Individual-Level Randomness in a Nonatomic Population

by

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Abstract

This paper provides a construction of an uncountable family of i.i.d. random vectors, indexed by the points of a nonatomic measure space, such that (a) samples are measurable functions from the index space, and (b) an exact analogue of the Glivenko-Cantelli theorem holds with respect to the measure on that space. That is, a sample possesses a.s. the same distribution as that of the random vectors from which it is drawn. Moreover, any subspace of the index space with positive measure inherits the same property. This homogeneity property is important for an application of the construction to mathematical economics.

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* Department of Economics, University of Minnesota, and Research Department, Federal Reserve Bank of Minneapolis. (Internet address: edgreen@atlas.socsci.umn.edu) A significant part of the research reported here was done during a visit to the Mathematical Sciences Research Institute in December, 1985. I would like to express my appreciation for the support and hospitality of the Institute. I would particularly like to thank William Zame for his comments and suggestions, and Jerome Keisler and Robert Anderson for their correspondence regarding an alternative approach via Loeb measure. This paper is available as ewp-ge/9402001 from the archive at econwpa.wustl.edu. It may be copied without fee, provided that copies are not made or distributed for direct commercial advantage and that credit to the source is given.

1. Introduction

The Glivenko-Cantelli theorem states that, almost surely, the sample distributions of i.i.d. random variables converge weakly to the statistical distribution of the random variables. (Parthasarathy (1967), Theorem II.7.1.) This paper provides a set-theoretic construction of an uncountable family of i.i.d. random vectors, indexed by the points of a nonatomic measure space, such that (a) samples are measurable functions from the index space, and (b) an exact analogue of the Glivenko-Cantelli theorem holds with respect to the measure on that space. That is, a sample can be viewed as a random vector by regarding the index space itself as a probability space, and a.s. the sample possesses the same distribution as that of the i.i.d. random vectors from which it is drawn. Moreover, any subspace of the index space with positive measure inherits the same property, if the measure of the subspace is normalized to be a probability measure. This homogeneity property is important for an application of the construction to mathematical economics which will be discussed below. The construction presented here is an alternative to the construction via Loeb measure, first presented by Keisler (1977) and subsequently simplified by Anderson (1991).

To understand what this construction accomplishes, first consider a more direct construction of a family of i.i.d. random variables for which an exact idealization of the strong law of large numbers holds. This construction begins with Kolmogorov's construction of a continuum of i.i.d. random variables $\{\phi_t \mid t \in [0,1]\}$ having a prescribed distribution with finite first moment. Then the measure on the sample space Ω so constructed is extended in such a way that $\int_0^1 \phi_t(\omega)\, dt = E(\phi_0)$ a.s. This equation idealizes the strong law of large numbers, with Lebesgue integration on $[0,1]$ playing the role of averaging over the sample. The techniques needed to prove the existence of a continuous-time i.i.d. process satisfying this integral equation were developed by Doob (cf. (1937), (1947), (1953) chapter II), and an existence proof has been given in full by Judd (1985).

This construction has been widely cited by economic theorists. In particular, the i.i.d. processes just discussed have been thought to provide the mathematical basis for tractable models of economies in which individual traders face idiosyncratic risks - - risks of gains or losses that are sizable for each individual trader, but that are independent across traders and accurately predictable in the aggregate. (The risk of death is tolerably close to being idiosyncratic, at least in populations where epidemic diseases are under control, but the risk of outbreak of war is not idiosyncratic. This contrast explains why life insurance is easily available but insurance against political risks is not.) In such models, $[0,1]$ represents the population of traders, ϕ_t represents the random gain or loss experienced by trader t, and $\int_0^1 \phi_t(\omega)\, dt$ represents the aggregate net gain or loss to the economy *ex post* when ω is the state of the world.

Feldman and Gilles (1985) provide documentation of the importance of models of this genre to current economic theory. They also argue that the construction just described is actually inadequate to provide a mathematical foundation for those models. The problem is that the models posit more than just the one integral equation $\int_0^1 \phi_t(\omega)\, dt = E(\phi_0)$ a.s. It is also assumed that, for any $\theta \in (0, 1]$, $\int_0^\theta \phi_t(\omega)\, dt = \theta\, E(\phi_0)$ a.s. This homogeneity assumption reflects the economic idea that any non-negligible fraction of the traders in a large economy could potentially form a risk-pooling coalition that would provide its members with virtually complete insurance against idiosyncratic risks, and that therefore the per-capita aggregate resources *ex post* of a large coalition in an economy of traders facing i.i.d. risks should not depend on the coalition. In the study of insurance, this assumption is required in order to demonstrate that equal sharing of resources is the unique cooperative arrangement (technically, the unique core allocation) in this economy.

Mathematically this strengthened assumption may seem innocuous, because the interval $[0, \theta]$ with normalized Lebesgue measure is isomorphic to $[0, 1]$. Thus the integral equation should be as plausible for any value of θ as it is for $\theta = 1$. However, Feldman and Gilles have shown that it is inconsistent to make the assumption for all θ, if ϕ is a Bernoulli process taking values *always* in $\{0, 1\}$.[1] In that case, the assumption is equivalent to $\int_0^\theta \phi_t(\omega)\, dt = \theta/2$ a.s. Considering this equation for rational values of θ (to assure measurability of the event that the equation holds for all values considered), the Radon-Nikodym theorem implies that a.s. $\{t \mid \phi_t(\omega) = 1/2\}$ has Lebesgue measure 1. This conclusion contradicts the restriction that has been imposed on the range of ϕ, though.

