A concretization of Mas-Colell and Zame's counter-example

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1 Introduction

The incomplete markets equilibrium existence theorem for economies with uncountably many states has been studied by several authors.However the following hypothesis is made:

$$\text{for every } \theta \geq v_i \; W_s^i + \theta A_s \geq 0 \text{ for almost every } s \in S$$

where S is the set of states,$W^i : S \to \Re_+^G$ is the initial endowments of consumer $i, 1 \leq i \leq I, \theta \in \Re^J$ is the vector of portfolios whose uncovered sales lower bound is $v_i \in -\Re_{++}^J$.Finally $A : S \to (\Re^G)^J$ is the matrix of real assets returns .This hypothesis is considered very strong. Mas-Colell and Zame(1991) by means of a counter-example demonstrated its indispensability .In the construction of theirs counter-example, they required the existence of utility functions satisfying several properties. However one may wonder which utilities functions satisfy their requirements.My purpose here is to specify such functions. The utilities functions I've got doesn't fit exactly in Mas-Colell and Zame's hypotheses.But there is not an equilibrium as well.

2 The utilities and endowments

First choose number $(\bar{x}, a, \beta, \phi)$:

1. $0 < \bar{x} < a < 1$
2. $\beta > (a+1)/a$,$\phi > \frac{1+a}{(a-\bar{x})(1+\bar{x})}$
3. $\frac{\beta}{\phi} + \frac{1-\beta(1+\bar{x})}{1+\bar{x}}(a-\bar{x}) > 0$
4. $\phi e^{\phi(\bar{x}-1)} > \beta e^{-\beta\bar{x}}\frac{1+\bar{x}}{a-\bar{x}}$
5. $-2+\beta(1+\bar{x}) \geq 0$

For example $\bar{x} = 3/4$,$a = 7/8$,$\phi = 121/14$,$\beta = 605/84$
The utilities(state independent) are as follows:
$u^1(x,y) = -e^{-\beta x} + g(y)$ and $u^2(x,y) = w(x) - e^{-\phi y}$ where

$$g(y) = \begin{cases} \beta\int_0^y e^{-\beta s}\frac{1+s}{a-s}ds & 0 \leq y \leq \bar{x} \\ \beta\int_0^{\bar{x}} e^{-\beta s}\frac{1+s}{a-s}ds + \beta e^{-\beta\bar{x}}\frac{1+\bar{x}}{a-\bar{x}}(y-\bar{x}) & \bar{x} \leq y \end{cases}$$

$$w(x) = \begin{cases} \phi\int_0^{1-\bar{x}} e^{-\phi s}\frac{a-1+s}{2-s}ds + \phi e^{-\phi(1-\bar{x})}\frac{a-\bar{x}}{1+\bar{x}}(x-1+\bar{x}) & 0 \leq x \leq 1-\bar{x} \\ \phi\int_0^{x} e^{-\phi s}\frac{a-1+s}{2-s}ds & 1-\bar{x} \leq x \leq 1 \\ \phi\int_0^{1} e^{-\phi s}\frac{a-1+s}{2-s}ds + \phi e^{-\phi}a(x-1) & 1 \leq x \end{cases}$$

The Pareto optimum set is as below. We have that u^i is concave and strictly

monotone(u^i is concave if (ii) is true).
Define $U^i(x_0,x,y) = x_0 + \int_0^1 u^i(x_s,y_s)ds$, $x_0,x_s,y_s \in \Re_+$. U^i is concave and strictly monotone.The endowments are $W_s^1 = (s,a)$, $W_s^2 = (1-s,1-a)$,$W^1+W^2 = (1,1)$.There is only one asset $A = (1,0)$.

