix are only for the cases of ratios of three-digit numerators and three-digit denominators, the interested reader can write a simple computer program to uncover yet more examples. (A sample program appears in Appendix B.) We can state here, however, that there are only four examples of ratios of two-digit numbers that support the “Theorem”:

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4}, \quad \frac{1\not{9}}{\not{9}5} = \frac{1}{5}, \quad \frac{2\not{6}}{\not{6}5} = \frac{2}{5}, \quad \frac{4\not{9}}{\not{9}8} = \frac{4}{8}. \qquad (1)$$

As stated above, and displayed in Table 1, there are over 100 examples of ratios of three-digit numbers without repeated digits that support the “Theorem.” Further, an investigation into ratios of four-digit numbers yields 612 examples of fractions $\frac{\mathrm{WXYZ}}{\mathrm{PQRS}}$, without repeated digits, that support the “Theorem.” However, we need not be restricted to examples in which the numerator and denominator have the same number of digits; consider:

$$\left.\begin{array}{l} \frac{\not{2}3}{1\not{2}65} = \frac{3}{165}, \quad \frac{\not{5}4}{1\not{5}12} = \frac{4}{112}, \quad \frac{\not{5}6}{4\not{5}92} = \frac{6}{492}, \quad \frac{\not{5}7}{1\not{5}96} = \frac{7}{196}, \quad \frac{\not{6}7}{3\not{6}85} = \frac{7}{385}, \\ \frac{2\not{6}7}{14\not{6}85} = \frac{27}{1485}, \quad \frac{5\not{0}1}{39\not{0}78} = \frac{51}{3978}, \quad \frac{6\not{1}2}{50\not{1}84} = \frac{62}{5084}, \quad \frac{8\not{4}6}{27\not{4}95} = \frac{86}{2795}, \quad \frac{9\not{3}4}{72\not{3}85} = \frac{94}{7285}. \end{array}\right\} \qquad (2)$$

Hopefully, these few examples, along with those given in the Appendix, suggest that there are many different types of examples to be found. Whatever examples may be chosen, however, the goal is the same: to discourage “proof by example.”

Appendix A

There are nine cases to consider for examples of the fraction $\frac{XYZ}{TUV}$ that support the "Theorem": X = T, X = U, X = V, …, Z = V. Each of these is considered in turn. In our search for suitable examples, i.e. examples that will support the ridiculous "Theorem," we will avoid certain problems and ambiguities by making the following two restrictions: there is only one digit which is common to the numerator and denominator, and furthermore, the digit that is shared by the numerator and denominator occurs exactly once in the numerator and exactly once in the denominator.[1] By permitting just one digit to be shared by the numerator and denominator, we avoid such "unsupportive" examples as $\frac{975}{195}$ (= 5), in which striking the 9's does *not* change the value of the fraction,

$$\frac{\cancel{9}75}{1\cancel{9}5} = \frac{75}{15},$$

but striking the 5's *does* change the value of the fraction,

$$\frac{97\cancel{5}}{19\cancel{5}} \neq \frac{97}{19}; \quad \text{and} \quad \frac{\cancel{9}75}{1\cancel{9}5} = \frac{75}{15} \ (= 5), \quad \text{but} \quad \frac{7\cancel{5}}{1\cancel{5}} \neq \frac{7}{1}. \tag{3}$$

By allowing the only shared digit to occur exactly once in the numerator and exactly once in the denominator, we avoid examples such as $\frac{656}{164}$ and $\frac{676}{169}$, for in these cases,

$$\left.\begin{aligned} \frac{\cancel{6}56}{1\cancel{6}4} &= \frac{56}{14} \ (= 4), \quad \text{but} \quad \frac{65\cancel{6}}{1\cancel{6}4} \neq \frac{65}{14}; \\ \frac{\cancel{6}76}{1\cancel{6}9} &= \frac{76}{19} \ (= 4), \quad \text{but} \quad \frac{67\cancel{6}}{1\cancel{6}9} \neq \frac{67}{19}. \end{aligned}\right\} \tag{4}$$

Our focus on examples admitting a single (i.e. unique) "cancellation" rules out 50 examples like (3) and the two examples in (4), none of which can support the claim of the "Theorem" in the mind of the skeptic who will be looking for "cancellations" that don't "work."