In view of this contradiction, the body of economic theory that has been formulated in terms of this model needs to be placed on a more secure foundation.[2] This task will be

[1] There are applications in which it is crucial that ϕ should take values only in $\{0, 1\}$. For instance, if ϕ is the characteristic function of some event, then 0 and 1 are the only values that it can meaningfully take.

[2] Feldman and Gilles (1985) cite a number of prominent contributions to economic theory that their argument shows to be inconsistent (at least if risk is parametrized as being Bernoulli). These authors, as well as Judd (1985), Uhlig (1987), and others, have proposed alternative definitions of integrals over the population in order to assure homogeneity. However, besides the general disadvantages of recourse to such alternatives cited by Doob (1947), there are economic arguments that employ Lebesgue integration with respect to countably additive measures on both the sample space and the population. (An example is Green (1987), where the results proved here are used.) This requirement motivates the present study.

accomplished here. It will be proved that there exists an i.i.d. family of random vectors that satisfy an *analogue* of the condition that, for each measurable set A in the range of ϕ_0 and for each $\theta \in [0, 1]$, $\lambda(\{t \mid \phi_t(\omega) \in A\} \bigcap [0, \theta]) = \theta\, P(\phi_0 \in A)$ a.s. (Here λ denotes Lebesgue measure on $[0, 1]$ and P denotes probability measure defined on the sample space.) This condition just stated would be an exact idealization of the Glivenko-Cantelli theorem, but in the case of integrable, $\{0, 1\}$-valued random variables it would imply the set of integral equations that has just been seen to be inconsistent. This problem will be avoided by indexing random vectors by elements of an abstract nonatomic probability space rather than by numbers in the unit interval. Correspondingly, sample functions will be integrated over measurable subsets of this space rather than over intervals. There is no distinction of economic realism between numbers in the unit interval and sample points of another probability space as names of idealized traders. Thus, the generalized stochastic process described here provides just as appropriate an economic model as does a process indexed by the unit interval.

The key to avoiding the contradiction derived by Feldman and Gilles is to represent the population as a probability space having a σ-algebra that is not countably generated. Then the strategy of restricting attention to the countable set of intervals $\{[0, \theta] \mid \theta$ is rational$\}$ and subsequently appealing to the Radon-Nikodym theorem cannot be emulated. Rather than starting with a given sample space Ω and population space Θ, a function ϕ will be defined on the Cartesian product of arbitrary sets Ω and Θ, and then these sets will be endowed with probability structure in a way that guarantees the required properties of ϕ. Consequently the function ϕ and the sets Ω and Θ with their respective σ-algebras will constitute a universal limit process. When appropriate probability measures are defined on these σ-algebras, the sections of ϕ will become an i.i.d. family of random vectors having any specified distribution, and satisfying the idealized Glivenko-Cantelli property relative to any measurable subset of the population.

Before carrying out the details of this program, two features of the limit process need to be discussed. First, it would be mathematically possible to endow the sample space and the population space with any combination of probability measures. This means that it would be possible for all of the random vectors to have one distribution, but for the sample functions a.s. to have another distribution. There seems to be no way to rule out this possibility by appeal to the kinds of separability or joint-measurability considerations that are usually invoked. However, the interpretation of the process as a limit object provides a strong reason to impose a connection between the two measures. This is made clear in the next section.

Second, although all of the sections of the function ϕ (defined by fixing either a sample point or a member of the population) are measurable, ϕ is not jointly measurable in its two arguments. At the end of the paper, it will be proved that no limit process for a nondegenerate distribution can be jointly measurable. Joint measurability might have been of interest for two reasons. First, it could serve as a selection criterion that would eliminate counterintuitive processes from consideration. It has been mentioned above that these counterintuitive processes can be identified as being pathological on other grounds. Second, joint measurability would justify the application of the Fubini theorem to the process. However, the marginal distributions of the process are a.s. constant, and the constants for the two variables are equal, so the conclusion of the Fubini theorem is satisfied even though the process is not jointly measurable. Thus the failure of joint measurability is not a serious problem here.

