EXISTENCE OF EQUILIBRIA WITH INFINITELY MANY GOODS, INCOMPLETE MARKETS AND BANKRUPTCY

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I. Introduction

The existence of equilibria with infinitely many goods and incomplete markets have been studied by many authors. The continuum of states case was studied by Mas-Colell and Monteiro (1990), Hellwig (1991), Mas-Colell and Zame (1991) and Monteiro (1991). Separable utilities and a lower bound on uncovered asset sales are common assumptions. However the strongest assumption is that portfolio returns can be covered by the initial endowments, i.e., $W_s^i + \sum_j y_j r_s^j \geq 0$ for almost every s and all admissible portfolios y. A counter-example in Mas-Colell and Zame (1991) shows that without this assumption (but still with a bound on uncovered sales) an equilibrium may not exist.

Our plan in this paper is to eliminate this awkward assumption with the introduction of bankruptcy. The idea of bankruptcy have been considered recently by many authors. Our context is closer to Dubey, Geanakoplos and Shubik (1988), which shows the advantage of the introduction of bankruptcy in many situations. Our model is also close to

Araujo-Páscoa (1993) which shows that, with bankruptcy, incomplete markets equilibria might be locally determinate. Others papers that work with bankruptcy includes Dubey, Geanakoplos (1989) and Zame (1993).

However, as shown in the paper, the introduction of bankrupty might introduce trivial equilibria that is: equilibria without transaction in the financial markets. To avoid this one has to impose high penalties which also might make the bankruptcy equilibria equal to the incomplete markets equilibria. This raise doubts about the usefulness of bankrupcty in the finite number of states of the world case. However, in the infinite case the situation is different since we are able to show existence even in cases where there is no equilibria in the incomplete markets case.

In the second section we introduce basic definitions and auxiliary propositions. In the third section we present the model and the fourth section is dedicated to the finite dimensional approximations. In the fifth section the main theorems proof is given. Finally in the sixth section the financial returns case is considered.

II. Some notation and auxiliary results

We denote by $L^\infty = L^\infty(0,1)$ the space of essentially bounded Lebesgue's measurable functions $f:[0,1] \to \mathbf{R}$. Analogously $L^1 = L^1(0,1)$.

If $x \in \mathbf{R}$, $x^+ = \max\{0,x\}$ is the positive part of x and $x^- = \max\{0,-x\}$ is the negative part of x. If $x \in \mathbf{R}^n$, $|x|_1$ denote $\sum_{i=1}^{n} |x_i|$ and if $f \in L^1$, $|f|_1 = \int |f(s)|\, ds$.

We define the simplex $\Delta = \{x \in \mathbf{R}^n_+; |x|_s = 1\}$.

The following proposition is a particular case of the multi-dimensional Fatou's lemma.

Proposition 1: Suppose $(f_n)_{n\geq 1}$ is an uniformly integrable sequence of functions $f_n\colon [0,1] \to \mathbf{R}^\ell$. Suppose also that $\lim_{n\to\infty} \int f_n(s)\,ds$ exists. Then there is an integrable function $g\colon [0,1] \to \mathbf{R}^\ell$ such that

i) $\int g(s)\,ds = \lim_{n\to\infty} \int f_n(s)\,ds$

ii) $g(s)$ is a limit point of $\{f_n(s)\}_{n\geq 1}$ for almost every $s \in [0,1]$.

Proof: I refer the reader to Arstein (1979) or Hildenbrand-Mertens (1971).

III. The model

The set of states is $S = [0,1]$. There are I consumers, J assets and G goods. The economy has two periods. In the first period there is no consumption, only buying and selling of assets $\theta \in \mathbf{R}^J$ at prices $\pi \in \mathbf{R}^J$. In the second period occurs the state of nature $s \in S$, there is consumption $x_s \in \mathbf{R}^G_+$ and delivery of real assets returns, A^j_s, $1 \leq j \leq J$.

If θ is the portfolio we write $\theta = y - z$, where $y_j \geq 0$ is the number of units of asset j the consumer bought and $z_j \geq 0$ is the number of units he sold. We allow that $y_j \cdot z_j \neq 0$. This may happen if the consumer want in some states to default more than the market default rate"$^{(*)}$"this hypotheses is easy to avoid if desired..