The "cost" of the restrictions is that we will not identify 61 examples which offer more than a single possible "cancellation" but nevertheless support the "Theorem":[2]

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[1] By imposing the restrictions, we also avoid double-counting pairs of examples which are either the same or reciprocals of each other. For example, XYZ/TUV = 335/938 = 35/98 would be uncovered for two cases: X = U and Y = U; also 199/597 = 19/57 would be found looking for Y = U examples that are less than one, while 597/199 = 57/19 would be found looking for Y = V examples.

[2] In addition, there are hundreds of uninteresting examples involving "trailing zeros" (Y = Z = V = 0, for example); we ignore these as well.

- 48 examples for which X = Y = U; e.g. $\frac{\not{1}12}{6\not{1}6} = \frac{1\not{1}2}{6\not{1}6} = \frac{12}{66}$.
- 4 examples for which X = U and Z = V = 0; e.g. $\frac{\not{6}40}{1\not{6}0} = \frac{4\not{0}}{1\not{0}} = \frac{4}{1}$.[3]
- 2 examples for which X = Y = V: $\frac{\not{7}75}{21\not{7}} = \frac{7\not{7}5}{21\not{7}} = \frac{75}{21}$ and $\frac{\not{9}96}{24\not{9}} = \frac{9\not{9}6}{24\not{9}} = \frac{96}{24}$.
- 7 examples for which Y = Z = U; e.g. $\frac{1\not{3}3}{5\not{3}2} = \frac{13\not{3}}{5\not{3}2} = \frac{13}{52}$.

We now turn to the individual cases, keeping in mind that for each example cited, the reciprocal "works" too; this may be helpful for those wanting to work solely with fractions less than one.

A.1. Examples of the form $\frac{\mathbf{XYZ}}{\mathbf{XUV}} = \frac{\mathbf{YZ}}{\mathbf{UV}}$ (X = T)

Save for the trivial instances in which X = T = 0 and the trivial instances in which the numerator and the denominator are the same (so that the value of the fraction is one), there are no examples for this case.

A.2. Examples of the form $\frac{\mathbf{XYZ}}{\mathbf{TXV}} = \frac{\mathbf{YZ}}{\mathbf{TV}}$ (X = U)

While the restrictions for a unique "cancellation" would not admit such examples, there are four examples with X = U (= Y = V) that "reduce" to the examples in (1), which can be further "reduced." Therefore, we present the following:

$$\frac{664}{166} = \frac{64}{16} = \frac{4}{1}, \quad \frac{665}{266} = \frac{65}{26} = \frac{5}{2}, \quad \frac{995}{199} = \frac{95}{19} = \frac{5}{1}, \quad \frac{998}{499} = \frac{98}{49} = \frac{8}{4}. \tag{5}$$

Allowing Y = Z in the numerator and/or T = V in the denominator yields four examples:

$$\frac{644}{161} = \frac{44}{11}, \quad \frac{655}{262} = \frac{55}{22}, \quad \frac{955}{191} = \frac{55}{11}, \quad \frac{988}{494} = \frac{88}{44}. \tag{6}$$

Finally, if the only equal digits are X and U, then we have the following twenty-two examples for the X = U case:

$$\left.\begin{array}{c} \frac{325}{130} = \frac{25}{10}, \quad \frac{340}{136} = \frac{40}{16}, \quad \frac{340}{238} = \frac{40}{28}, \quad \frac{345}{138} = \frac{45}{18}, \quad \frac{648}{162} = \frac{48}{12}, \quad \frac{652}{163} = \frac{52}{13}, \\ \frac{670}{268} = \frac{70}{28}, \quad \frac{670}{469} = \frac{70}{49}, \quad \frac{672}{168} = \frac{72}{18}, \quad \frac{756}{270} = \frac{56}{20}, \quad \frac{960}{192} = \frac{60}{12}, \quad \frac{965}{193} = \frac{65}{13}, \\ \frac{970}{194} = \frac{70}{14}, \quad \frac{970}{291} = \frac{70}{21}, \quad \frac{975}{390} = \frac{75}{30}, \quad \frac{980}{196} = \frac{80}{16}, \quad \frac{980}{294} = \frac{80}{24}, \quad \frac{980}{392} = \frac{80}{32}, \\ \frac{982}{491} = \frac{82}{41}, \quad \frac{985}{197} = \frac{85}{17}, \quad \frac{985}{394} = \frac{85}{34}, \quad \frac{986}{493} = \frac{86}{43}. \end{array}\right\} \tag{7}$$

[3]These are just the ratios of two-digit numbers from (1) with a zero appended to the ends of the numerator and the denominator.