2. The sample-distribution limit of an i.i.d. sequence

Suppose that $\Omega = (\Omega, \mathcal{B}, \pi)$ and $R = (R, \mathcal{R}, \mu)$ are probability spaces and that $\{\phi_n : \Omega \to R\}_{n \in \mathbf{N}}$ is a sequence of independent random vectors, each having distribution μ. The finite sample $(\phi_k(\omega))_{k<n}$ can be regarded as a random vector on the probability space $\Theta_n = (\{0, \ldots, n-1\}, \mathcal{F}_n\ \nu_n)$, where $\mathcal{F}_n$ is the power set of $\{0, \ldots, n-1\}$ and ν_n is normalized counting measure. In the case that R is a separable metric space, the finite sample distributions ν_n converge weakly to μ almost surely as n tends to infinity. (Parthasarathy (1967), Theorem II.7.1.) In view of this fact, it is natural to look for a probability space $\Theta = (\Theta, \mathcal{F}, \nu)$ and a function $\phi : \Omega \times \Theta \to R$ such that

(1) $\{\phi_\theta : \Omega \to R\}_{\theta \in \Theta}$ are independent random vectors having distribution μ, and

(2) the sample functions $\phi_\omega : \Theta \to R$ are measurable and have distribution μ a.s.[3]

If $\phi = (\Omega, \Theta, \phi)$ satisfies these conditions, then it will be called a *sample-distribution limit* for R. This paper proves the existence of a sample-distribution limit that is homogeneous in the sense that restricting ν to any set $A \in \mathcal{F}$ such that $\nu(A) > 0$ and normalizing so that $\nu(A) = 1$ yields again a sample-distribution limit for R. (The impossibility of this, if Θ is the unit interval with Lebesgue measure, is the result of Feldman and Gilles (1985)

[3] $\phi_\omega(\theta) = \phi_\theta(\omega) = \phi(\omega, \theta)$. An infinite family of random variables is independent iff every finite subfamily is independent. "Almost surely" will always refer to events in $\mathcal{B}$ rather than in $\mathcal{F}$.

that has been discussed above.) However it will be proved here that, subject to a mild restriction, no sample-distribution limit is measurable with respect to $\mathcal{B} \times \mathcal{F}$.

The construction of a homogeneous sample-distribution limit will rely heavily on Kolmogorov's construction of a set of independent random vectors having distribution $R = (R, \mathcal{R}, \mu)$. The relevant details of Kolmogorov's construction are now reviewed. Throughout this paper it will be assumed that

(3) $\mathcal{R} \neq \{\emptyset, R\}$, and Θ and Ω are disjoint infinite sets.

Kolmogorov's construction specifies R^Θ as the sample space. To define a σ-algebra and probability measure, first define

(4) $\mathcal{A}^0_\mathcal{B} = \{\alpha \mid \alpha : \Theta \to \mathcal{R}\backslash\{\emptyset\}$ and $\Theta\backslash\alpha^{-1}(R)$ is finite$\}$.

Every $\alpha \in \mathcal{A}^0_\mathcal{B}$ can be regarded as specifying a subset $\alpha^\#$ of R^Θ by

(5) $x \in \alpha^\#$ iff $\forall\theta\ x(\theta) \in \alpha(\theta)$.

The sets $\alpha^\#$ defined by (5) are called *cylinders.* For $X \subseteq R^\Theta$, define $A \subseteq \mathcal{A}^0_\mathcal{B}$ to be a *cylindrical partition* of X iff

(6) A is a finite subset of $\mathcal{A}^0_\mathcal{B}$ and $\{\alpha^\#\}_{\alpha \in A}$ is a partition of X.

Define $\mathcal{A}_\mathcal{B}$ to be the set of subsets of R^Θ having a cylindrical partition. That is,

(7) $X \in \mathcal{A}_\mathcal{B}$ iff $\exists A\,[A$ satisfies (6) w.r.t. $X]$.

Now, begin to define the Kolmogorov extension measure κ_π, by defining it on $\mathcal{A}_\mathcal{B}$.

(8) $\kappa_\pi(X) = \Sigma_{\alpha \in A}\,[\Pi_{\theta \in \Theta}\ \mu(\alpha(\theta))]$ if A satisfies (6) w.r.t. X.

Let $\mathcal{E}_\mathcal{B}$ be the smallest σ-algebra containing $\mathcal{A}_\mathcal{B}$. The definition of κ_π will be extended to $\mathcal{E}_\mathcal{B}$. The following lemma summarizes results of a series of arguments and constructions that are described in Halmos (1974), §33, §37, §38.[4]

Lemma 1 (Kolmogorov): The definition of $\kappa_\pi(X)$ in (8) does not depend on which cylindrical partition of X is used. $\mathcal{A}_\mathcal{B}$ is an algebra of subsets of R^Θ. There is a unique probability measure κ_π defined on $\mathcal{E}_\mathcal{B}$ that satisfies (8) for every $X \in \mathcal{A}_\mathcal{B}$ and for every A that satisfies (6) w.r.t. X. The projections $p_\theta : R^\Theta \to R$ defined by

(9) $p_\theta(x) = x(\theta)$

[4] Halmos deals explicitly only with the case that $\Theta = \mathbf{N}$, but the argument is completely general.

are independent random vectors having distribution μ.

Note that the last assertion follows directly from (8).

It is useful to know that a set in $\mathcal{E}_{\mathcal{B}}$ is defined in terms of restrictions on only a countable set of coordinates. That is,

Lemma 2: If $X \in \mathcal{E}_{\mathcal{B}}$, then there exists $\Theta_X \subseteq \Theta$ such that

(10) Θ_X is countable and $\forall y \in R^{\Theta}$ $[y \in X \equiv \exists x \in X \quad \forall \theta \in \Theta_X\ [x(\Theta) = y(\Theta)]]$.