The consumer problem is

$$\max\ x_0 + \int_0^1 u^i(x_s,y_s)ds$$

subject to $w^i \geq x_0 + \pi\theta$, $\theta \geq v_i$ and to $(1, q_s)(W_s^i + \theta A) \geq x_s + q_s y_s$ This problem is equivalent to

$$\max -\pi\theta + \int_0^1 v_s^i((1, q_s)(W_s^i + \theta A))ds$$

subject to $w^i \geq \pi\theta$ and $\theta \geq v_i$ and to $(1, q_s)(W_s^i + \theta A) \geq 0$
where $v^i(R)$ is the indirect utility at prices $(1, q_s)$ and income R. An equilibrium is a function $q : S \rightarrow \Re_+, \pi \in \Re$ and numbers $(\bar{\theta}^i)_{i=1,2}$ such that $\bar{\theta}^1 + \bar{\theta}^2 = 0$ and $W_s^1 + W_s^2 = \sum_{i=1}^2 x_i((1, q_s), (1, q_s)(W_s^i + \bar{\theta}^i A))$ where $x_i(\cdot, \cdot)$, is the Marshallian demand,solves $\max x^0 + \int_0^1 u^i(x_s)ds$ subject to $x^0 + \pi\theta \leq w^i, \theta \geq v_i, (1, q_s)(W_s^i + \theta A) \geq 0$.

3 The Pareto optimum points of (u^1, u^2).

Suppose $\lambda \in (0, 1)$. I want to find the solutions of

$$\max_{0 \leq x \leq 1, 0 \leq y \leq 1} \lambda(-e^{-\beta x} + g(y)) + (1 - \lambda)(w(1 - x) - e^{-\phi(1-y)})$$

Define $\phi(x) = -\lambda e^{-\beta x} + (1 - \lambda)w(1 - x)$ and $\psi(y) = \lambda g(y) - (1 - \lambda)e^{-\phi(1-y)}$. It is easy to check that both functions are strictly concave.Suppose α is the solution of $\max_{0 \leq x \leq 1} \phi(x)$. We have:
$\alpha \leq \bar{x} \Leftrightarrow \phi'(\bar{x}) \leq 0 \Leftrightarrow \frac{\lambda\beta}{(1-\lambda)\phi} \leq e^{(\beta+\phi)\bar{x}-\phi}\frac{a-\bar{x}}{1+\bar{x}}$
The first order conditions are the same for ϕ and ψ if $\frac{\lambda\beta}{(1-\lambda)\phi} \leq e^{(\beta+\phi)\bar{x}-\phi}\frac{a-\bar{x}}{1+\bar{x}}$.Therefore in this case ϕ and ψ have the same maximizer $\alpha \leq \bar{x}$.If $\frac{\lambda\beta}{(1-\lambda)\phi} \geq e^{(\beta+\phi)\bar{x}-\phi}\frac{a-\bar{x}}{1+\bar{x}}$ then call u the maximizer of ϕ and v the maximizer of ψ. We have that both are not smaller than $\bar{x}$.The first order condition imply:
$\lambda\beta e^{-\beta u} = (1 - \lambda)\phi e^{-\phi(1-\bar{x})}\frac{a-\bar{x}}{1+\bar{x}}$, $\beta\lambda e^{-\beta\bar{x}}\frac{1+\bar{x}}{a-\bar{x}} = \phi(1 - \lambda)e^{-\phi(1-v)}$.
Then dividing one by the other and cancelling terms in common and finally taking logarithms we have

$$\frac{\beta}{\phi}(u - \bar{x}) + \bar{x} = v$$

The supporting price of the allocation (x, x) is $p(x) = \frac{1+x}{a-x}$(I take the first price coordinate equal to one).For $x \geq \bar{x}$ we have $p(x) = e^{\beta(x-\bar{x})}\frac{1+\bar{x}}{a-\bar{x}}$. For $x = -1$ let Y be the meeting point of the line passing through (u, v) with inclination $-p(u)$: $(1, p(u))(-1, Y) = (1, p(u))(u, \frac{\beta}{\phi}(u - \bar{x}) + \bar{x})$. Then

$$Y(u) \quad = \quad \frac{(u + 1)(a - \bar{x})}{(1 + \bar{x})}e^{\beta(u-\bar{x})} + \frac{\beta}{\phi}(u - \bar{x}) + \bar{x}$$

$$
\begin{aligned}
Y'(u) &= \frac{a-\bar{x}}{1+\bar{x}}\{e^{-\beta(u-\bar{x})}-(u+1)\beta e^{-\beta(u-\bar{x})}\}+\beta/\phi \\
Y''(u) &= \frac{a-\bar{x}}{1+\bar{x}}\beta e^{-\beta(u-\bar{x})}\{-2+\beta(u+1)\}>0
\end{aligned}
$$