A.3. Examples of the form $\frac{\mathbf{XYZ}}{\mathbf{TUX}} = \frac{\mathbf{YZ}}{\mathbf{TU}}$ (X = V)

The only nontrivial examples for this case are the four from (5) and the following three which possess a leading zero in the denominator (T = 0):

$$\frac{325}{13} = \frac{25}{1}, \quad \frac{756}{27} = \frac{56}{2}, \quad \frac{975}{39} = \frac{75}{3}. \tag{8}$$

A.4. Examples of the form $\frac{\mathbf{XYZ}}{\mathbf{YUV}} = \frac{\mathbf{XZ}}{\mathbf{UV}}$ (Y = T)

These are just the reciprocals of the examples in Section A.2.

A.5. Examples of the form $\frac{\mathbf{XYZ}}{\mathbf{TYV}} = \frac{\mathbf{XZ}}{\mathbf{TV}}$ (Y = U)[4]

Once again we find the four examples of (5) for this case. But if we maintain the restrictions for a unique "cancellation," and a value less than one, we find thirty-six examples with X = Z *and* T = V:

$$\left.\begin{array}{cccc} \frac{101}{202} = \frac{11}{22}, \quad \frac{101}{303} = \frac{11}{33}, & \dots, & \frac{101}{808} = \frac{11}{88}, & \frac{101}{909} = \frac{11}{99}, \\ \frac{202}{303} = \frac{22}{33}, & \dots, & \frac{202}{808} = \frac{22}{88}, & \frac{202}{909} = \frac{22}{99}, \\ & \ddots & \vdots & \vdots \\ & & \frac{707}{808} = \frac{77}{88}, & \frac{707}{909} = \frac{77}{99}, \\ & & & \frac{808}{909} = \frac{88}{99}; \end{array}\right\} \tag{9}$$

seventeen with X = Z *or* T = V (but not both):

$$\left.\begin{array}{cccccc} \frac{134}{737} = \frac{14}{77}, & \frac{143}{242} = \frac{13}{22}, & \frac{156}{858} = \frac{16}{88}, & \frac{165}{363} = \frac{15}{33}, & \frac{178}{979} = \frac{18}{99}, & \frac{187}{484} = \frac{17}{44}, \\ \frac{242}{341} = \frac{22}{31}, & \frac{264}{363} = \frac{24}{33}, & \frac{286}{484} = \frac{26}{44}, & \frac{363}{462} = \frac{33}{42}, & \frac{363}{561} = \frac{33}{51}, & \frac{385}{484} = \frac{35}{44}, \\ \frac{484}{583} = \frac{44}{53}, & \frac{484}{682} = \frac{44}{62}, & \frac{484}{781} = \frac{44}{71}, & \frac{536}{737} = \frac{56}{77}, & \frac{737}{938} = \frac{77}{98}; & \end{array}\right\} \tag{10}$$

ninety-five for which X $\neq$ Z *and* T $\neq$ V:

[4]Examples for the Y = U case come in "reciprocal pairs." Without some restriction, a search for Y = U examples would identify both $\frac{134}{536} = \frac{14}{56}$ *and* $\frac{536}{134} = \frac{56}{14}$, for example. To avoid double-counting the reciprocals, we consider only the examples that are less than one.

