Existence of Competitive Equilibrium with a system of Complete Prices

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Abstract

In classical equilibrium analysis it is assumed that consumers may purchase a given amount of good in any fraction he (she) wishes from any of a fixed number of firms. Sometimes, as with ski-lifts, at any one time it is only possible for consumers to purchase from one firm. An equilibrium price system is a complete price system. In this paper, a complete price system is a system of prices that varies between firms even though the product each firm sells may be identical. In addition as any consumer switches the firm that he (she) purchases from the price faced by all consumers may change and it is sometimes necessary to impose lump sum transfers between consumers and between firms and between consumers and firms.

Congestion is allowed. Even if there are no non-convexities associated with the choice of firm by consumers then lump sum transfers may be needed to ensure that first best allocations are supported.

The presence of non-convexities and congestion ensures that proof of existence of equilibrium is difficult. A weak proof of existence is given.

1. Introduction

In classical equilibrium analysis it is assumed that consumers may purchase a given amount of good in any fraction he (she) wishes from any of a fixed number of firms. Sometimes, as with ski-lifts, at any one time it is only possible for consumers to purchase from one firm. An equilibrium price system is a complete price system. In this paper, a complete price system is a system of prices that varies between firms even though the product each firm sells may be identical.

Barro and Romer (1987) give a simple model with a fixed and finite number of institutions providing an excludable and rival good. Unlike classical private goods, consumers may only consume the output of one institution.

In Section 2, I present the class of models for which First and Second Welfare theorems are presented in Section 4 and an existence result is presented in Section 5.

Two candidate dual spaces are considered. Each is associated with a different characterisation of the commodity space.

In the first dual space each consumer is charged the same price for each good, whatever his (her) choice of firm or the choice of firm of other consumers. The second dual space is characterised by each consumer being charged a price for each good that varies as the consumers' firm of choice varies. Further, the price each consumer faces may vary as the firm of choice of any other consumer varies.

Barro and Romer's analysis assumes that there are fewer consumer types than institutions and that consumers may be fractionated. That Barro and Romer constrain attention to markets with fewer preference types than institutions is characteristic of Bewley (1981). That Barro and Romer allow consumers to be fractionated is characteristic of Schweizer (1985, 1986). The implications of relaxing the constraint on the number of preference types and recognizing that consumers and institutions may only be indexed by integers is the focus of our work. Barro and Romer do not take into account the possibility that consumers of different preference type may choose to consume from the same institution.

Because of the quantity constraint, in the Barro and Romer model, as the integer constraint is recognized and the number of preference types is allowed to vary freely, per unit and access prices must be complete in equilibrium.

The class of private goods I introduce allows for any form of congestion in production or consumption. For instance, the utility I attain by eating at a restaurant may be increased if my girlfriend has a meal with me in the same restaurant. On the other hand the presence of my girlfriend in the restaurant may distract the waiter and lower his productivity.

Section 3 gives an example of a model that illustrates the importance of complete prices in supporting first best allocations.

Section 6 characterises complete prices and all proofs are presented in an Appendix.

2. The Models

To simplify the notation the following conventions are adopted except where confusion may arise. Denote $i \in I$ by I, $j \in J$ by J, $m \in M$ by M and $S \in Z$ by Z.

Each model consists of a finite number of firms, indexed by j in J. There are M private goods, vectors of which are denoted by $\underline{x}_i$ in $\mathbb{R}^M$.

2.1 Consumers

There are a finite number of consumers, indexed by i in I.

For each commodity there is a partition of I, I_m. Consumers may choose to purchase each commodity from only one firm but we also allow consumers to purchase different commodities from different firms. Therefore, each partition, I_m, will differ for each commodity. An *allocation* of consumers consists of the collection of partitions $\{I_m\}_M$. Denote the allocation of consumers by S. The set of all such partitions is denoted by Z.

Each consumer i has a consumption set $X_i \subset \mathbb{R}^M \times Z$. Let the aggregate consumption set be $X \equiv \sum_I X_i$. Define $X_i(S)$ and $X(S)$ to be the allocations in X_i and X associated with the partition S. Define the consumption set of consumer i and the aggregate consumption set, when commodities are indexed by the allocation of consumers, to be $X_i \equiv \prod_Z X_i(S)$ and $X \equiv \prod_Z X(S)$ where $X(S) \equiv \sum_I X_i(S)$. The consumption vector of consumer i is $x_i \equiv (\underline{x}_i; S)$. When commodities are indexed by the allocation of consumers x_i is written $x_i(S)$. Denote aggregate consumption by x.

