A Note on the Core of a Monetary Economy

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Abstract

This note shows that there are monetary equilibria in the model of overlapping generations that are in the core. These equilibria have positive stocks of outside money in every generation, and they support Pareto-optimal equilibrium allocations. These equilibria are thus self-enforcing, and introducing money into an economy with infinitely many agents need not be tantamount to contriving a new social institution designed to enforce sequential contracts between generations. *Journal of Economic Literature* Classification Numbers: E40, E60, C71.

1. Introduction

This note shows that there are monetary equilibria in the model of overlapping generations with positive stocks of outside money in every period that are in the core. Since the coalition of the whole cannot improve upon the equilibrium allocations, they are Pareto optimal. Thus introducing positive stocks of an outside asset may entail that some agent in a future generation will be taxed to service the generational debt. I show below that there are economies in which the trade between generations is sufficiently important so that infinitely many future generations are willing to be taxed in order to service a prior debt.

In the simplest versions of models of overlapping generations, positive stocks of an outside asset may support Walrasian equilibrium allocations that are not in the core.[1] The (loose) intuition is that agents in later generations have an incentive to renege on any generational debt they have inherited and then issue their own inside money as an asset for future generations. But this note presents an example of a monetary economy in which each generation's temptation to renege on the stock of past debt is weaker than the gains from trade between the generations alive in each period. This example is robust with respect to changes in endowments, preferences, and monetary policies, and it extends the results of Esteban and Millan [7] to economies in which the present value of the stock of debt is not constant. Those authors show that Walrasian equilibria with constant positive stocks of outside assets cannot support equilibria in the core, and they conclude that Gale [10] was correct in emphasizing that introducing money into a Walrasian system was analogous to creating a new social institution. This note shows that such a conclusion may be true for large economies, but it is not uniformly true for economies with only a few agents in each generation.

2. Notation and Definitions

The set of index agents is $H = \bigcup_{t=0}^{\infty} H_t$, where H_t is the set of agents belonging to generation t. Each agent $h \in H \setminus H_0$ lives for two periods,[2] but agent $h \in H_0$ lives for only one

[1] Hendricks, Judd, and Kovenock [11] showed that there are Walrasian equilibrium that are not in the core. Using a model with two agents per generation, Kovenock [13] showed that there are Pareto optimal Walrasian equilibrium that are not in the (empty) core.
[2] This assumption is not at all as restrictive as it may seem. Balasko, Cass, and Shell [2] give an algorithm for converting an arbitrary demographic structure into the one described here.

period. In each period $t \geq 1$, there are ℓ commodities that cannot be stored. Hence the endowment of agent $h \in H_t$ is $\omega_h = (0,...,\omega_h^t, \omega_h^{t+1}, 0,...)$, with $\omega_h^t \in \Re_+^\ell$,[3] and the consumption bundle of agent $h \in H_t$ is $x_h = (0,\ldots,x_h^t, x_h^{t+1}, 0,\ldots)$, where again $x_h^t \in \Re_+^\ell$.[4] The commodity space is $X = \Re_+^m \times \Re_+^m \times \ldots$ The preferences of agent $h \in H_t$ are summarized by $u_h \colon \Re_+^{2\ell} \to \Re$ whose rule is $u_h(x_h) = u_h(x_h^t, x_h^{t+1})$.[5]

Agent $h \in H_t$ receives lump-sum transfers of a fiat asset; thus $m_h = (0,...,m_h^t, m_h^{t+1}, 0,...)$, with $m_h^t \in \Re$.[6] If $m_h^t < 0$, then h is taxed at time t. This agent's incremental asset demands are $y_h = (0,...,y_h^t, y_h^{t+1}, 0,...)$, with $y_h^t \in \Re$.[7] If $y_h^t > 0$, then agent $h \in H_t$ holds the asset as a store of value in the first period of his life.

Let $p^t \in \Re_+^\ell$ be the vector of commodity prices at time t. The sequence of such prices is $p = (p^1, p^2,...)$, and the normalization is $p^{1,1} = 1$. Now let $q^t \in \Re_+$ be the price of the fiat asset at time t; then $q = (q^1, q^2,...)$ is the sequence of present prices of the asset.

Agent $h \in H_t$ solves:

$$\text{maximize } u_h(x_h) \tag{1}$$

subject to

(i) $p \cdot x_h + q \cdot y_h \leq p \cdot \omega_h + q \cdot m_h$;

(ii) $x_h^t \geq 0,\ x_h^{t+1} \geq 0$; and

(iii) $y_h^t + y_h^{t+1} \geq 0$.

Let $Y_h = (0,...,y_h^t, y_h^t + y_h^{t+1}, 0,...)$ be the cumulative asset holdings of agent $h \in H_t$. Consider also $M^t = \sum_{s=1}^{t} \sum_{h \in G_{s-1} \cup G_s} m_h^s$, the sum of all injections of the fiat asset that have occurred up to time t. Then $M = (M^1, M^2,...)$ is the profile of the stock of the fiat asset.