Suppose the prices at state s are $p_s \in \Delta$. The consumer debt will be $p_s A^j_s z_j$. Of this debt, he pays D_{sj}, $0 \leq D_{sj} \leq p_s A^j_s z_j$.

For each state s and asset j there is a market payment rate $t^j_s \in [0,1]$.

Consumer's i utility function is composed of two parts: his pleasure, $\int u^i_s(x_s)ds$, derived from consumption of goods $x\colon S \to \mathbf{R}^G_+$ and his penalty $\sum_{j=1}^{J} \int \lambda^j_s (p_s A^j_s z_j - D_{sj})^+ \, ds$, which increases in proportion of his unpaid debt $p_s A^j_s z_j - D_{sj}$. Precisely the economy $\mathcal{E} = (U^i, W^i, A^j, t^j)_{\substack{i \leq I \\ j \leq J}}$ is as folllows:

(Lebesgue's measurability is supposed)

i) $W^i\colon S \to \mathbf{R}^G_+$, $1 \leq i \leq I$

ii) $A^j\colon S \to \mathbf{R}^G_+$, $1 \leq j \leq J$

iii) $t\colon S \to [0,1]^J$

iv) $U^i(x,z,p,D) = \int_S u^i_s(x_s)ds - \sum_{j=1}^{J} \int \lambda^j_j (p_s A^j_s z_j - D_{sj})^+ \, ds$ defined

for $x \in (L^\infty_+)^G$, $z \in \mathbf{R}^J_+$, $p: S \to \Delta$, $D: S \to \mathbf{R}^J_+$ where $\lambda^j \in L^\infty_{++}$ is the penalty rate.

There is a lower bound on assets short sales, $v_i \in -\mathbf{R}^J_{++}$. An equilibrium for $\mathcal{E}$ is a vector $(\overline{p}, \overline{\pi}, \overline{t}, \overline{y}^i, \overline{z}^i, \overline{D}^i, \overline{x}^i)_{i \le I}$ such that

a) $\overline{p}: S \to \Delta$

b) $\overline{\pi} \in \mathbf{R}^J$, $|\overline{\pi}|_1 = 1$

c) $\sum_i (\overline{y}^i - \overline{z}^i) = 0$

d) $\overline{D}^i: S \to \mathbf{R}^J_+$

e) $\overline{t}: S \to [0,1]^J$

f) $\overline{x}^i: S \to \mathbf{R}^G_+$, $\sum_i (\overline{x}^i - W^i) = 0$

g) $t_s \cdot \sum_i \overline{p}_s A^j_s \overline{y}^i_j = \sum_i \overline{D}^i_{sj} \quad 1 \le j \le J$, a.e. s

h) $(\overline{x}^i, \overline{y}^i, \overline{z}^i, \overline{D}^i)$ maximize $U^i(x, z, \overline{p}, D)$ subject to

1) $x \in (L^\infty_+)^G$, $y, z \in \mathbf{R}^J_+$, $v_i \le -z$

2) $\overline{p}_s(x_s - W^i_s) \le \sum_j t^j_s \overline{p}_s A^j_s y_j - \sum_j D_{sj}$ a.e. s

3) $\overline{\pi}(y - z) \le 0$

i) $\{s; t^j_s > 0\}$ is non-null.

The condition (i) is to avoid the trivial equilibrium $t^j = 0$, $\overline{y}^i = \overline{z}^i = 0$, $\overline{D}^i = 0$.

We say that an equilibrium is a non-trade equilibrium if $\overline{y}^i = \overline{z}^i = 0$ for every i. An equilibrium will be called non-trivial if there is trade or if it is a non-trade equilibrium with $t^j \equiv 1$ for every j.

Theorem 1: Suppose that

1) $u^i_s: \mathbf{R}^G_+ \to \mathbf{R}$ is concave, continuous and monotone

2) there is a $\delta \in 0$ such that $W^i_s \ge (\delta, \ldots, \delta)$ for every i, a.e. s

3) for every r, $s \mapsto u^i_s(r)$ is bounded

4) A^j, W^i are bounded

5) $\{s \in S; \lambda_s^j > \max_{1 \leq i \leq I} u_s^i(\sum_\ell W_s^\ell)/\delta\}$ is non-null, $1 \leq j \leq J$.