To keep the analysis simple consumers are only endowed with one good. One might think of the endowment good as money or labor. The endowment of consumer i is represented by w_i in X_i. The aggregate endowment is w.

Preferences for each consumer i are represented by the preordering $\succeq_i$ over X_i.

2.2 Production

The production set of firm j, under S, is denoted by $Y_j(S)$ and the aggregate production set, under S, by $Y(S)$. Net output is denoted by a vector $y(S)$ in $Y(S)$. Define the production set, when commodities are indexed by the allocation of consumers, by

$Y \equiv \prod_Z Y(S)$.

Each consumer i has a shareholding θ_{ij} in firm j so the total income of consumer i is

$$I_i = pw_i + \sum_J \theta_{ij}\pi_j$$

where π_j is the profit of firm j and p is the price consumer i faces.

2.3 Prices

Each consumer faces a price $p_m \in I\!R$ for the private good m. A price vector is complete if the price each consumer faces depends upon the partition of consumers. A system of complete prices is denoted by $p = \{p_m(S)\}_{M,Z}$. The endowment good is assumed to be numeraire.

Sometimes it proves necessary to impose lump sum transfers between consumers and between firms. These lump sum transfers may change as the partition changes. Let the lump sum transfer to consumer i be τ_i. A system of lump sum transfers between consumers is a sequence $\tau^C(S) \equiv \{\tau_i(S)\}_I$, where each element of the sequence may change as the allocation of consumers changes. A system of lump sum transfers between firms is a sequence $\tau^F(S) \equiv \{\tau_j(S)\}_J$, where each element of the sequence may change as the allocation of consumers to firms changes. Of course, $\sum_I \tau_i(S) = -\sum_J \tau_j(S)$. We call such a system a system of *complete prices with lump sum transfers.*

2.4 Equilibrium

Two concepts of equilibrium are defined.

Any allocation $(\{x_i^\star\}_I, \{y_j^\star\}_J)$ is associated with some partition $S^\star$. An allocation $(\{x_i^\star\}_I, \{y_j^\star\}_J)$ is an *equilibrium relative to a price system* $p^\star$ if and only if

(1) for all $y_j \in Y_j$, $\quad p^\star y_j^\star \geq p^\star y_j$, $\quad$ for every j,

(2) for all i, if $\quad x_i \succ_i x_i^\star \quad$ then $\quad p^\star x_i > p^\star x_i^\star$,

(3) $x^\star = y^\star + w$.

An allocation $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J)$ is an *equilibrium relative to a complete price system* $\{p^\star(S)\}_Z$ with lump sum transfers $\tau^C(S)$ and $\tau^F(S)$ if and only if

(1) for all $y_j(S) \in Y_j(S)$, $\quad p^\star(S^\star)y_j^\star(S^\star) + \tau_j(S^\star) \geq p^\star(S)y_j(S) + \tau_j(S)$, $\quad$ for every j,

(2) for all i, if $\quad x_i(S) \succ_i x_i^\star(S^\star) \quad$ then $\quad p^\star(S)x_i(S) + \tau_i(S) > p^\star(S^\star)x_i^\star(S^\star) + \tau_i(S^\star)$,

(3) $x^\star(S^\star) = y^\star(S^\star) + w.$

2.5 Pareto Optimal Allocations

The *preferred, worse than* and *strictly preferred* sets of consumer i at allocation x are respectively defined as

$$R_i(x) \equiv \{z \in X_i |\ z \succeq_i x\}, \quad L_i(x) \equiv \{z \in X_i |\ x \succeq_i z\}, \quad P_i(x) \equiv \{z \in X_i |\ z \succ_i x\}.$$

Let $R(x^\star(S^\star)|S) \equiv \sum_I R_i(x_i^\star(S^\star)|S)$. Define

$$G_i'(S) \equiv P_i(x_i^\star(S^\star)|S) + \sum_{I \setminus i} R_i(x_i^\star(S^\star)|S) - \sum_J Y_j(S).$$

A *Pareto optimal* allocation is given by a list $(\{x_i^\star\}_I, \{y_j^\star\}_J)$ which meets the conditions

(1) for all$j, y_j^\star \in Y_j$ and for all $i,\ x_i^\star \in X_i$,

(2) $z \in G_i'(S)$, for some i, implies $z \notin Y(S) + \{w\}$,

(3) $x^\star(S^\star) = y^\star(S^\star) + w.$

Define $R_i(x|S)$, $L_i(x|S)$ and $P_i(x|S)$ to be the allocations in $R_i(x)$, $L_i(x)$ and $P_i(x)$ associated with the partition S.

