Tracking error: a multistage portfolio model

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Abstract

We study multistage tracking error problems. Different tracking error measures, commonly used in static models, are discussed as well as some problems which arise when we move from static to dynamic models. We are interested in dynamically replicating a benchmark using only a small subset of assets, considering transaction costs due to rebalancing and introducing a liquidity component in the portfolio. We formulate and solve a multistage tracking error model in a stochastic programming framework. We numerically test our model by dynamically replicating the MSCI Euro index. We consider an increasing number of scenarios and assets and show the superior performance of the dynamically optimized tracking portfolio over static strategies.

1 Introduction

Tracking error models have been widely used in recent years in the investment industry. The classical tracking error problem focuses on minimizing the deviations from a benchmark portfolio under some restrictions.

From an investor and regulatory point of view the benchmark represents an objective parameter which can help to evaluate the risk profile of the investment or the fund and it can guarantee transparency. From both an active and passive management point of view targeting a benchmark may result in a more objective evaluation of the performance obtained relating to the current market conditions.

There are many different definitions of tracking error and as a consequence different tracking portfolio models. Rudolf et al. [26] propose a comparison between four different tracking error linear models in a static framework. Roll [25] proposes a mean-variance analysis of tracking error. A positive expected tracking error is equivalent to an average excess return over the benchmark while the variance of the difference between the managed portfolio returns and the benchmark returns equals the variance of the tracking error. This approach yields the *TEV portfolio criterion*, which is the minimization of tracking error variance for a given expected tracking error. The author also discusses the relation between TEV portfolio and mean-variance Markowitz portfolio.

For a relation between tracking error models and tactical asset allocation see for example [1] [8] and for the use of benchmarking in stock selection [24].

Some authors introduce a scenario approach in tracking error models. This allows use of more general distributions and non-linear instruments such as options. Moreover it allows to introduce subjective views on future developments of the market conditions and easily blend them with quantitative forecasts from traditional models. Worzel et al. [29] introduce an integrated simulation and optimization model for tracking fixed-income indexes, where through the simulation part of the model they generate scenarios for the holding period returns of the securities considered. D'Ecclesia et al. [11] considered the problem of tracking a fixed-income index in the Italian market. Dembo and Rosen [12] propose a single-period portfolio replication model in a scenario framework. The method is applied to static replication of barrier options and to capital allocation problems. Dempster and Thompson [13] extend the methodology presented in [12] to a dynamic portfolio replication model in the context of stochastic dynamic programming. They consider the problem of tracking a portfolio of European options and defining a dynamic replicating strategy. For a dynamic tracking error problem and a brief discussion on the limitation of the number of assets to include (i.e. the introduction of cardinality constraints) see [15]. For a comparison of benchmarking with other dynamic asset allocation strategies in Monte Carlo simulation framework see [7].

We are interested in dynamically replicating a benchmark portfolio or index using only a small subset of assets considering transaction costs due to rebalancing and introducing a liquidity component in the model.

In section 2 we analyze different approaches to the concepts of tracking error measures and replication models. In section 3 we discuss some topics arising when we move from a static to a dynamic framework; we present two dynamic models in multistage stochastic programming framework. In section 4 we discuss a solution approach for the second model and in section 5 we test the model by tracking MSCI Euro index. Section 6 concludes.

2 Tracking error measures and models

The goal of a tracking error portfolio model is to replicate, as close as possible, a given benchmark.

The task of properly defining a benchmark is complex since it involves not only regulatory and transparency issues but also constraints on the investing strategies and the risk profile. We assume that the benchmark has been assigned and that we can observe its value at predetermined dates.

Different tracking strategies are used in practice according to the style of investment adopted and to the risk profile assumed, see for example [28].