Then $\mathcal{E}$ has an equilibrium. Moreover if (5) is valid for a.e. s, the equilibrium is non-trivial.

Remark 1: Suppose for example that $\mathcal{E}$ has not an incomplete markets equilibrium (like in Mas-Colell and Zame's counterexample. See also Monteiro (1994)). Then if λ is large enough, $\mathcal{E}$ has a bankruptcy equilibrium with trade.

If the consumer's are too pessimistic about the market repayment rate the equilibrium may be a non trade equilibrium. This is the point of the

Theorem 2: Suppose U^i:$(\mathbf{R}_+^G)^{S+1} \to \mathbf{R}$, is concave, continuous and monotonic. $W_s^i \gg 0$, $1 \leq i \leq I$. Then there exists a bankruptcy equilibrium such that $\overline{t} > 0$ and $\overline{y}_j^i = \overline{z}_j^i = 0$ for every j, i, where $\overline{t}$ is the market repayment rate.

Proof: Take $(p_s)_{s \in S'}$, $S' = S \cup \{0\}$ and $(\overline{x}_x^i)_{s \in S'}$ an equilibrium of the incomplete markets economy without assets. Take $(\mu_s^i)_{s \in S'}$ Lagrange multipliers of the problem:

$$\max\ U^i(x)$$

$$p_s(x_s - W_s^i) \leq 0 \quad \text{for every} \quad s \in S'.$$

That is: $U^i(x) - U^i(\overline{x}^i) \leq \sum_{s \in S'} \mu_s^i p_s(x_s - W_s^i)$ and $\mu_s^i p_s(\overline{x}_s^i - W_s^i) = 0$.

We want to choose $\pi \in \mathbf{R}^J$ and $\overline{t} > 0$ such that

$$U^i(x) - \sum_{j,s} \lambda_s p_s(A_s^j z_j - D_s^j) - \mu_o^i(p_o(x_o - W_o^i) + \pi(y - z)) -$$

$$- \sum_s \mu_s^i p_s(x_s - W_s^i - \sum_j (A_s^j \overline{t} y_j - D_s^j)) \leq U^i(\overline{x}^i)$$

for every $y \geq 0$, $-v^i \geq z \geq 0$, $x_s \geq 0$, $0 \leq D_s^j \leq A_s^j z_j$.
In this case the maximum in D_s^j is for $D_s^j = 0$ if $\lambda_s < \mu_s^i$ and $D_s^j = A_s^j z_j$ if $\lambda_s > \mu_s^i$. Substituting this value we have that the left hand side is not bigger than:

$$\alpha = U^i(x) - \sum_{s \in S'} \mu_s^i p_s (x_s - W_s^i) - \sum_{j,s} \min\{\lambda_s, \mu_s^i\} p_s A_s^j z^j - \mu_o^i \pi_o (y - z) + \\ + \sum_{j,s} \mu_s^i p_s A_s^j y_j .$$

Therefore

$$\alpha \leq U^i(\overline{x}^i) + \sum_j z_j \left\{ - \sum_s \min\{\lambda_s, \mu_s^i\} A_s^j p_s + \mu_o^i \pi_j \right\} + \\ + \sum_j y_j \left(\sum_s \mu_s^i p_s A_s^j \overline{t} - \mu_o^i \pi_j \right) .$$

Choose $\pi_j > 0$, such that $\mu_o^i \pi_j < \sum_s \min\{\lambda_s, \mu_s^i\} p_s A_s^j$ for every i and choose $\overline{t} > 0$ such that $\overline{t} \sum_s \mu_s^i p_s A_s^j < \mu_o^i \pi_j$. Therefore $\alpha \leq U^i(\overline{x}^i)$ and the maximum is achieved at $y_j = 0 = z_j$. QED