3. Example

In Example 3.1, supporting prices are found for a Pareto optimal allocation in a model with two consumers. It is shown that "migration", from one firm to another, will necessitate a change in the relative per unit prices faced by at least one consumer and may necessitate a change in the relative per unit prices faced by both consumers.

Example 3.1 There are two consumers, indexed by $i \in I \equiv \{1,2\}$; two firms, indexed by $j \in J \equiv \{1,2\}$ and two private goods; first leisure, denoted by $l \in I\!R$, and ski-runs, denoted by $x \in I\!R$. The consumption vector of consumer i is

$$x_i = (\{(x_{ij}, l_{ij})\}_J; S)$$

where x_{ij} is consumer i's consumption of ski-runs from institution j and l_{ij} is consumer i's consumption of leisure when consuming ski-runs from institution j.

The model is endowed with two units of leisure. The preference preordering of consumer 1 is represented by $U^1(x,l) = x^2 l$ and the preference preordering of consumer 2 is represented by $U^2(x,l) = xl^2$.

Without loss of generality, consider the two partitions S_1 and S_2; respectively associated with both consumers purchasing from the first firm and consumer 1 purchasing from firm 1 and consumer 2 purchasing from firm 2. Let $Z \equiv \{S_1, S_2\}$. Each consumer has a consumption set $X_i = I\!R_+ \times [0,1] \times Z$.

The corresponding production sets are

$$Y(S_1) + w = \{y \in I\!R^4 \times Z |\ y = ((x_1, 2), (0,0); S_1) \text{ and } x_1 = 1\},$$

$$Y(S_2) + w = \{y \in I\!R^4 \times Z |\ y = ((x_1, 1), (x_2, 1); S_2) \text{ and } x_1 = 1, x_2 = 1\},$$

where $Y = \prod_Z Y(S_k)$.

A weakly efficient allocation is

$$x_1^\star = ((1,1),(0,0); S_2), \quad x_2^\star = ((0,0),(1,1); S_2).$$

At the weakly efficient allocation $(x_1^\star, x_2^\star)$ consumers enjoy utility of $U^1(x_1^\star) = 1, U^2(x_2^\star) = 1$. The weakly preferred set is $P(x_1^\star, x_2^\star) = \prod_Z P(x_1^\star, x_2^\star | S_k)$ where

$$P(x_1^\star, x_2^\star | S_1) = \{x \in X(S_1) | \; x = x_1 + x_2; \; x_{11}^2 l_{11} > 1, x_{21} l_{21}^2 > 1\},$$

$$P(x_1^\star, x_2^\star | S_2) = \{x \in X(S_2) | \; x = x_1 + x_2; \; x_{11}^2 l_{11} > 1, x_{22} l_{22}^2 > 1\}.$$

Restrict attention to the set of all complete prices with no lump sum transfers. Let the set of admissible separating prices under each partition; $S_k, \; k = 1,2$; be denoted by $\triangle^C(S_k)$. Pick leisure to be numeraire.

$$\triangle^C(S_k) \equiv \{p \in I\!R^4 | \; p = ((p_1, 1), (p_2, 1))\}.$$

Clearly,

$$p(S_1) = ((2,1),(2,1)) \in \triangle^C(S_1)$$

$$\triangle^C(S_2) = \{p \in I\!R^4 | \; p = ((4/3, 1), (2/3, 1))\}.$$

Therefore $\triangle^C(S_1) \cap \triangle^C(S_2) = \emptyset$ and complete prices are needed to support the allocation $(x_1^\star, x_2^\star)$.

4. Welfare Theorems

In this section First and Second Welfare Theorems are given for the class of models described in Section 2.