Therefore since $Y'(\bar{x})>0$ we have that $Y(u)>Y(\bar{x})=a$ if $u>\bar{x}$.So every budget line supporting a Pareto point after $\bar{x}$ do not touch the line $x=-1$ below a[1]

Now suppose $(\bar{x}_0^i,\bar{x}^i,\bar{y}^i,\bar{\theta}^i,\pi,q_s)$ is an equilibrium.We have that $\bar{\theta}^1+\bar{\theta}^2=0$, $\bar{x}_0=w^i-\pi\bar{\theta}^i$,$(1,q_s)(\bar{x}_s^i,\bar{y}_s^i)=(1,q_s)(W_s^i+\bar{\theta}^iA)$ a.e. s. We have also $u^i(\bar{x}_s^i,\bar{y}_s^i)=\nu_s^i((1,q_s)(W_s^i+\bar{\theta}^iA))$,and $q_s=\frac{\partial u^i/\partial y}{\partial u^i/\partial x}$. Necessarily $W_{1s}^1+\bar{\theta}^1>-1$ a.e. s since there are not supporting prices passing through $(x,a),x<-1$. Therefore $\bar{x}_s^i>\bar{x}$,$\bar{y}_s^i>\bar{x}_i$ a.e. s. Define

$$h^i(\theta)=-\pi\theta+\int\nu_s^i((1,q_s)(W_s^i+\theta A))ds$$

$$Dh^i(\theta)=-\pi+\int\frac{\partial\nu_s^i}{\partial R}ds$$

$$Dh^i(\bar{\theta}^i)=-\pi+\int\frac{\partial u^i}{\partial x}(\bar{x}_s^i)ds$$

At the optimum we have:

$$Dh^1(\bar{\theta}^1)=-\pi+\int\beta e^{-\beta\bar{x}_s^1}ds\leq-\pi+\int\beta e^{-\beta\bar{x}}ds=-\pi+\beta e^{-\beta\bar{x}}$$

and

$$Dh^2(\bar{\theta}^2)=-\pi+\int w'(\bar{x}_s^2)ds=-\pi+\phi e^{-\phi(1-\bar{x})}\frac{a-\bar{x}}{1+\bar{x}}>Dh^1(\bar{\theta}^1)$$

The proof now is divided in two cases:

a) $\bar{\theta}^1\geq 0$

Then $Dh^1(\bar{\theta}^1)=0$ or $Dh^1(\bar{\theta}^1)\geq 0$ and $w^1-\pi\bar{\theta}^1=0$. In any case this implies $Dh^2(\bar{\theta}^2)>0$.So consumer 2 wants to increase $\bar{\theta}^2$. This is possible if $\bar{\theta}^2\leq 0$. A contradiction.

b) $\bar{\theta}^1<0$.

The income inequality is not binding,since at $(\bar{x},\bar{x})$ the income has a strictly positive lower bound.The inequality $w^1-\pi\bar{\theta}^1\geq 0$ is not binding also.Therefore

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[1] for $x=1,\frac{\beta}{\phi}(1-\bar{x})+\bar{x}\leq y\leq 1,$ we have that $Y\geq Y(1)>a$.

we have $Dh^1(\bar{\theta}^1) = 0$, hence $\pi = \beta e^{-\beta\bar{x}}$. Also $Dh^2(\bar{\theta}^2) > 0$.So $w^2 - \pi\bar{\theta}^2 = 0$,so $\bar{\theta}^1 = -w^2/(\beta e^{-\beta\bar{x}})$. Now we choose $v_1 < \frac{-w^2}{\beta e^{-\beta\bar{x}}}$ and w^2 such that

$$-\frac{w^2}{\beta e^{-\beta\bar{x}}} < -1$$

This contradicts the fact that $s + \bar{\theta}^1 > -1$ for almost every s.

References

Mas-Colell,A,W.Zame(1991),The existence of security markets equilibria with a non-atomic state space,to appear in Journal of Mathematical Economics.