IV. The truncated economy

Our proof will be by finite dimensional approximations as in Monteiro (1991). So we now define the truncated economy $\mathcal{E}^N = (U^{iN}, W_N^i, A_N^j, t_N^j)_{\substack{i \leq I \\ j \leq J}}$:

i) $W_N^i = (W_{nN}^i)_{n \leq N}$ $W_{nN}^i = N \int_{\frac{n-1}{N}}^{\frac{n}{N}} W_s^i \, ds$

ii) $A_{nN}^j = N \int_{\frac{n-1}{N}}^{\frac{n}{N}} A_s^j \, ds$

iii) $t_N^j \in [0,1]^N$

iv) $p_N \in \Delta^N$

v) $\pi \in \mathbf{R}^J$, $|\pi|_1 = 1$

vi) $U^{iN}(x_1, \ldots, x_N, z, p_N, D) = \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} u_s^i(x_n)\, ds - \sum_n \sum_j \lambda_{nN}^j (p_{nN} A_{nN}^j z_j - D_{nN}^j)^+$ where $\lambda_{nN}^j = \int_{\frac{n-1}{N}}^{\frac{n}{N}} \lambda_s^j\, ds$.

The equilibrium for $\mathcal{E}^N$ is the same as before with n for s. The economy $\mathcal{E}^N$ has an equilibrium since it satisfy the hypothesis of Dubey-Geanakoplos-Shubik (1988).

Proposition 2: Suppose $(p_N, \pi_N, y^{iN}, \overline{D}^{iN}, t^N, \overline{x}_N^i)$ is an equilibrium of $\mathcal{E}^N$. Then there are Lagrange multipliers $(\mu_{nN}^i)_{n \le N} \ge 0$, $K_N^i \ge 0$ such that

$$
\begin{aligned}
1) \quad & \sum_n \int_{\frac{n-1}{N}}^{\frac{n}{N}} (u_s^i(x_n) - u_s^i(\overline{x}_{nN}^i))\, ds \le \sum_{j,n} \lambda_{nN}^j [p_{nN} A_{nN}^j (z_j - \overline{z}_j^{iN}) - (D_n^j - \overline{D}_{nj}^{iN})] \\
& + \sum_n \mu_{nN}^i \{p_{nN}(u_n - \overline{x}_{nN}^i) - \sum_j t_{nN}^j p_{nN} A_{nN}^j (y_j - \overline{y}_j^{iN}) + \sum_j (D_n^j - \overline{D}_{jn}^{iN})\} \\
& + K_N^i \pi^N (y - \overline{y}^{iN} + \overline{z}^{iN} - z) \qquad (*)
\end{aligned}
$$

for every $x_n \in \mathbf{R}_+^G$, $y \ge 0$, $v_i \le -z \le 0$, $0 \le D_n^j \le p_{nN} A_{nN}^j z_j$

2) $K_N^i \pi_N (\overline{y}^{iN} - \overline{z}^{iN}) = 0$

3) $\mu_{nN}^i p_{nN} (\overline{x}_{nN}^i - W_{nN}^i) = \mu_{nN}^i \left(\sum_j p_{nN} A_{nN}^j \overline{y}_j^{iN} - \sum_j \overline{D}_j^{iN} \right)$

4) $t_{nN}^j \sum_i p_{nN} A_{nN}^j \overline{y}_j^{iN} = \sum_i \overline{D}_{jn}^{iN}$

5) $t_{nN}^j = 1$ if $\lambda_{nN}^j > \max_{1 \le i \le I} \mu_{nN}^i$.

Proof: Items 1), 2), 3), 4) are a direct consequence of the Kuhn-Tucker theorem and Slater's condition. Item 5) follows, if $\sum_i \overline{z}_j^{iN} \neq 0$ from the fact that $\overline{D}_{nj}^{iN} = p_{nN} A_{nN}^j \overline{z}_j^{iN}$ whenever $\lambda_{nN}^j > \mu_{nN}^i$. And if $\sum_i \overline{z}_j^{iN} = 0$, it follows from the inequalities in Dubey-Geanokoplos-Shubik.