Example 3.1 illustrated how the introduction of complete prices ensures that first best allocations may be implemented. The efficiency of equilibrium relative to a complete price system, is demonstrated for the class of economies that satisfy the following assumption, in Theorem 1;

Assumption 1 *for every i and for every S, $X_i(S)$ is convex*

Assumption 2 *for every i and for every S, if $x_i^1(S)$ and $x_i^2(S)$ are two points of $X_i(S)$ and if t is a real number in (0,1) then $x_i^1(S) \succ_i x_i^2(S)$ implies $t x_i^1(S) + (1-t) x_i^2(S) \succ_i x_i^1(S)$.*

Theorem 1 *(First Welfare Theorem) Under 1 and 2, any equilibrium relative to a price system, a complete price system or a complete price system with lump sum transfers is Pareto optimal.*

That any weakly efficient allocation $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J)$ may be implemented as an equilibrium relative to a complete price system is demonstrated for the class of economies that satisfy the following assumptions, in Theorem 2:

Assumption 3 *for every S,* $R_i(x_i^\star(S^\star)|S)$ *and* $L_i(x_i^\star(S^\star)|S)$ *are closed in* $X_i(S)$.

Assumption 4 *for every S, Y(S) is convex.*

Theorem 2 *(Second Welfare Theorem) Under 1 through 4, if* $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J)$ *is Pareto optimal, there exists a price vector* $\{p^\star(S)\}_Z$, $\{\tau^C(S)\}_Z$ *and* $\{\tau^F(S)\}_Z$ *such that* $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J, \{p^\star(S)\}_Z, \{\tau^C(S)\}_Z, \{\tau^F(S)\}_Z)$ *is an equilibrium relative to a complete price system with lump sum transfers.*

Let $A(Y)$ be the asymptotic cone of Y, that is, $A(Y) \equiv \{y|\ \alpha y \in Y \text{ for all } \alpha \geq 0\}$.

Assumption 5 *for every S,* $R(x^\star(S^\star)|S) \cap \{A(Y(S)) + w\} \neq \emptyset$.

Corollary 3 *(Second Welfare Theorem) Under 1 through 5, if* $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J)$ *is Pareto optimal, there exists a price vector* $\{p^\star(S)\}_Z$, *and lump sum transfers* $\{\tau^C(S)\}_Z$ *and* $\{\tau^F(S)\}_Z$ *such that* $\sum_I \tau_i(S) = 0$ *and* $\sum_J \tau_j(S) = 0$ *and such that* $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J, \{p^\star(S)\}_Z, \{\tau^C(S)\}_Z, \{\tau^F(S)\}_Z)$ *is an equilibrium relative to a complete price system with lump sum transfers.*

If, in addition, assumption 6 holds we can constrain the dual space to a system of prices that is independant of partition.

Assumption 6 *for every S,* $Y(S) \subseteq Y(S^\star)$ *and* $R(x^\star(S^\star)|S) \subseteq R(x^\star(S^\star)|S^\star)$.

Corollary 4 *(Second Welfare Theorem) Under 1 through 4 and 6 if* $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J)$ *is Pareto optimal, there exists a price vector* $p^\star$, *such that* $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J, p^\star)$ *is an equilibrium relative to a price system.*

5. Existence of Equilibrium

An allocation $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J)$ is an *equilibrium* at complete prices $\{p^\star(S)\}_Z$ if and only if

(1) for all $y_j(S) \in Y_j(S)$, $p^\star(S^\star)y_j^\star(S^\star) + \tau_j(S^\star) \geq p^\star(S)y_j(S) + \tau_j(S)$, for every j,

(2) for all i, $p^\star(S^\star)x_i^\star(S^\star) + \tau_i(S^\star) \leq w_i + \sum_J \theta_{ij} p^\star(S^\star)y_j^\star(S^\star) + \tau_j(S^\star)$
and for all $x_i(S)$ such that

$x_i(S) \succ_i x_i^\star(S^\star), \quad p^\star(S)x_i(S) + \tau_i(S) > p^\star(S^\star)x_i^\star(S^\star) + \tau_i(S^\star),$

(3) $x^\star(S^\star) = y^\star(S^\star) + w.$

Generally, the proof of existence for the class of models in Section 2 is difficult. The reasons are presented in Manning (1993a) and (1993b) in the context of a class of models with public goods. They are repeated here, in part, for completeness. Commodities are indexed by the partition with which they are associated. In this commodity space the projection of the net production set, $\Pi_Z Y(S)$, into the private good subspace is star convex relative to the endowment point. Production may never occur under more than one partition. This generates discontinuities in the demand and supply correspondences as prices change. As prices change consumers can change the firm they purchase from. As a consumer "migrates" his (her) demand correspondence for the output of the former firm takes on the value zero. Therefore the aggregate demand and supply correspondences under the former partition take on the value zero.