A first distinction can be made between *full replication* and *partial replication* strategies. The first one consists in building a portfolio with exactly the same composition of the benchmark, that is shares are included with the same market proportions. The second one, partial replication, is done using a subsets of the universe of assets composing the index. In the first case we can have a perfect match of the index behavior but with high transactions cost due to initial composition of the portfolio and rebalancing over time. In the second case we will not have an exact match but there will be lower transaction costs. The objective is to minimize the tracking error, i.e. deviation or distance of the chosen portfolio from the benchmark index, limiting in the same time the transaction costs due to rebalancing.

Let x_t denote the portfolio weights, r_t be the vector of returns on assets, in portfolio at time t, and y_t denote the benchmark return. The tracking error over period $[t_0, T]$ can be defined as a distance measure between the managed portfolio and the benchmark. A generic distance measure is given by the α-norm of the vector of deviations

$$TE(t_0, T) = \left(\sum_{t=t_0}^{T} |r_t' x_t - y_t|^{\alpha} \right)^{\frac{1}{\alpha}}. \tag{1}$$

Specifying the considered norm we obtain different tracking error measures. For example the 1-norm and the ∞-norm give origin to Mean Absolute Deviation and MinMax measures respectively, see for example Rudolf et al. [26], and Konno and Yamazaki [17]. The Euclidean norm yields a quadratic tracking error measure, see for example [25] and [18].

Traditional tracking error measures are symmetric distance measures, i.e. they penalize both positive and negative deviations from the benchmark, and thus can be more suitable for passive strategies where the final objective is to mimic the benchmark. In active tracking strategies positive deviations are usually not only allowed but also desired. This can be achieved introducing a class of asymmetric risk measures which allow to treat separately positive and negative deviations, see for example [14] and [26]. Starting from the previously presented measures, we can define the Mean Absolute Downside Deviation and the Downside MinMax [26].

The Mean Absolute Deviation, the MinMax and the corresponding asymmetric tracking measures give origin to linear programming problems, see [26].

Dembo and Rosen [12] propose a tracking model where the objective function is a weighted average between positive and negative deviations, that is a trade-off between the maximization of expected overperformance and the minimization of expected underperformance. Let $R^-(x; t_0, T)$ be the expected underperformance with respect to the benchmark, i.e. the regret of a portfolio over the period (t_0, T), and $R^+(x; t_0, T)$ the expected overperformance. The objective function of the tracking portfolio model is

$$\min \left[R^-(x; t_0, T) - \lambda R^+(x; t_0, T) \right] \tag{2}$$

where $\lambda \geq 0$ is a weighting parameter. This yields an efficient frontier of replicating portfolios.

Many other characterizations of the tracking problems can be given. For example, it is possible to consider particular quantiles or lower partial moments of the distribution of index and portfolio returns and general distributions with fat-tails, infinite moments and asymmetry. For a description of the lower partial moment framework see [4], and for a relation between some of the tracking error measures and lower partial moments see [26].

Browne [6] considers some nonstandard objective functions related to the achievement of performance goals and shortfalls in the tracking problem. The author considers a model based on the maximization of the probability of beating the benchmark and the correlated objective of minimizing the expected time to reach predetermined goals or shortfall targets. These approaches can lead to chance-constrained optimization problems and to quantile based objective function.

In static tracking error problems an optimal portfolio minimizes the tracking error over a set of past observations and the portfolio is kept for the subsequent period (see for example [25][29]). In this framework the interval $[t_0, T]$ denotes a set of past observations.

For a scenario approach in static models see [12][29][30]. This approach is an improvement on the static models based on past history since it allows forecasts of future realizations and blending of forecasts and subjective views. In this way we move from what can be called a backward perspective, based on realized past returns, to a forward one which is the first step towards dynamic models.

Given a tracking error measure and the corresponding model we can reformulate it in a scenario approach. Below we present an example. In static scenario approach we assume that at the end of the period a finite number of events (scenarios) can occur. The objective of the optimization problem is to minimize the tracking error over all the scenarios (see for example [10][12][29]).