Lemma 1: The following inequality is true:

$$\int_{\frac{n-1}{N}}^{\frac{n}{N}} u_s^i(x) - \int_{\frac{n-1}{N}}^{\frac{n}{N}} u_s^i(\overline{x}_{nN}^i) ds \leq \mu_{nN}^i p_{nN}(x - \overline{x}_{nN}^i) \;\; x \in \mathbf{R}_+^G, \;\; 1 \leq n \leq N \tag{1}$$

Proof: Fix $m \leq N$. Substitute in inequality (*), $D_{nN}^j = \overline{D}_{nN}^{ij}$, $y = y^{iN}$, $z = z^{iN}$ for every j, n and $x_n = \overline{x}_{nN}^i$ for every $n \neq m$. After simplification we obtain the desired inequality for $n = m$. Since m is arbitrary the proof is done. QED

Lemma 2: The following inequality is true for every i, j, n

$$0 \leq (\mu_{nN}^i - \lambda_{nN}^j)(D - \overline{D}_{jn}^{iN}), \qquad 0 \leq D \leq p_{nN} A_{nN}^j \overline{z}_j^{iN}.$$

Proof: Substitute in (*) $x_n = \overline{x}_{nN}^{iN}$ for every n, $y = y^{iN}$, $z = z^{iN}$ and $D_{nN}^j = \overline{D}_{jn}^{iN}$ for every $(n, j) \neq (m, k)$. After cancellation of terms we obtain (2).

We need one more inequality:

Lemma 3: The next inequality is true for every $v_i \leq -z \leq 0$, $y \geq 0$

$$0 \leq \sum_{j,n} \min\{\lambda_{nN}^j, \mu_{nN}^i\}(p_{nN} A_{nN}^j (z_j - \overline{z}_j^{iN}) - \sum_{n,j} \mu_{nN}^i t_{nN}^j p_{nN} A_{nN}^j (y_j - y_j^{iN}) +$$

$$+ K^{iN} \pi^N (y - \overline{y}^{iN} + \overline{z}^{iN} - z).$$

Proof: Substitute in (*) $x_n = \overline{x}^i_{nN}$ for every n. And choose $D^j_{nN} = 0$ if $\mu^i_{nN} > \lambda^j_{nN}$ and $D^j_{nN} = p_{nN} A^j_{nN} z_j$ if $\mu^i_{nN} < \lambda^j_{nN}$. And note that this is the best choice for D. Therefore $\overline{D}^{iN}_{jn} = 0$ if $\mu^i_{nN} > \lambda^j_{nN}$ and $\overline{D}^{iN}_{jn} = p_{nN} A^j_{nN} \overline{z}^{iN}_j$ if $\mu^i_{nN} < \lambda^j_{nN}$.

This ends the proof.

Lemma 4: Define $\mu^{iN}(t) = N\mu^i_{nN}$ if $\frac{n-1}{N} \leq t < \frac{n}{N}$, $1 \leq n \leq N$. Then $\{\mu^{iN}\}_N$ is a bounded sequence.

Proof: Fix n and $t \in \left[\frac{n-1}{N}, \frac{n}{N}\right)$. If $\mu^i_{nN} \leq \lambda^j_{nN}$, $\mu^{iN}(t) \leq N\lambda^j_{nN} \leq |\lambda^j|_\infty$. If $\mu^i_{nN} > \lambda^j_{nN}$ for every j. From lemma 2 it follows that $\overline{D}^{iN}_{jn} = 0$ for every j. Now, proposition 2(3) imply

$$\mu^i_{nN} p_{nN} \overline{x}^i_{nN} = \mu^i_{nN} p_{nN} W^i_{nN} + \sum_j \mu^i_{nN} p_{nN} A^j_{nN} t^j_{nN} \overline{y}^{iN}_j \geq \mu^i_{nN} p_{nN} W^i_{nN} \geq \delta \mu^i_{nN}.$$

Also, lemma 1 for $x = 0$ imply

$$\delta \mu^{iN}(t) \leq N \mu^i_{nN} p_{nN} \overline{x}^i_{nN} \leq N \int_{\frac{n-1}{N}}^{\frac{n}{N}} u^i_s(\overline{x}^i_{nN})\, ds \leq$$

$$\leq N \int_{\frac{n-1}{N}}^{\frac{n}{N}} u^i_s \left(\sum_\ell W^\ell_{nN} \right) ds \leq N \int_{\frac{n-1}{N}}^{\frac{n}{N}} u^i_s \left(\left| \sum_\ell W^\ell_s \right|_\infty \right) ds \leq \left| u^i_s \left(\left| \sum_\ell W^\ell_s \right|_\infty \right) \right|_\infty .$$

This ends the proof.