However, a technique for avoiding such discontinuities suggests itself. To prove the existence of an Equilibrium at complete prices, given the Second Welfare Theorem holds, it is sufficient to prove that, for some Pareto optimal allocation $\{x_i^\star(S^\star)\}_I$,

$$\text{for every } i, \quad p^\star(S)x_i^\star(S^\star) \leq w_i + \sum_J \theta_{ij} p^\star(S^\star) y_j^\star(S^\star). \tag{1}$$

$\{p^\star(S)\}_Z$ is a selection from the set of complete prices that support the Pareto optimal allocation $\{x_i^\star(S^\star)\}_I$ in the sense defined in Section 2.

If all Pareto optimal allocations are associated with one partition, the discontinuities associated with the search for a Pareto optimal allocation that satisfy (1) can be avoided.

For an equilibrium at complete prices to exist it is sufficient that 2 through 5 and the following assumptions hold.

Assumption 7 *$S^\star$ is unique.*

Assumption 8 *For every i, $X_i(S^\star)$ is bounded below in $\leq$.*

Assumption 9 *For every i, $X_i(S^\star)$ is closed.*

Assumption 10 *For every i and for every $x_i^\star(S^\star)$ in $X_i(S^\star)$, there is an $x_i(S^\star)$ in $X_i(S^\star)$ preferred to $x_i^\star(S^\star)$.*

Assumption 11 *The relative interiors of $\{Y(S^\star)+\{w\}\}$ and $X(S^\star)$ have a non-empty* *intersection.*

Assumption 12 *$Y(S^\star)$ is closed.*

Assumption 13 $Y(S^\star) \cap I\!R^M_+ = \{0\}$.

Theorem 5 *(Existence) Under through, there exists an allocation $(\{x^\star_i(S^\star)\}_I, \{y^\star_j(S^\star)\}_J)$ that is an equilibrium at a price $p^\star$.*

Appendix

Proof of Theorem 1: Let $G(S) \equiv \sum_I R_i(x^\star_i(S^\star)|S) - \sum_J Y_j(S)$. Let $(\{x^\star_i(S^\star)\}_I, \{y^\star(S^\star)\}_J, \{p^\star(S)\}_Z$ be an equilibrium relative to a complete price system.

(i) Since the conditions of (2) of 4.9 (Debreu 1959) are satisfied it follows from (1) that $x^\star_i(S^\star)$ minimises $p.a$ on $R_i(x^\star_i(S^\star)|S^\star)$. Further, from (2), $-y^\star_j(S^\star)$ minimises $p.a$ on $-Y_j(S^\star)$. Therefore, by (1) of 3.4 (Debreu 1959), $\sum_I x^\star_i(S^\star) - \sum_J y^\star_j(S^\star), (= w)$, minimises $p.a$ on $G(S^\star)$ (and so w is on the boundary of $G(S^\star)$).

Let $(\{x_i(S)\}_I, \{y_j(S)\}_J)$ be an attainable state such that $x_i(S) \succ_i x^\star_i(S^\star)$, for all i.

Since $x(S) - y(S) = w$, the point $\sum_I x_i(S) - \sum_J y_j(S)$ has no value no greater than the minimum of $p(S).a(S)$ on $G(S)$. $p(S) \sum_I x_i(S) \leq p(S^\star) \sum_I x^\star_I(S^\star)$, by (1). The lump sum transfers are chosen such that $p(S)x_i(S) \leq p(S^\star)x^\star_i(S^\star)$ and so, by (2), $x^\star_i(S^\star) \succ_i x_i(S)$.

(ii) Conditions (1) and (3) are implied directly by the conditions of an equilibrium relative to a complete price system.