Proof: It is easily verified that the set of all subsets of R^{Θ} for which there exists $\Theta_X \subseteq \Theta$ satisfying (10) is a σ-algebra containing $\mathcal{A}_{\mathcal{B}}$. Thus $\mathcal{E}_{\mathcal{B}}$ is a sub σ-algebra, since it is the smallest σ-algebra containing $\mathcal{A}_{\mathcal{B}}$. ∎

3. Construction of a sample-distribution limit from a free function

A *free function* is defined to be a function $\phi : \Omega \times \Theta \to R$ that satisfies

(11) $\forall h \in R^{\mathbf{N}}$ $[\forall f \in \Theta^{\mathbf{N}}\ \exists \omega\ [f$ is $1-1 \supset \forall n\ \phi(\omega, f(n)) = h(n)]$ and
$\forall g \in \Omega^{\mathbf{N}}\ \exists \theta\ [g$ is $1-1 \supset \forall n\ \phi(\omega, g(n), \theta) = h(n)]]$.

Note that f, Θ and ω are dual to g, Ω and θ in (11).

Given a free function ϕ, Ω is now constructed. The idea that guides this construction is that (11) guarantees enough diversity in the behavior of ϕ on Ω so that a σ-algebra isomorphic to that constructed by Kolmogorov is required to make every ϕ_θ measurable. This isomorphism commutes in an appropriate sense with ϕ and the corresponding stochastic process (9) of Kolmogorov's construction. Using this fact, Kolmogorov's construction can be pulled back to Ω to define Ω in such a way that the ϕ_θ are i.i.d. with distribution μ. Then, by the duality in (11), Θ can be constructed analogously so that the ϕ_ω are i.i.d. with distribution μ. Thus ϕ is a sample-distribution limit for R. The independence of the random vectors ϕ_ω will be further exploited to show that ϕ is homogeneous.

Ω is given as a Cartesian factor space of the domain of the free function ϕ. $\mathcal{B}$ is now defined as the range of a mapping $\Phi : \mathcal{E}_{\mathcal{B}} \to \mathcal{P}(\Omega)$, where $\mathcal{E}_{\mathcal{B}}$ continues to denote the σ-algebra on R^{Θ} obtained in Lemma 1. Specifically, for all $X \in \mathcal{E}_{\mathcal{B}}$,

(12) $\Phi(X) = \{\omega \mid \phi_\omega \in X\}$.

Next, define

(13) $\mathcal{B} = \Phi(\mathcal{E}_{\mathcal{B}})$.

Lemma 3: If ϕ is free, then $\mathcal{B}$ is a σ-algebra of subsets of Ω, and $\Phi : \mathcal{E}_{\mathcal{B}} \to \mathcal{B}$ is an isomorphism of σ-algebras.

Proof: It is clear that $\mathcal{B}$ is a σ-algebra and that Φ is a homomorphism onto $\mathcal{B}$. To show that Φ is an isomorphism, it is only necessary to show that it is $1-1$. Suppose that $X \in \mathcal{E}_{\mathcal{B}}$ and $Y \in \mathcal{E}_{\mathcal{B}}$ and $X \neq Y$. Without loss of generality, assume that $x \in X \backslash Y$. Let Θ_X and Θ_Y be countable sets possessing the property (10) for X and Y respectively, which are guaranteed to exist by Lemma 2, and let $f : \mathrm{N} \to \Theta$ be $1-1$ and $\Theta_X \bigcup \Theta_Y \subseteq f(\mathrm{N})$. Define $h : \mathrm{N} \to R$ by $h(n) = x(f(n))$. Then, by (11), there exists an ω such that $\forall n\ \phi(\omega, f(n)) = h(n)$. By (10), then, $\phi_\omega \in X \backslash Y$. By (12), $\omega \in \Phi(X) \backslash \Phi(Y)$, so $\Phi(X) \neq \Phi(Y)$. ∎

In view of Lemma 3, it is clear how to define π. Namely,

$$(14)\quad \pi(X) = \kappa_\pi(\Phi^{-1}(X)).$$

Lemma 4: If ϕ is free, then $\Omega = (\Omega, \mathcal{B}, \pi)$ defined by (12) - (14) is a probability space on which the random vectors ϕ_θ are i.i.d. with distribution R.

Proof: Lemma 1 and Lemma 3 show that Ω is a probability space. The independence assertion also follows from Lemma 1, using the equations $\phi_\theta(\omega) = p_\theta(\phi_\omega)$ and $\Phi(\alpha^{\#}) = \{\omega \mid \forall\theta\ \phi_\omega(\theta) \in \alpha(\theta)\}$, which are consequences of the definitions above. ∎

As mentioned after equation (11), this argument can be dualized with respect to Ω and Θ. Let (n') be the dual of equation (n), and let Lemma n' be the dual of Lemma n. The only term in the preceding argument that cannot be dualized simply by substitution (e.g., of θ for ω whenever ω occurs either as a term or as a subscript) is Φ; let its dual be Ψ. That is,

$$(12')\quad \Psi(X) = \{\theta \mid \phi_\theta \in X\}.$$

The following result is an immediate consequence of Lemma 4 and Lemma $4'$.

Theorem 1: If ϕ is free, then $\phi = (\Omega, \Theta, \phi)$ is a sample-distribution limit for R.