Lemma 5: $(K^i_N)_N$ is bounded.

Proof: Substitute $-z = v_i$, $y = 0$ in lemma 3 and use that $K^i_N \pi^N(y^{iN} - z^{iN}) = 0$ to obtain

$$K^i_N \pi^N(-v_i) \leq \sum_{j,n} \min\{\lambda^j_{nN}, \mu^i_{nN}\} p_{nN} A^j_{nN}(-v_i - z^{iN}_j) + \sum_{n,j} \mu^i_{nN} t^j_{nN} p_{nN} A^j_{nN} y^{iN}_j$$

$$\leq \frac{1}{N} \left(\sum_{j,n} |\mu^{iN}|_\infty |A^j|_\infty |v_i| + \sum_{j,n} |\mu^{iN}|_\infty |A^j|_\infty \sum_i (-v^i) \right) \leq \widetilde{M} < \infty.$$

V. Theorem's 1 proof

Without loss of generality we suppose that $(\pi_N, K_N^i, y^{iN}, z^{iN}) \to (\overline{\pi}, \overline{K}^i, \overline{y}^i, \overline{z}^i)$. Moreover we may suppose (taking a subsequence if necessary) that for every j, either $\sum_i \overline{z}_j^{iN} = 0$ for every N or $\sum_i \overline{z}_j^{iN} > 0$ for every N.

Define $(\overline{x}^{iN}(s), p^N(s), t_N^j(s), \lambda^{jN}(s)) = (\overline{x}_{nN}^i, p_{nN}, t_{nN}^j, N\lambda_{nN})$ for $s \in \left[\frac{n-1}{N}, \frac{n}{N}\right)$ $1 \le n \le N$. Now we define $f^N \colon S \to \mathbf{R}^\ell$ (we do not specify ℓ):

$$f^N = (\mu^{iN}, \overline{x}^{iN}, p^N, \mu^{iN} A_N^j p^N, \overline{D}_N^{ij}, t_N^j, p^N A_N^j \min\{\lambda^{jN}, \mu^{iN}\}, \mu^{iN} p^N A_N^j t_N^j, t_N^j,$$
$$p^N A_N^j \overline{z}_j^{iN}, p^N A_N^j \min_i\{\lambda^{jN}, \mu^{iN}\}, p^N A_N^j t_N^j \max_{1\le k\le I} \mu^{kN})_{\substack{i\le I \\ j\le J}}$$

It shall be clear that $(f^N)_N$ is uniformly bounded, hence uniformly integrable. Without loss of generality there exists $\lim_{N\to\infty} \int f^N(s)\, ds$. By Fatou's lemma, there is a $\overline{f} \colon S \to \mathbf{R}^\ell$ such that

i) $\overline{f}(s) = (\overline{\mu}_s^i, \overline{x}_s^i, \overline{p}_s, \overline{\mu}_s^i A_s^j \overline{p}_s, \overline{D}_s^{ij}, t_s^j, \overline{p}_s A_s^j \min\{\lambda_s^j, \overline{\mu}_s^i\},$

$$\overline{\mu}_s^i \overline{p}_s A_s^j t_s^j, \overline{p}_s A_s^j \overline{z}_j^i, \overline{p}_s A_s^j \min\{\lambda_s^i, \overline{\mu}_s^i\} \overline{p}_s A_s^j t_s^j \max_{1\le k\le I} \overline{\mu}_s^k)_{\substack{i\le I \\ j\le J}}$$

ii) $\int \overline{f}(s)\, ds = \lim_{N\to\infty} \int f^N(s)\, ds.$

Now we check that $(\overline{p}, \overline{\pi}, \overline{y}^i, \overline{D}^i, t^j, \overline{x}^i)_{\substack{j\le J \\ i\le I}}$ is an equilibrium of $\mathcal{E}$. The conditions a), b), c), d), e), f), g) are pointwise and derives directly by a passage to the limit and (i) above.