Let $s = 1, \ldots, S$ denote the set of scenarios and π_s their probability of occurrence, r_{ts} and y_{ts} are the vector of returns of the assets in portfolio and the benchmark return at the end of the period under scenario s. Let $TE(t_0, t; s)$ be the generic tracking error measure under scenario s. The objective function is

$$\sum_{s=1}^{S} \pi_s TE(t_0, t; s). \tag{3}$$

In dynamic approaches we need to specify a scenario tree structure and optimize the tracking error over the scenario tree.

One of the main issue to be considered in the forward looking perspective is the relation between the index and the assets used to track the benchmark.

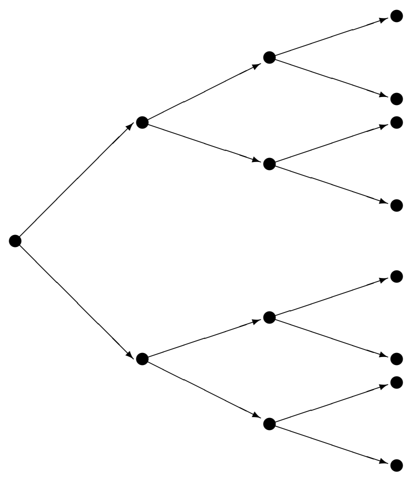

Figure 1: A 3-period, 2-branches scenario tree.

3 Multistage models

Static models usually assume a backward perspective that is they allow to find the portfolio that best tracks the performance of a given benchmark during a past period and then keep it for a subsequent period. Introducing scenarios in a static model improves this approach since the optimal portfolio is composed at the beginning of the period considering different possible future realizations (scenarios) and not only past history.

In moving from a static to a dynamic tracking strategy we need to consider portfolio rebalancing and associated transaction costs.

We present two dynamic models. The first considers a penalty term in the objective function related to the amount of portfolio turnover. The second model extends the formulation, explicitly modelling transaction costs as a percentage of the traded amount, and allows for a cash component in the model which absorbs the transaction costs effects and can also manage other cash inflows or outflows related to the instruments considered in the portfolio. For a discussion on the introduction of cash management in index tracking see [9].

In a dynamic framework each scenario corresponds to a path over the horizon of interest $[t_0, T]$, where $t_0 = 0$ denotes the current date. A set of scenarios, $s = 1, \ldots, S$, is a collection of paths from $t = 0$ to T, with associated probabilities π_s; according to the information structure assumed this collection can be represented as an event tree where the current state corresponds to the root of the tree and each scenario is represented as a path from the origin to a leaf of the tree. In figure 1 we present an example of scenario tree.

Let $\varphi(x_{ts}, y_{ts})$ denote a generic tracking error measure related to deviations of x from the benchmark y at time t under scenario s; $\phi(x_{ts}, x_{t+1\,s})$ be a penalty function for portfolio turnover between two subsequent periods under the same scenario and γ

a penalty parameter. The tracking portfolio objective function of a general multistage problem over the period $[0, T]$ is

$$\min_{\substack{x_{1s}, \ldots, x_{Ts} \\ s=1,\ldots,S.}} \quad \sum_{s=1}^{S} \pi_s \sum_{t=0}^{T-1} [\varphi(x_{t+1\,s}, y_{t+1\,s}) + \gamma\phi(x_{ts}, x_{t+1\,s})]. \tag{4}$$

We want to minimize the sum of the deviations along the horizon with a penalty for turnover which is introduced in order to obtain portfolio weights stability. A slightly modified objective function can be obtained considering the weighted sum of the deviations

$$\min \sum_{s=1}^{S} \pi_s \sum_{t=0}^{T-1} [\beta_{t+1}\varphi(x_{t+1\,s}, y_{t+1\,s}) + \gamma\phi(x_{ts}, x_{t+1\,s})] \tag{5}$$

where $\beta_t,\ t = 1, \ldots, T$ is a sequence of weights which can account for example for preference on the time necessary to reach predetermined goals. A decreasing sequence can also be used to reflect the difficulty of obtaining reliable forecasts for subsequent periods.