4. Homogeneity of ϕ

Define ϕ to be a *homogeneous* sample-distribution limit for R if it is a sample-distribution limit for that distribution and also

$$(15)\ \forall A \in \mathcal{F} \quad \forall B \in \mathcal{R} \quad [\nu(A \bigcap \phi_\omega^{-1}(B)) = \nu(A)\mu(B) \text{ a.s.}]$$

If positive-measure sets in $\mathcal{F}$ are analogous to infinite subsets of $\mathbf{N}$, then (15) intuitively ought to hold in the limit because the sequential-convergence result cited at the beginning of the paper applies to every infinite subsequence of $\{\phi_n\}_{n \in \mathbf{N}}$.

It will now be shown that the sample-distribution limit ϕ just constructed is homogeneous, that is, that ϕ satisfies (15). For any $A \in \mathcal{F}$ and $B \in \mathcal{R}$, there will be a countable subset of Ω where the condition asserted by (15) to hold a.s. is violated. Thus, it must be shown that a countable subset is a subset of a probability-zero event of $\mathcal{B}$. The next two lemmas establish this.

Lemma 5: If ϕ is free, $r \in R$, and $\omega \in \Omega$, then the cardinal of $\{\theta \,|\, \phi(\omega, \theta) = r\}$ is at least 2^a, where a is the cardinal of $\mathbf{N}$.

Proof: Define $H = \{h \,|\, h : \mathbf{N} \to R$ and $h(0) = r\}$. By (3), R has at least two distinct elements, so the cardinal of H is at least 2^a. Let $g : \mathbf{N} \to \Omega$ be $1-1$ with $g(0) = \omega$. (g also exists by (3).) By (11), for every $h \in H$ there exists θ_h such that $\forall n\ [\phi(g(n), \theta_h) = h(n)]$. Note that, if $h \neq j$, then $\theta_h \neq \theta_j$, so the cardinal of $\{\theta_h \,|\, h \in H\}$ is at least 2^a. Setting $n = 0$ yields $\forall h \in H\ [\phi(\omega, \theta_h) = r]$. ∎

Lemma 6: If ϕ is free, then for every $\omega \in \Omega$ there is an event $B \in \mathcal{B}$ satisfying $\omega \in B$ and $\pi(B) = 0$. Consequently, for every countable $C \subseteq \Omega$ there is an event $B \in \mathcal{B}$ satisfying $C \subseteq B$ and $\pi(B) = 0$.

Proof: By countable additivity of π it is sufficient to establish, for arbitrary $\omega \in \Omega$ and for every rational $x > 0$, that there is an event $B \in \mathcal{B}$ satisfying $\omega \in \mathcal{B}$ and $\pi(B) < x$. By (3), there exist $r \in R$ and $X \in \mathcal{R}$ such that $r \in X$ and $\mu(X) < 1$. Let $f : \mathbf{N} \to \Theta$ be $1-1$ with $f(\mathbf{N}) \subseteq \{\theta \,|\, \phi(\omega, \theta) = r\}$. Such a function exists by Lemma 5. For $n \in \mathbf{N}$, define $\alpha_n : \Theta \to \mathcal{R}$ by $\alpha_n(f(m)) = X$ for all $m < n$, and $\alpha_n(\theta) = R$ for every other θ. By (8), $\kappa_\pi(\alpha_n^\#) = (\mu(X))^n$. For n sufficiently large, $(\mu(X))^n < x$. ∎

Theorem 2: If ϕ is free, then ϕ is a homogeneous sample-distribution limit for R.

Proof: ϕ is a sample-distribution limit by Theorem 1, so only (15) has to be verified. That is, it must be shown that if $A \in \mathcal{F}$ and $B \in \mathcal{R}$, then the set of ω satisfying

$$(16)\quad \nu(A \bigcap \phi_\omega^{-1}(B)) = \nu(A)\,\mu(B)$$

is an event of $\mathcal{B}$ having probability 1.

That (16) holds a.s. will first be proved in the case that $A \in \Psi(\mathcal{A}_\mathcal{F})$. (Recall that Ψ was defined by $(12')$ at the end of section 3.) By Lemma 6, it is sufficient to prove that (16) holds for all but countably many ω. Moreover by $(6')$, $(7')$ and $(12')$, it is sufficient to prove (16) on this complement for $A = \Psi(\alpha^\#)$, where $\alpha \in \mathcal{A}_\mathcal{F}^0$. That is, any element A of $\Psi(\mathcal{A}_\mathcal{F})$ is a finite disjoint union of sets $\Psi(\alpha^\#)$. Let $A = \Psi(\alpha^\#)$, then, and define $F = \{\omega \,|\, \alpha(\omega) \neq R\}$. By $(4')$, F is finite. Suppose that $\omega^* \notin F$, and define $\beta \in \mathcal{A}_\mathcal{F}^0$ by $\beta(\omega^*) = B$ and $\beta(\omega) = R$ otherwise. Also define $\gamma \in \mathcal{A}_\mathcal{F}^0$ by $\gamma(\omega^*) = B$ and $\gamma(\omega) = \alpha(\omega)$ otherwise. Note that $\alpha^\# \bigcap \beta^\# = \gamma^\#$ that $\kappa_\nu(\gamma^\#) = \kappa_\nu(\alpha^\#)\kappa_\nu(\beta^\#)$, and that $\phi_{\omega^*}^{-1}(B) = \Psi(\beta^\#)$.