$$\left.\begin{array}{lllllll}
\frac{102}{306}=\frac{12}{36}, & \frac{102}{408}=\frac{12}{48}, & \frac{103}{206}=\frac{13}{26}, & \frac{104}{208}=\frac{14}{28}, & \frac{134}{536}=\frac{14}{56}, & \frac{134}{938}=\frac{14}{98}, & \frac{136}{238}=\frac{16}{28}, \\
\frac{154}{253}=\frac{14}{23}, & \frac{154}{352}=\frac{14}{32}, & \frac{165}{264}=\frac{15}{24}, & \frac{165}{462}=\frac{15}{42}, & \frac{176}{275}=\frac{16}{25}, & \frac{176}{374}=\frac{16}{34}, & \frac{176}{473}=\frac{16}{43}, \\
\frac{176}{572}=\frac{16}{52}, & \frac{187}{286}=\frac{17}{26}, & \frac{187}{385}=\frac{17}{35}, & \frac{187}{583}=\frac{17}{53}, & \frac{187}{682}=\frac{17}{62}, & \frac{195}{390}=\frac{15}{30}, & \frac{196}{294}=\frac{16}{24}, \\
\frac{196}{392}=\frac{16}{32}, & \frac{196}{490}=\frac{16}{40}, & \frac{197}{394}=\frac{17}{34}, & \frac{198}{297}=\frac{18}{27}, & \frac{198}{396}=\frac{18}{36}, & \frac{198}{495}=\frac{18}{45}, & \frac{198}{594}=\frac{18}{54}, \\
\frac{198}{693}=\frac{18}{63}, & \frac{198}{792}=\frac{18}{72}, & \frac{201}{603}=\frac{21}{63}, & \frac{201}{804}=\frac{21}{84}, & \frac{203}{406}=\frac{23}{46}, & \frac{203}{609}=\frac{23}{69}, & \frac{204}{306}=\frac{24}{36}, \\
\frac{206}{309}=\frac{26}{39}, & \frac{234}{936}=\frac{24}{96}, & \frac{253}{451}=\frac{23}{41}, & \frac{264}{561}=\frac{24}{51}, & \frac{268}{469}=\frac{28}{49}, & \frac{275}{374}=\frac{25}{34}, & \frac{275}{473}=\frac{25}{43}, \\
\frac{275}{671}=\frac{25}{61}, & \frac{286}{385}=\frac{26}{35}, & \frac{286}{583}=\frac{26}{53}, & \frac{286}{781}=\frac{26}{71}, & \frac{297}{396}=\frac{27}{36}, & \frac{297}{495}=\frac{27}{45}, & \frac{297}{594}=\frac{27}{54}, \\
\frac{297}{693}=\frac{27}{63}, & \frac{297}{891}=\frac{27}{81}, & \frac{298}{596}=\frac{28}{56}, & \frac{301}{602}=\frac{31}{62}, & \frac{302}{604}=\frac{32}{64}, & \frac{302}{906}=\frac{32}{96}, & \frac{304}{608}=\frac{34}{68}, \\
\frac{306}{408}=\frac{36}{48}, & \frac{352}{451}=\frac{32}{41}, & \frac{374}{572}=\frac{34}{52}, & \frac{374}{671}=\frac{34}{61}, & \frac{385}{682}=\frac{35}{62}, & \frac{385}{781}=\frac{35}{71}, & \frac{392}{490}=\frac{32}{40}, \\
\frac{394}{591}=\frac{34}{51}, & \frac{395}{790}=\frac{35}{70}, & \frac{396}{495}=\frac{36}{45}, & \frac{396}{594}=\frac{36}{54}, & \frac{396}{792}=\frac{36}{72}, & \frac{396}{891}=\frac{36}{81}, & \frac{398}{597}=\frac{38}{57}, \\
\frac{398}{796}=\frac{38}{76}, & \frac{401}{802}=\frac{41}{82}, & \frac{402}{603}=\frac{42}{63}, & \frac{403}{806}=\frac{43}{86}, & \frac{462}{561}=\frac{42}{51}, & \frac{473}{572}=\frac{43}{52}, & \frac{473}{671}=\frac{43}{61}, \\
\frac{495}{693}=\frac{45}{63}, & \frac{495}{792}=\frac{45}{72}, & \frac{495}{891}=\frac{45}{81}, & \frac{532}{931}=\frac{52}{91}, & \frac{536}{938}=\frac{56}{98}, & \frac{572}{671}=\frac{52}{61}, & \frac{583}{682}=\frac{53}{62}, \\
\frac{583}{781}=\frac{53}{71}, & \frac{594}{693}=\frac{54}{63}, & \frac{594}{792}=\frac{54}{72}, & \frac{594}{891}=\frac{54}{81}, & \frac{596}{894}=\frac{56}{84}, & \frac{602}{903}=\frac{62}{93}, & \frac{603}{804}=\frac{63}{84}, \\
\frac{682}{781}=\frac{62}{71}, & \frac{693}{792}=\frac{63}{72}, & \frac{693}{891}=\frac{63}{81}, & \frac{792}{891}=\frac{72}{81}; & & &
\end{array}\right\} \tag{11}$$

and, finally, ten more with a leading zero (X = 0 in each case so that the numerator is a two-digit number):

$$\left.\begin{array}{lllll}
\frac{34}{136}=\frac{4}{16}, & \frac{34}{238}=\frac{4}{28}, & \frac{67}{268}=\frac{7}{28}, & \frac{67}{469}=\frac{7}{49}, & \frac{96}{192}=\frac{6}{12}, \\
\frac{97}{194}=\frac{7}{14}, & \frac{97}{291}=\frac{7}{21}, & \frac{98}{196}=\frac{8}{16}, & \frac{98}{294}=\frac{8}{24}, & \frac{98}{392}=\frac{8}{32}.
\end{array}\right\} \tag{12}$$