[3] Note that $h \in H_0$ is endowed with commodities only in the (first and) last period of her life.
[4] Again, $h \in H_0$ consumes commodities only in the last period of her life.
[5] Agent $h \in H_0$ has a utility function $u_h \colon \Re_+^\ell \to \Re$ whose rule is $u_h(x_h) = u_h(x_h^1)$
[6] Agent $h \in H_0$ receives tax-transfers only in the last period of her life
[7] It is implicit that $y_h^0 = 0$ for $h \in H_0$.

A *perfect-foresight equilibrium* is a sequence of goods prices and asset prices and a corresponding list of equilibrium allocations and asset demands:

$$\{(p^t, q^t)\}_{t=1}^{\infty} \quad \text{and} \quad \{(x_h, y_h)\}_{h\in H}$$

such that

(i) (x_h, y_h) solves (1);

(ii) $\sum_{h\in H} x_h \leq \sum_{h\in H} \omega_h$, and

(iii) $\sum_{h\in H} Y_h \leq M$.

In a perfect foresight equilibrium, the present price of the fiat asset is a constant; hence, $q = (q^0, q^0, ...)$ for some $q^0 \in \Re_+$. Thus the real effects of each agent's lump-sum tax-transfers depend only on the sum $\mu_h = m_h^t + m_h^{t+1}$ because $q \cdot m_h = q^0(m_h^t + m_h^{t+1}) = q^0\mu_h$. Let $\mu = (\mu_h)_{h\in H}$ be the government's *generational policy*.[8] A *monetary equilibrium* is a perfect foresight equilibrium in which $q^0 > 0$ and $\mu \neq 0$. A *monetary economy* is a four-tuple $\mathcal{E} = \langle H,\ \omega = (\omega_h)_{h\in H},\ \mu = (\mu_h)_{h\in H},\ u = (u_h)_{h\in H} \rangle$ that has a monetary equilibrium. Let $\mathcal{E}(k)$ be the *k-fold replication* of $\mathcal{E}$, consisting of k identical agents for each $h \in H$. This definition includes replicating the endowment and tax-transfers of each agent in the original economy.

A *coalition* is a non-empty subset of *H*. For any coalition S and assignment $x = (x_h)_{h\in H}$,

$$F_S(x) = \{r \in X: \sum_{h\in S} z_h = r \ \textit{ for some assignment } z \textit{ such that } u_h(z_h) \geq u_h(x_h) \textit{ for all } h \in S \textit{ with strict inequality for some } h \in S\}$$

is the set of resources that allow these agents to improve upon x. A coalition V is *relevant* to another coalition S if for any feasible assignment $x = (x_h)_{h\in H}$ there exists some $a > 0$ such that $a\sum_{h\in V} \omega_h + \sum_{h\in S} x_h \in F_S(x)$. A coalition S is *irreducible* if, for any two coalitions U and V forming a partition of S, U is relevant to V. Finally, a feasible assignment $x = (x_h)_{h\in H}$ belongs to the *core* if $\sum_{h\in S} \omega_h \notin F_S(x)$ for any coalition S.

[8] This apt terminology is that of Auerbach, Gokhale, and Kotlikoff [1].

3. The Core of a Monetary Economy

I can now state my main result.

Proposition:. *There are monetary economies with generational policies satisfying* $\sum_{s=0}^{t}\sum_{h\in H_s}\mu_h > 0$ *for all* $t \geq 0$ *that support equilibrium allocations in the core.*

Proof: I will give an example of such an economy and construct a generational policy satisfying $\sum_{s=0}^{t}\sum_{h\in H_s}\mu_h > 0$ for all $t \geq 0$ that supports equilibrium allocations in the core.

Let $\# H_t = 1$ for all $t \geq 0$, and set $\ell = 2$. Let $\omega_h = ((2,1),(0,0),\ldots)$ if $h \in H_0$ and $\omega_h = ((0,0),\ldots,(0,1),(2,1),(0,0),\ldots)$ if $h \in H_t$ with $t \geq 1$. Let preferences be summarized by $u_h(x_h) = (1/2)\log(x_h^{1,1}) + (1/2)\log(x_h^{1,2})$ if $h \in H_0$ and $u_h(x_h) = (1/4)\log(x_h^{t,1}) + (1/4)\log(x_h^{t,2}) + (1/4)\log(x_h^{t+1,1}) + (1/4)\log(x_h^{t+1,2})$ if $h \in H_t$ with $t \geq 1$. Since there is only one agent per generation, I can write $\mu_t = \mu_h$ for $h \in H_t$ without confusion. Let $\mu = (1/2, -1/3, -1/9, \ldots)$ and fix $0 < q^0 < 2$. Then the unique[9] perfect foresight equilibrium is supported by $p = (p^{1,1}, p^{1,2}, \ldots, p^{t,1}, p^{t,2}, \ldots)$, with $p^{1,1} = p^{1,2} = 1$ and $p^{t,1} = p^{t,2} = (3 + q^0(2t-4)/3)(1/3)^t$ for $t \geq 2$.[10] Note the equilibrium allocations $x = (x_h)_{h\in H}$ are such that $x_h >> 0$.

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[9] Because of the assumption of log-linear preferences, uniqueness follows from arguments in Balasko and Shell [5] or Kehoe and Levine [12].