To formulate the optimization problem we specify functions $\varphi(\cdot)$ and $\phi(\cdot)$, and introduce possible constraints. We consider two particular specifications of this model which lead to quadratic optimization problems.

Let $x_{ts} \in \mathbb{R}^n$ denote the vector of portfolio weights at time t under scenario s. Let $\varphi(x_{ts}, y_{ts}) = (r'_{ts}x_{ts} - y_{ts})^2$ and $\phi(x_{ts}, x_{t+1\,s}) = (x_{t+1\,s} - x_{ts})'(x_{t+1\,s} - x_{ts})$. The optimization model is

$$\min_{\substack{x_{1s}, \ldots, x_{Ts} \\ s=1,\ldots,S}} \quad \sum_{s=1}^{S} \pi_s \sum_{t=0}^{T-1} [\beta_{t+1}(r'_{t+1\,s}x_{t+1\,s} - y_{t+1\,s})^2 + \gamma(x_{t+1\,s} - x_{ts})'(x_{t+1\,s} - x_{ts})] \tag{6}$$

$$\text{s.t.} \quad x_{ts} \geq 0 \tag{7}$$

$$\sum_{i=1}^{n} x_{its} = 1 \tag{8}$$

$$x_{0\,s} = x_0 \qquad s = 1 \ldots, S; \qquad t = 1, \ldots, T. \tag{9}$$

To introduce transaction costs incurred in portfolio rebalancing we extend the previous model introducing new variables, considering a slightly modified penalty term and minimizing the difference between the values of the portfolio and the index. To quantify the variation in assets holdings we introduce variables $a_{ts} = (a_{1ts}, \ldots, a_{nts})$ and $v_{ts} = (v_{1ts}, \ldots, v_{nts})$, denoting respectively the value of assets purchased and sold at time t under scenario s. We model transaction costs as a constant percentage κ of the traded value.

Moreover we introduce a riskless asset which acts as a liquidity component in the model. The cash asset is the $(n+1)$-th asset in portfolio. Transaction costs reflect on the dynamics of the riskless asset.

Let $z_{ts} = (z_{1ts}, \ldots, z_{n+1\,ts})$ denote the composition of the managed portfolio at time t under scenario s and let y_{ts} be the value of the benchmark under the same conditions. The model is

$$\min_{\substack{a_{0s},\ldots,a_{T-1\,s} \\ v_{0s},\ldots,v_{T-1\,s} \\ s=1,\ldots,S.}} \quad \sum_{s=1}^{S} \pi_s \sum_{t=0}^{T-1} [(\underline{1}' z_{t+1\,s} - y_{t+1\,s})^2 + \gamma\kappa \underline{1}'(a_{ts} + v_{ts})] \tag{10}$$

$$\text{s.t.} \quad z_{it+1\,s} = (1 + r_{i\,t+1\,s})[z_{its} + a_{its} - v_{its}] \qquad i = 1, \ldots, n \tag{11}$$

$$z_{n+1\,t+1\,s} = (1 + r_{n+1\,t+1\,s})[z_{n+1\,ts} - \sum_{i=1}^{n}(1+\kappa)a_{its} + \sum_{i=1}^{n}(1-\kappa)v_{its}] \tag{12}$$

$$z_{its} + a_{its} - v_{its} \geq 0 \qquad i = 1, \ldots, n \tag{13}$$

$$z_{n+1\,ts} - \sum_{i=1}^{n}(1+\kappa)a_{its} + \sum_{i=1}^{n}(1-\kappa)v_{its} \geq 0 \tag{14}$$

$$a_{ts} \geq 0 \quad v_{ts} \geq 0 \tag{15}$$

$$z_{0\,s} = z_0 \tag{16}$$

$$t = 0, \ldots, T-1 \qquad s = 1, \ldots, S.$$

Equations (11)-(12) describe the dynamics of the value of the managed portfolio over time and account for transaction costs. Constraint (16) gives the initial portfolio composition; finally only self-financing strategies are allowed (13)-(14). Taking into account the objective function of the problem and the non negativity constraints (15), constraints (13) can be rewritten as $z_{its} - v_{its} \geq 0$. To ensure non-anticipativity of the optimal decision we should add to problem (10)-(16) a set of non-anticipativity constraints which can be derived from the information structure of the scenario tree.