Similarly for any equilibrium relative to non-complete prices and prices. *Q.E.D.*

Proof of Theorem 2:

Since the state $(\{x^\star_i(S^\star)\}_I, \{y^\star_j(S^\star)\}_J)$ is Pareto optimal, w does not belong to $G'_i(S)$, for every S. It follows from 1, 2 and 3 that the sets $P_i(x^\star_i(S^\star)|S)$ and $R_i(x^\star_i(S^\star)|S)$ are convex, for every S. Hence $G'_i(S)$ is convex as the sum of convex sets, for every S. Thus, by Minkowski's theorem, there is a hyperplane $H(S)$ through w, bounding for $G'_i(S)$, for every S i.e. there is a $p(S)$ in $I\!R^M$ different from 0 such that $p(S)a(S) \geq w$ for every $a(S)$ in $G'_i(S)$ and every S.

The set $G(S)$ is contained in the adherence of $G'_i(S)$ and hence in the closed half-space above the hyperplane $H(S)$, for every S. Since the point w belongs to $G(S^\star)$ it minimises $p(S^\star)a(S^\star)$ on $G(S^\star)$. It follows from $w = \sum_I x^\star_i(S^\star) - \sum_J y^\star_j(S^\star)$ that $x^\star_i(S^\star)$ minimises $p(S^\star)a(S^\star)$ on $R_i(x^\star_i(S^\star)|S^\star)$ for every i and $y^\star_j(S^\star)$ maximises $p(S^\star)a(S^\star)$ on $Y_j(S^\star)$ for every j.

We may write

$$p(S^\star)x^\star(S^\star) = p(S^\star)y^\star(S^\star) + w \quad \text{and} \quad p(S)x(S) \geq p(S)y(S) + w,$$

for every $S \neq S^\star$. There exists an $r(S)$ such that $p(S)x(S) \geq r(S)$ and $p(S)y(S) + w \leq r(S)$, for every S.

If $Y(S)$ is a cone and $p(S)y(S) > 0$ then $y(S)$ can be expended indefinitely implying an unbounded profit. Therefore, $p(S)y(S) \leq 0$. Since, in addition, $\sum_I \tau_i(S) = 0$ and $\sum_J \tau_j(S) = 0$, for every S, we can pick $r(S) = w$ for every S.

If $Y(S)$ is not a cone, then it may not be possible to pick prices under each partition such that the total profit of all firms under each partition is the same. If the total profit differs between partitions then it is necessary to impose lump sum transfers between consumers and firms. If it is possible to pick prices under each partition such that total profit is the same under each partition then we can set $r(S) = \pi + w$, for every S.

Therefore

$$p(S)x(S) \geq p(S^\star)x^\star(S^\star) \quad \text{and} \quad p(S)y(S) \leq p(S^\star)y^\star(S^\star).$$

To show that

$$p(S)x_i(S) \geq p(S^\star)x_i^\star(S^\star) \quad \text{and} \quad p(S)y_j(S) \leq p(S^\star)y_j^\star(S^\star)$$

note that the net payments $\{\tau_i(S)\}_I$ and $\{\tau_j(S)\}_J$ may be of the same sign. Therefore, condition (1) and (2) of an equilibrium relative to a complete price system with lump sum transfers is satisfied.

Condition (3) of an equilibrium relative to a complete price system with lump sum transfers is immediate by the definition of Pareto optimality. Q.E.D.

Proof of Corollary 3: Since $R(x^\star(S^\star)|S) \cap \{A(Y(S)) + w\} \neq \emptyset$ for every S, it is possible to pick prices such that total profit is zero under each partition. In this case we can let $r(S) = w$ for every S and the Corollary follows. Q.E.D.

Proof of Corollary 4: If assumption 6 holds there exists a common hyperplane through w, bounding for $G_i'(S)$, for every S. Q.E.D.

Proof of Theorem 5: Under 1 through 4 the Pareto optimal allocation $(\{x_i^\star(S^\star)\}_I, \{y_j^\star(S^\star)\}_J)$ has a supporting price. By assumption the value of each consumers endowment of the private good is invariant to the partition with which his (her) residence is associated.

By 7 any Pareto optimal allocation is associated with a unique partition $S^\star$. Therefore, to prove the existence of an equilibrium it is sufficient to prove the existence of an equilibrium under the partition $S^\star$.

Existence under the partition $S^\star$ follows as 1 through 4, 6 adn 8 through 13 imply the assumptions used to prove existence by Debreu (1962). 8 implies a.1, 1 and 9 imply a.2, 10 implies b.1, 3 implies b.2, 2 implies b.3, 11 implies c.1, 4 and 12 imply d.1 and 13 implies d.2. Q.E.D.

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