These facts establish (16) for $\alpha^\#$, B, and all $\omega^* \notin F$.

Now using the facts that (16) holds a.s. on $\Psi(\mathcal{A}_\mathcal{F})$, it will be shown that (16) holds a.s. for arbitrary $A \in \mathcal{F}$. Since $\mathcal{A}_\mathcal{F}$ generates $\mathcal{E}_\mathcal{F}$, Lemma $3'$ asserts that $\Psi(\mathcal{A}_\mathcal{F})$ generates $\mathcal{F}$. Therefore, by Halmos ((1974), §13, Theorem D) and Lemma $1'$, there exists a sequence $\{X_n\}_{n \in \mathbf{N}} \subseteq \mathcal{A}_\mathcal{F}$ such that

$$(17)\quad \forall n\ [\nu(\Psi(X_n)\ \Delta\ A) < 1/n].$$

Replacing A in (16) by $\Psi(X_n)$ and applying (17) yields

$$(18)\quad |\,\nu(A \bigcap \phi_\omega^{-1}(B)) - \nu(A)\,\mu(B)\,| < 1/n$$

which implies (16) since n is arbitrary. Since (16) has been shown to hold a.s. for each $\Psi(X_n)$, it then holds a.s. for A as well. ∎

5. Existence of a free function

A free function with range R is now proved to exist for suitably chosen sets Θ and Ω. This function is constructed by transfinite recursion, using some basic results of cardinal arithmetic.[5] a will denote the cardinal of $\mathbf{N}$; Ψ, ρ, σ and τ will denote ordinal

[5] These topics are covered, for instance, in Takeuti and Zaring (1982). The facts about cardinal arithmetic that will be used are (a) exact analogues of rules for manipulating sums, products and exponents of natural numbers, (b) generalization of the distributive law to

numbers; and q, r, s and t will denote cardinal numbers. Addition, multiplication and exponentiation will refer to cardinal operations.

A free function $\phi : \Omega \times \Theta \to R$ will be obtained as the union of a transfinite nested sequence of partial functions. The sequence must be chosen to that ϕ will be a total function that satisfies (11). At each stage of the sequence, either $\phi(\omega, \theta)$ will be defined for some specified element (ω, θ) of the domain in order to ensure that ϕ will be total, or else an instance of one of the implications in (11) will be satisfied. These characteristics of the function to be determined will be called *features.* The set T of features is given by

(19) $T = \{(h, f) \in R^{\mathbf{N}} \times \Theta^{\mathbf{N}} \mid f \text{ is } 1-1\} \bigcup \{(h, g) \in R^{\mathbf{N}} \times \Omega^{\mathbf{N}} \mid g \text{ is } 1-1\} \bigcup [\Omega \times \Theta].$

Let N be an *enumeration* of T. That is suppose that τ is an ordinal and that,

(20) $N : \tau \to T$ is onto T.

Let P denote the set of partial functions from $\Omega \times \Theta$ to R. That is, P is given by

(21) $P = \{p \mid \exists A\, [A \subseteq \Omega \times \Theta \text{ and } p : A \to R]\}.$

Define the domain of a partial function, and the projections of the domain on Θ and Ω, by

(22) $D(p) = \{(\omega, \theta) \mid \exists r\; p(\omega, \theta) = r\};\; D_\Omega(p) = \{\omega \mid \exists \theta\; (\omega, \theta) \in D(p)\};$
$D_\Theta(p) = \{\theta \mid \exists \omega\; (\omega, \theta) \in D(p)\}.$

A partial function *specifies* a feature if it defines the values of the function to which the feature refers. To be precise,

(23) p specifies $(h, f) \in R^{\mathbf{N}} \times \Theta^{\mathbf{N}}$ iff $\exists \omega\, \forall n\, [p(\omega, f(n)) = h(n)].$

(24) p specifies $(h, g) \in R^{\mathbf{N}} \times \Omega^{\mathbf{N}}$ iff $\exists \theta\, \forall n\, [p(g(n), \theta) = h(n)].$

(25) p specifies $(\omega, \theta) \in \Omega \times \Theta$ iff $\exists r\, [p(\omega, \theta) = r].$

A *fully specified sequence* of partial functions is a transfinite nested sequence such that every feature is eventually specified. that is, a function $S : \tau \to P$ is a fully specified sequence if

(26) $\forall \rho\, \forall \sigma\, [\rho < \sigma < \tau \supset [D(S(\rho)) \subseteq D(S(\sigma)) \text{ and } S(\sigma) \mid D(S(\rho)) = S(\rho)]],$

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transfinite addition, and (c) the facts that the sum and the product of two infinite cardinals are both equal to the maximum of the two operands, and that the cardinal of the set of functions from one set to another is the cardinal of the range taken to the exponent of the cardinal of the domain.

and

(27) $\forall \sigma < \tau\ [S(\sigma+1)$ specifies $N(\sigma)]$.