4 Solution approach

The solution of problem (10)-(16) together with non anticipativity constraints can proceed using stochastic programming techniques. We discuss a solution approach based on a double decomposition. According to this method we can decompose the stochastic dynamic problem both with respect to scenario (stochastic component), and with respect to time (dynamic component), obtaining smaller and easier to solve problems. The decomposition is obtained jointly applying the Progressive Hedging Algorithm (PHA) [23], and a discrete version of Pontryagin Maximum Principle [3][22][27].

In the following we briefly present the solution approach. Problem (10)-(16) is not separable with respect to scenarios, the linking constraints are due to the information structure, i.e. the structure of the event tree, and assure optimal decision to be non-anticipative. To obtain scenario separability Rockafellar and Wets propose to relax the non-anticipativity constraints with an augmented lagrangian approach [23]. The resulting problem can be split into S scenario subproblems which can be solved separately; convergence to the solution of the original stochastic problem is obtained through an iterative procedure which progressively ensures that non-anticipativity constraints are satisfied (Progressive Hedging Algorithm).

Applying this framework to our problem we obtain that each scenario problem is a deterministic dynamic optimization problem that can be solved applying a discrete version of Maximum Principle [21], which leads to a further decomposition with respect to time. For other solution approaches based on stochastic programming see for example [5].

To apply this approach we need to make explicit the non-anticipativity constraints, i.e. the constraints related to the information structure of the problem. We require that the portfolio values z_{ts} in different scenarios, which share a common history up to time t, are equal, in order not to use in the decision process unavailable information

$$z_{tj} = z_{tk} \qquad \forall j,\ k \quad \text{equivalent up to time t.} \tag{17}$$

Using the equations of the dynamics of the risky assets (11) and given the initial portfolio composition (16), which is constant over all the scenarios, constraints (17) are equivalent to

$$a_{tj} - v_{tj} = a_{tk} - v_{tk} \qquad \forall j,\ k \quad \text{equivalent up to time t.} \tag{18}$$

Let $\mathcal{P}_t$ be the partition of the set of scenarios at time t, and denote with $A_t \in \mathcal{P}_t$ a set of scenarios equivalent up to time t, according to the PHA. The non-anticipativity constraints, formulate according to [23], are

$$a_{ts} - v_{ts} = \left(\widehat{a_{ts} - v_{ts}}\right)_{A_t} \quad \forall s \in A_t. \tag{19}$$

where $\left(\widehat{a_{ts} - v_{ts}}\right)_{A_t} = \frac{\sum_{s \in A_t} \pi_s (a_{ts} - v_{ts})}{\sum_{s \in A_t} \pi_s}$.
For a more detailed discussion on the non-anticipativity constraints issue see [5] and [23]. Relaxing non-anticipativity constraints, adding a quadratic penalty term yields

$$\min_{\substack{a_{0s},\dots,a_{T-1\,s}\\ v_{0s},\dots,v_{T-1\,s}\\ s=1,\dots,S.}} \sum_{s=1}^{S}\pi_s\sum_{t=0}^{T-1}[(\underline{1}'z_{t+1\,s}-y_{t+1\,s})^2+\gamma\kappa\underline{1}'(a_{ts}+v_{ts})+$$

$$-W'_{ts}(a_{ts}-v_{ts})+\frac{1}{2}\rho||(a_{ts}-v_{ts})-(\widehat{a_{ts}-v_{ts}})||^2] \tag{20}$$