Let's verify (h). Therefore suppose $x \in (L_+^\infty)^G$, $y \in \mathbf{R}_+^J$, $v_i \le -z \le 0$, $D^j \colon S \to \mathbf{R}_+$ are such that

$$\overline{\pi}(y - z) \le 0 \quad \text{and} \quad \overline{p}_s(x_s - W_s^i) \le \sum_j t_s^j \overline{p}_s A_s^j y_j - \sum_j D_{sj}$$

for a.e. s and $D_{sj} \le p_s A_s^j z_j$. We have

$$\alpha = U^i(x,y,\overline{p},D) - U^i(\overline{x}^i,\overline{y}^i,\overline{p},\overline{D}^i) = \int_S (u_s^i(x_s) - u_s^i(\overline{x}_s^i))\,ds - \sum_j \int \lambda_s^j(\overline{p}_s A_s^j z_j - D_{sj})\,ds$$

$$+ \sum_j \int \lambda_s^j(\overline{p}_s A_s^j \overline{z}_j^i - \overline{D}_{sj}^i)\,ds.$$

From lemma 1 we have, in the limit,

$$\alpha \le \int_S \overline{\mu}_s^i \overline{p}_s (x_s - \overline{x}_s^i)\,ds - \sum_j \int \lambda_s^j(\overline{p}_s A_s^j(z_j - \overline{z}_j^i) - (D_{sj} - \overline{D}_{sj}^i))\,ds \le$$

$$\le \sum_j \int_S \overline{\mu}_s^i(\overline{p}_s A_s^j(y_j - \overline{y}_j^i) - (D_{sj} - \overline{D}_{sj}^i))\,ds - \sum_j \int \lambda_s^j(\overline{p}_s A_s^j(z_j - \overline{z}_j^i) - (D_{sj} - \overline{D}_{sj}^i))\,ds$$

$$= \sum_j \int_S \overline{\mu}_s^i \overline{p}_s A_s^j(y_j - \overline{y}_j^i)\,ds - \sum_j \int [\lambda_s^j \overline{p}_s A_s^j(z_j - \overline{z}_j^i) + (-\lambda_s^j + \overline{\mu}_s^i)(D_{sj} - \overline{D}_{sj}^i)]\,ds.$$

Now using that $\overline{D}_{sj}^i = 0$ if $\lambda_s^j < \overline{\mu}_s^o$ and $\overline{D}_{sj}^i = \overline{p}_s A_s^j \overline{z}_j^i$ if $\lambda_s^j > \overline{\mu}_s^i$ and that $\min\limits_{0 \le D_{sj} \le \overline{p}_s A_s^j z^j} (\lambda_s^j \overline{p}_s A_s^j z_j + (-\lambda_s^j + \overline{\mu}_s^i) D_{sj}) = \min\{\lambda_s^j, \overline{\mu}_s^i\}(\overline{p}_s A_s^j z_j)$.

Hence

$$\alpha \le \sum_j \int_S \overline{\mu}_s^i \overline{p}_s A_s^j(y_j - \overline{y}_j^i)\,ds - \sum_i \int \min\{\lambda_s^j, \overline{\mu}_s^i\}(\overline{p}_s A_s^j z_j - \overline{p}_s A_s^j \overline{z}_j^i)\,ds$$

and lemma 3 (in the limit) imply

$$\alpha \le \overline{K}^{iN} \overline{\pi}(y - \overline{y}^i + \overline{z}^i - z) \le 0.$$

This proves (h). Now verify (i).

If $t_s^j = 0$ a.e. s, we have $\overline{D}_{sj}^i = 0$ a.e. s and that $\overline{p}_s \overline{x}_s^i = \overline{p}_s W_s^i \ge \delta$.

Therefore by lemma 4 $\delta \mu^{iN}(s) \le N \int_{\frac{n-1}{N}}^{\frac{n}{N}} u_s^i \left(\sum_\ell W_{nN}^\ell \right) ds$, so in the

limit $\delta\overline{\mu}^i_s \leq u^i_s \left(\sum_\ell W^\ell_s\right)$ for a.e. s. Therefore $\lambda^j_s > \delta\overline{\mu}^i_s$ and from this we have that $\lambda^j_{nN} > \delta\mu^i_{nN}$ for every i. Hence $t^j_{nN} = 1$ and then $t^j_s = 1$. A contradiction.