Lemma 7: If S is a fully specified sequence and $\phi = \bigcup_{\sigma < \tau} S(\sigma)$, then ϕ is a free function.

Proof: That ϕ is a total function satisfying (11) follows directly from (19) – (27). ∎

It can be shown that if S satisfies (26), then the range of S cannot have larger cardinality than does $\Omega \times \Theta$. However, (19) implies that the cardinality of T may be larger than this. This inequality of size would present a problem for the construction of ϕ that is described in Lemma 7, because the domain of ϕ would be exhausted before all features had been specified. The next lemma establishes that Θ and Ω can be taken to be of a cardinality such that this problem will not arise.

Lemma 8: Let r be the cardinal of R, and define $t = r^a$.[6] Then $t^a = t$. If Θ and Ω are of cardinality t, then T is also of cardinality t.

Proof: The first assertion is true because $a^2 = a$, so that $t^a = (r^a)^a = r^{a^2} = r^a = t$. To prove the second assertion, note first that $\{(h, f) \in R^{\mathbf{N}} \times \Theta^N \mid f \text{ is } 1-1\}$ is a subset of T. This subset has at cardinality at least t, since t is the cardinal of $R^{\mathbf{N}}$. T is a subset of $[R^{\mathbf{N}} \times (\Theta^{\mathbf{N}} \bigcup \Omega^{\mathbf{N}})] \bigcup [\Omega \times \Theta]$, the cardinal of which is $[t \cdot (t^a + t^a)] + [t \cdot t] = t$ also. ∎

Lemma 9: Let r be the cardinal of R, $t = r^a$, Θ and Ω be disjoint sets of cardinality t, and τ be the initial ordinal of t. Then there exist an enumeration $N : \tau \to T$ and a completely specified sequence $S : \tau \to P$.

Proof: The existence of N follows from Lemma 8. S will now be described recursively. That is, at each stage σ, the graph of $S(\sigma) \in P$ will be described. Let σ be the first ordinal for which S has not been defined, and let s be the cardinal of σ. Consider the induction hypothesis that, if $\Psi < \sigma$ and q is the cardinal of Ψ, then $D(S(\Psi))$ has cardinality no

[6] Note that, if $r = 2^a$, then $t = r$. $r = 2^a$ if R is a standard Borel space.

greater than $q+a$, as well as that (26) and (27) are satisfied (with Ψ replacing τ). If $\sigma = 0$, define $S(\sigma) = \emptyset$. At 0, (26) and (27) are satisfied trivially. If $\sigma = \rho + 1$ and ρ satisfies the induction hypothesis, then $D(S(\rho))$, $D_\Theta(S(\rho))$ and $D_\Omega(S(\rho))$ all have cardinality no greater than $s + a < t$. Therefore $\Theta \backslash D_\Theta(S(\rho))$ and $\Omega \backslash D_\Omega(S(\rho))$ are nonempty. Let $\theta^* \in \Theta \backslash D_\Theta(S(\rho))$, $\omega^* \in \Omega \backslash D_\Omega(S(\rho))$, and $r^* \in R$. If $N(\rho) = (h, f) \in R^{\mathbf{N}} \times \Theta^{\mathbf{N}}$, define $S(\sigma) = S(\rho) \bigcup \{(\omega^*, f(n), h(n)) \mid n \in \mathbf{N}\}$. If $N(\rho) = (h, g) \in R^{\mathbf{N}} \times \Omega^{\mathbf{N}}$, define $S(\sigma) = S(\rho) \bigcup \{(g(n), \theta^*, h(n)) \mid n \in \mathbf{N}\}$. If $N(\rho) = (\omega, \theta) \in \Omega \times \Theta$, define $S(\sigma) = S(\rho)$ if $(\omega, \theta) \in D(S(\rho))$ and define $S(\sigma) = S(\rho) \bigcup \{(\omega, \theta, r^*)\}$ otherwise. For each of the three types of feature, a $S(\rho)$ is extended to a countable set outside $D(S(\rho))$ to obtain $S(\sigma)$. Thus (26) and (27) hold, and $D(S(\sigma))$ has cardinality no greater than $s + a$. If S has been defined up to σ and σ is a limit ordinal, then define $S(\sigma) = \bigcup_{\rho < \sigma} S(\rho)$. (26) and (27) continue to hold. $S(\sigma)$ is a union of s sets of cardinality no greater than $s + a$, so its cardinality is no greater than $s \cdot (s + a) = s + a$. By transfinite induction, then $S : \tau \to P$ is defined and satisfies (26) and (27). ∎

Theorem 3: A free function exists, and R possesses a sample-distribution limit.