A.6. Examples of the form $\frac{XYZ}{TUY} = \frac{XZ}{TU}$ (Y = V)

For this case we find four examples which have X = Z *and* T = U:

$$\frac{464}{116}=\frac{44}{11}, \quad \frac{565}{226}=\frac{55}{22}, \quad \frac{595}{119}=\frac{55}{11}, \quad \frac{898}{449}=\frac{88}{44}; \tag{13}$$

fifteen with X ≠ Z *and* T ≠ U:

$$\left.\begin{array}{l} \frac{265}{106}=\frac{25}{10},\quad \frac{298}{149}=\frac{28}{14},\quad \frac{365}{146}=\frac{35}{14},\quad \frac{465}{186}=\frac{45}{18},\quad \frac{596}{149}=\frac{56}{14},\\ \frac{695}{139}=\frac{65}{13},\quad \frac{698}{349}=\frac{68}{34},\quad \frac{732}{183}=\frac{72}{18},\quad \frac{765}{306}=\frac{75}{30},\quad \frac{854}{305}=\frac{84}{30},\\ \frac{864}{216}=\frac{84}{21},\quad \frac{865}{346}=\frac{85}{34},\quad \frac{895}{179}=\frac{85}{17},\quad \frac{965}{386}=\frac{95}{38},\quad \frac{976}{427}=\frac{96}{42}; \end{array}\right\} \qquad (14)$$

and, finally, four more with a leading zero (T = 0 in each case so that the denominator is a two-digit number):

$$\frac{195}{39}=\frac{15}{3},\quad \frac{196}{49}=\frac{16}{4},\quad \frac{392}{49}=\frac{32}{4},\quad \frac{395}{79}=\frac{35}{7}. \qquad (15)$$

A.7. Examples of the form $\frac{XYZ}{ZUV}=\frac{XY}{UV}$ (Z = T)

These are just the reciprocals of the examples in Section A.3.

A.8. Examples of the form $\frac{XYZ}{TZV}=\frac{XY}{TV}$ (Z = U)

These are just the reciprocals of the examples in Section A.6.

A.9. Examples of the form $\frac{XYZ}{TUZ}=\frac{XY}{TU}$ (Z = V)

Save for the possibilities involving Z = 0, there are no nontrivial examples for this case.

Appendix B

Following is the SAS program used to uncover the examples given in Section A.5; the others would be obvious modifications of this one.

```
DATA FRACTION;
TITLE1 ’XYZ/TYU = XZ/TU:  NO REPEATS’;
TITLE2 ’                  LEADING ZERO ALLOWED’;
COUNTER=0;
DO X=0 TO 9;
   DO Y=0 TO 9;
      DO Z=0 TO 9;
         DO T=0 TO 9;
            DO U=0 TO 9;
               NUMER = 100*X + 10*Y + Z;
               DENOM = 100*T + 10*Y + U;
               NEWTOP = 10*X + Z;
               NEWBOT = 10*T + U;
* ---------------------------------------------------------------------;
            IF NEWBOT NE 0  AND  NUMER LT DENOM
               AND  X NE T  AND  X NE Y  AND  X NE U
               AND  Y NE T               AND  Y NE U
               AND  Z NE T  AND  Z NE Y  AND  Z NE U
               THEN  DO;
                     FIRST = NUMER/DENOM;
                     SECOND = NEWTOP/NEWBOT;
                     IF FIRST=SECOND THEN DO;
                        COUNTER=COUNTER+1;
                        PUT NUMER ’/ ’ DENOM ’= ’ NEWTOP ’/ ’ NEWBOT
                        ’ : ITEM NUMBER = ’ COUNTER
                        ’, CANCELLED DIGIT = ’ Y;
                        END;
            END;
* ---------------------------------------------------------------------;
            END;
         END;
      END;
   END;
END;
```