Finally we prove the last conclusion of theorem 1. Suppose that $\lambda^j_s > \max_{1\leq i\leq I} u^i_s \left(\sum_\ell W^\ell_s\right)/\delta$ for a.e. s. If there is no-trade $\overline{y}^i = \overline{z}^i = 0$ we have that $\overline{D}^i_{sj} = 0$ a.e. s. By the reasoning above we conclude that $t^j_s = 1$ a.e. s. QED

VI. The nominal returns case

The nominal returns case allows us to dispense with the lower bound on short sales. Suppose $r^j : S \to \mathbf{R}_+$ is the nominal return of asset j, $1 \leq j \leq J$. Define $A^j = r^j(1, \ldots, 1)$. This reduce the nominal case to the real goods case.

Theorem 2: The economy $\mathcal{E}$ with nominal returns $\{r^j\}_{1\leq j\leq J}$ has an equilibrium with unrestricted short sales if $\{r^j; 1 \leq j \leq J\}$ is a linearly independent setIn this theorem we will ask for equilibria for which $y^i_j \cdot z^i_j = 0$. This will simplifty the proof..

Proof: Define the truncated economy $\mathcal{E}^N$ as before except that we substitute $-N(1, \ldots, 1)$ for v_i, $1 \leq i \leq I$, $N \geq 1$. The only additional difficulty is to prove that $h_N = (y^{iN}, z^{iN})_{1\leq i\leq I}$ converges (at least for a subsequence). Suppose, to obtain a contradiction that $(h_N)_N$ doesn't have a converging subsequence. So there is some i such that $\left(|y^{iN}|_1 + |z^{iN}|_1\right)_N$ is unbounded. We define $k_N = \max_i \left(|y^{iN}|_1 + |z^{iN}|_1\right)$, and suppose, without loss of generality, that $k_N \to \infty$ and $\frac{1}{k_N}(y^{iN}, z^{iN}) \to (u^i, v^i)$,

$1 \leq i \leq I$, $u_j^i \cdot v_j^i = 0$. At least for one i, $|u^i|_1 + |v^i|_1 = 1$. The following is true:

(1) $\sum_i \hat{D}_{sj}^i = t_s^j r_s^j \sum_i u_j^i$ almost every s.

(2) $0 = \sum_j r_s^j u_j^i - \sum_j \hat{D}_{sj}^i$ almost every s.

(3) $\sum_j v_j^i \int r_s^j \min\{\lambda_s^j, \mu_j^i\}\, ds \leq \sum_j u_j^i \int \mu_s^i t_j^i r_j^i\, ds$

Equations (1) and (2) follows directly from proposition 2 ($\hat{D}_{sj}^i = \lim \frac{1}{k_N} \overline{D}_N^{ij}(s)$), item (3) from lemma 3 for $y = z = 0$ after division by k_N and taking of limits. From $\sum_j u_j^i t_s^j r_s^j \leq \sum_j u_j^i r_s^j = \sum_j \hat{D}_{sj}^i$ we get

$$\sum_j v_j^i \int r_s^j \min\{\lambda_s^j, \mu_s^i\}\, ds \leq \sum_j u_j^i \int \mu_s^i t_s^j r_j^j\, ds \leq \sum_j \int \mu_s^i \hat{D}_{sj}^i =$$

$$= \sum_j \int_{\lambda_s^j \geq \mu_s^i} \mu_s^i r_s^i v_j\, ds.$$

Therefore $t_s^j = 1$ a.e. s. Also $\hat{D}_{sj}^i = r_s^j v_j^i$. Hence

$$0 = \sum_j r_s^j (u_j^i - v_j^i) \qquad 1 \leq i \leq I.$$

Since there is an i, $(u^i, v^i) \neq 0$, we have for this i, $u_j^i - v_j^i \neq 0$ for some j. A contradiction with independence. QED

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