Proof: The existence of a free function follows from Lemma 7 and Lemma 9. Given this function, the existence of a sample-distribution limit for R follows from Theorem 2 ∎

6. Nonatomicity and nonmeasurability of homogeneous sample-distribution limits

The construction of ϕ has guaranteed that all of the sections of ϕ, both with respect to Θ and to Ω, are measurable. However, the measurability of ϕ with respect to the product σ-algebra $\mathcal{B} \times \mathcal{F}$ has not been asserted. In this section it will be shown that, under a mild restriction, no homogeneous sample-distribution limit can be jointly measurable in its two variables. The restriction in that $\mathcal{R}$ should contain a set of μ-measure strictly between 0 and 1.[7]

[7] If $\mathcal{R}$ is contained in the completion by measure-zero sets of the σ-field of invariant sets of an ergodic transformation on R, then (by definition) the restriction is not satisfied. This seems to be the only nontrivial case of practical interest in which the restriction would not be satisfied. Doob (1937) has proved the nonmeasurability of ϕ when Θ is taken to be the unit interval with Lebesgue measure, but his proof does not generalize to the situation where Θ is not countably generated.

The proof of this result makes use of the fact that, under the restriction, every homogeneous sample-distribution limit has a nonatomic population measure (in the sense of Halmos ((1974), §40): that every set of positive measure has a subset of strictly smaller positive measure). This fact, which is of some independent interest, is now proved. Note that Lemma $6'$ fails to imply that the population measure is nonatomic because the σ-algebra on which it is defined is not countably generated. (cf. the example at the end of this section.)

Lemma 10: If ϕ satisfies (15), and if $\mathcal{R}$ contains a set of μ-measure strictly between 0 and 1, then ν is nonatomic.

Proof: Suppose that $B \in \mathcal{R}$ and $0 < \mu(B) < 1$. Then (15) implies that if $\nu(A) > 0$, then there is an element of $\mathcal{F}$ having measure strictly between 0 and $\nu(A)$. Thus ν cannot have an atom. ∎

Theorem 4: Suppose that ϕ satisfies (1) and satisfies (15), and that $\mathcal{R}$ contains a set of μ-measure strictly between 0 and 1. Then ϕ is not measurable with respect to $\mathcal{B} \times \mathcal{F}$.

Proof: Suppose that $A \in \mathcal{R}$, $\mu(A) = a$, and $0 < a < 1$. It will be assumed that $\phi^{-1}(A)$ is measurable, and this assumption will be shown to lead to a contradiction. Define $\Psi = \pi \times \nu$ and $b = (a - a^2)/2$. Since $0 < a < 1$, $b > 0$. If $\phi^{-1}(A)$ is measurable, then there exist a finite set I and sets $\{B_i\}_{i \in I} \subseteq \mathcal{B}$ and $\{F_i\}_{i \in I} \subseteq \mathcal{F}$ such that $i \neq j \supset [(B_i \times F_i) \bigcap (B)_j \times F_j) = \emptyset]$ and $\Psi(G) < b$, where
$G = \phi^{-1}(A) \Delta \bigcup_{i \in I}(B_i \times F_i)$. (Halmos (1974),§33, Theorem E and §13, Theorem D).
By Fubini's theorem (Halmos (1974), §36, Theorem B)
$\Psi(G) = \int_\Theta \pi(\phi_\Theta^{-1}(A) \Delta \bigcup\{B_i \mid \theta \in F_i\})\, d\nu(\theta)$. Therefore, for some $H \in \mathcal{F}$, $\nu(H) > 0$ and $\forall \theta \in H\ \pi(\phi_\theta^{-1}(A) \Delta \bigcup\{B_i \mid \theta \in F_i\}) < b$. Define an equivalence relation $\approx$ on H by $\eta \approx \theta$ iff $\forall i \in I\ \eta \in F_i \equiv \theta \in F_i$. This relation induces a finite measurable partition of H, and H is infinite because ν is nonatomic by Lemma 10. Thus there exist distinct η and θ such that $\eta \approx \theta$. Define $J = \bigcup\{B_i \mid \eta \in F_i\} = \bigcup\{B_i \mid \theta \in F_i\}$. Then

(27) $\pi(\phi_\eta^{-1}(A) \Delta J) < b$ and $\pi(\phi_\theta^{-1}(A) \Delta J) < b$.

Since $\pi(\phi_\eta^{-1}(A)) = \pi(\phi_\theta^{-1}(A)) = a$ by (1), (27) implies that

(28) $\pi(J \bigcap \phi_\eta^{-1}(A) \bigcap \phi_\theta^{-1}(A)) > a - 2b = a^2$.

This contradicts the independence of ϕ_η and ϕ_θ, which requires that $\pi(\phi_\eta^{-1}(A) \bigcap \phi_\theta^{-1}(A)) = a^2$. ∎

The hypothesis of Theorem 4 concerning $\mathcal{R}$ (ensuring that μ is nonatomic) evidently cannot be dropped. That is, if $\mu(\{r\}) = 1$ and $\forall\omega\ \forall\theta\ \phi(\omega, \theta) = r$, then ϕ is measurable and satisfies (1) and (16). A further example will show that the hypothesis cannot be weakened to the statement that μ is not concentrated at a single point. Let $\mathcal{E}$ be the σ-algebra of countable and co-countable subsets of $[0, 1]$, and define $\varepsilon(A)$ to be 0 if A is countable and 1 if $X \backslash A$ is countable. Let $\Omega = \Theta = R = ([0, 1], \mathcal{E}, \varepsilon)$. Then any $1-1$ function $\phi : [0, 1]^2 \to [0, 1]$ is product measurable and satisfies (1) and (15).

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