$$\tag{21}$$

$$\text{s.t.}\quad z_{it+1\,s}=(1+r_{i\,t+1\,s})[z_{its}+a_{its}-v_{its}] \qquad i=1,\dots,n \tag{22}$$

$$z_{n+1\,t+1\,s}=(1+r_{n+1\,t+1\,s})[z_{n+1\,ts}-\sum_{i=1}^{n}(1+\kappa)a_{its}+\sum_{i=1}^{n}(1-\kappa)v_{its}] \tag{23}$$

$$z_{its}-v_{its}\geq 0 \qquad i=1,\dots,n \tag{24}$$

$$z_{n+1\,ts}-\sum_{i=1}^{n}(1+\kappa)a_{its}+\sum_{i=1}^{n}(1-\kappa)v_{its}\geq 0 \tag{25}$$

$$a_{ts}\geq 0 \quad v_{ts}\geq 0 \tag{26}$$

$$z_{0s}=z_0 \tag{27}$$

$$t=0,\dots,T-1 \qquad s=1,\dots,S.$$

Applying the Progressive Hedging Algorithm yields deterministic scenario problems which we reformulate according to the following general scheme of a deterministic discrete time optimal control problem with mixed constraints

$$\min_{\{u_0,\dots,u_{T-1}\}} \{\sum_{t=0}^{T-1}L_t(z_t,u_t)+L_T(z_T)\} \tag{28}$$

$$z_{t+1}=z_t+A_tz_t+B_tu_t+q_t \tag{29}$$

$$z_0=\bar{z}_0 \tag{30}$$

$$G_tz_t+H_tu_t+p_t\geq 0 \tag{31}$$

$$u_t\geq 0 \tag{32}$$

$$t=0,\dots,T-1$$

where $z_t = z_{ts}$ represent the state variables of the control problem and $u_t = u_{ts} = (a_{ts}, v_{ts}) = (a_t, v_t)$ are the control variables. Since in each scenario sub-problem s is fixed for a simpler notation we drop the s subscript.

Let λ_t be the vector of lagrangian multipliers associated with the mixed constraints, ψ_t the adjoint variables associated with the dynamics of the state variables and W_t the multiplier associated with the non-anticipativity constraints in the Progressive Hedging Algorithm. The matrices involved in the problem are defined as

$$A_t=diag(r_{i\,t+1}) \quad i=1,\dots,n+1 \tag{33}$$

$$B_t = \begin{pmatrix} diag(1+r_{i\,t+1}) & -diag(1+r_{i\,t+1}) \\ -(1+r_{n+1\,t+1})(1+\kappa)\underline{1}' & (1+r_{n+1\,t+1})(1-\kappa)\underline{1}' \end{pmatrix} \tag{34}$$

$$G_t = I_{n+1} \qquad\qquad H_t = \begin{pmatrix} 0_n & -I_n \\ -(1+\kappa)\underline{1}' & (1-\kappa)\underline{1}' \end{pmatrix} \tag{35}$$

$$q_t = \underline{0} \qquad\qquad p_t = \underline{0}. \tag{36}$$

where $diag(x_i)$ denotes a diagonal matrix whose diagonal elements are x_i. Moreover $L_T(z_T) \equiv 0$ and

$$\begin{aligned} L_t(z_t,u_t) \quad &= (\underline{1}'z_{t+1} - y_{t+1})^2 + \gamma\kappa\underline{1}'(a_t+v_t) + \\ &\qquad -W_t'(a_t-v_t) + \frac{1}{2}\rho||(a_t-v_t) - (\widehat{a_t-v_t})||^2 \\ &= (\underline{1}'[z_t + A_t z_t + B_t u_t] - y_{t+1})^2 + \gamma\kappa\underline{1}'(a_t+v_t) + \\ &\qquad -W_t'(a_t-v_t) + \frac{1}{2}\rho||(a_t-v_t) - (\widehat{a_t-v_t})||^2. \end{aligned} \tag{37}$$