[10] Here is how to derive the supporting prices explicitly. The strong symmetry in the utility functions implies $p^{t,1} = p^{t,2}$, and the material balances condition at time $t \geq 2$ implies $p^{t,1} = c_1 + c_2(1/3)^t + \dfrac{\sum_{i=0}^{t-2} Z_{t-i} - (1/3)\sum_{i=0}^{t-2}(1/3)^i Z_{t-i}}{2/3}$, where $Z_t = -q^0(\mu_{t-1} + \mu_t)/3$ and q^0 is the constant present price of the fiat asset The constants c_1 and c_2 are determined by two initial conditions. The normalization $p^{1,1} = 1$ and material balances in period $t = 1$ implies that $c_1 = -q^0(\mu_0 + \mu_1/2)$ and $c_2 = 3(1 - c_1)$. Note that any generational policy μ satisfying $\lim_{t\to\infty}\sum_{s=0}^{t}\mu_s = 0$ and whose asymptotic behavior is dominated by the sequence $\{(1/3)^t\}_{t=0}^{\infty}$ is a candidate for a *bona fide* (see Balasko and Shell [4]) generational policy for this economy. I will use this fact in the discussion below.

Since $\lim_{t\to\infty} \inf_{s\geq t} \left\| p^s \right\| = 0$, the equilibrium allocations are Pareto optimal and cannot be improved upon by the coalition of the whole. Consider some $h \in H_t$, with $t \geq 1$, in the leading generation of some coalition $S \subset H$. Since the $(t,1)-th$ element of $\sum_{h\in S} \omega_h$ is 0, $\sum_{h\in S} \omega_h \notin F_S(x)$. Hence the equilibrium allocations are in the core. *Q.E.D.*

The importance of the example in the proof is that it is robust. The economy has a non-monetary equilibrium supporting allocations that are Pareto optimal;[11] hence, any generational debt that is introduced must satisfy $\lim_{t\to\infty} \sum_{s=0}^{t} \sum_{h\in H_s} \mu_h = 0$. But this condition does not imply that the outstanding stock of generational debt must be zero in every period (or even after finitely many periods). The basic intuition behind the proposition is that positive stocks of generational debt entail that future generations will be taxed in order to finance the current debt.[12] This is a robust and extremely important property of any economy whose non-monetary equilibrium supports Pareto optimal allocations. Hence, in an economy with only one good per period, any generation being taxed on net can block the equilibrium allocations by forming a coalition consisting of its members only. But in an economy with two goods per period, such a coalition may not improve upon the equilibrium allocations if trade between the generations within each period is sufficiently important to outweigh (slight) generational taxes needed to service the debt. Since the example consists of an economy with irreducible coalitions, the proof points towards an extension of Chae and Esteban [6, Theorem 1].

The example used in the proof is robust with respect to changes in endowments and preferences. A monetary equilibrium is in the core if the gains from trade between generations are sufficiently important.[13] Also, the example shows that the present price of the asset matters; the well-being of the initial generation is increasing in the present value of the stock of debt of which it is the beneficiary. Further, the example is robust with respect to the generational policy underlying the monetary equilibrium. Thus there is nothing special about the monetary economy described in the

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[11] The non-monetary equilibrium is a generalization of Gale's [9] classical equilibrium. It is in what Esteban and Millan [7] call the *classical set*.

[12] Note that $|q^0\mu_h / p\cdot\omega_h| \geq \varepsilon > 0$ for every $h \in H$ in the example in the proof. Hence, these taxe-transfers can be an important portion of each generation's aggregate resources.

[13] Fisher [8] has argued that gains from trade between agents in the same generation are also important in describing monetary equilibria supporting allocations in the core.

proof. Indeed, economies in which trade between generation is important and whose long-run real interest rates exceed the rate of growth of resources may be the norm, not the exception, in our world.

The arguments in Esteban and Millan [7, Proposition 7] can be used to show that the monetary equilibrium described in the proof of the proposition will not be in the core of a sufficiently large economy. But there are monetary economies with equilibria in the core that do survive replication. Consider an economy with agents, endowments, and generational policies identical to the one given in the proof of the Proposition, but let $u_h(x_h) = \log(\min\{x_h^{1,1}, x_h^{1,2}\})$ if $h \in H_0$ and $u_h(x_h) = (1/2)\log(\min\{x_h^{t,1}, x_h^{t,2}\}) + (1/2)\log(\min\{x_h^{t+1,1}, x_h^{t+1,2}\})$ if $h \in H_t$ with $t \geq 1$. An equilibrium for this economy is supported by the same sequence of prices given in the proof above, and it is possible to show that the equilibrium allocations are in the core of any replication of this economy. Such an economy does not contradict Esteban and Millan [7, Proposition 7] because the Gaussian curvature on each consumer's indifference curve evaluated at the equilibrium allocations is not bounded.[14] But these preferences and endowments create such a strong inter-dependence on trade between generations that no coalition consisting of a subset of agents from two different generations can improve upon the equilibrium allocations. Of course, this particular example is not robust because it depends so crucially on the fact that goods within each period are perfect complements in each agent's preferences.

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