To solve this problem we apply a discrete version of Maximum Principle [21] [27], and decompose the set of optimality conditions, according to [3], into two main blocks

$$\begin{aligned} min_{u_t} \quad & \{(\underline{1}'(z_t + A_t z_t + B_t u_t) - y_{t+1})^2 + \gamma\kappa\underline{1}'(a_t+v_t) - W_t'(a_t-v_t) + \\ & \qquad + \frac{1}{2}\rho||a_t - v_t - (\widehat{a_t-v_t})||^2 - \psi_{t+1}' B_t u_t\} & (38) \\ \text{s.t.} \quad & H_t u_t \geq -G_t z_t & (39) \\ & u_t \geq 0 & (40) \end{aligned}$$

$$\begin{aligned} max_{\lambda_t} \quad & \{-[G_t z_t]'\lambda_t\} & (41) \\ \text{s.t.} \quad & H_t'\lambda_t \leq \frac{\partial L_t(z_t,u_t)}{\partial u_t} - B_t'\psi_{t+1} & (42) \\ & \lambda_t \geq 0 & (43) \end{aligned}$$

and

$$\begin{aligned} & z_{t+1} = z_t + A_t z_t + B_t u_t & (44) \\ & z_0 = \bar{z}_0 & (45) \end{aligned}$$

$$\psi_t = \psi_{t+1} + A_t \psi_{t+1} - \frac{\partial L_t(z_t, u_t)}{\partial z_t} + G_t' \lambda_t \qquad (46)$$

$$\psi_T = -\frac{\partial L_T(z_T)}{\partial z_T} \qquad (47)$$

$$t = 0, \ldots, T-1$$

For each scenario conditions (38)-(47) can be solved applying an iterative fixed-point method (see [3][16][19]), the obtained solution are then aggregated according to PHA.

5 Testing the model

We test the dynamic tracking model on market data with a rolling simulation procedure to dynamically replicate the MSCI Euro index, a widely used equity index.

The MSCI Euro index[1] is a free float-adjusted market capitalization index. It comprises large and liquid securities with the goal of capturing 90% of the capitalization of the MSCI EMU index, a broader market capitalization index designed to measure equity market performance within European Economic and Monetary Union. The MSCI Euro index is reviewed annually in November.

In figure 2 we present the MSCI Euro Index composition on October 16, 2003. In more detail, at that date, the index comprised 122 securities across 10 countries (Belgium, Finland, France, Germany, Greece, Ireland, Italy, Netherlands, Portugal, Spain).

In order to use only a limited number of assets to track the MSCI Euro index we decide to choose among the MSCI equity indexes of the countries eligible for inclusion in the Euro index. In more detail the dataset has end-of-week values on the MSCI equity indexes for Euro, Belgium, Finland, France, Germany, Ireland, Italy, Netherlands, Portugal, Spain[2] from October 15, 1998 to October 16, 2003[3].

As a first experiment we consider all the 9 different country indexes to track the benchmark, subsequently on the basis of the obtained results we compare the tracking performance of portfolios built on a limited number of indexes.

In order to assess and point up the contribution of the optimization process to the tracking performances of the optimized tracking portfolios we test them against the performances of equally-weighted portfolios built on the corresponding indexes. These portfolios have the same information content of the optimized ones, since they use the same indexes, but lack the optimization process.

The equally-weighted portfolios are built at the beginning of the simulation period without any further rebalancing, we only record their values at the dates of interest.

[1] For a more detailed description of the index composition, methodology and constituents we refer to the publicly available technical documentation [20].

[2] In order to have a wide enough common period we exclude the MSCI Greece index.

[3] According to the MSCI technical documentation the MSCI Euro index has a base date in December 31, 1998 and has initially a daily history back to January 1, 1997.

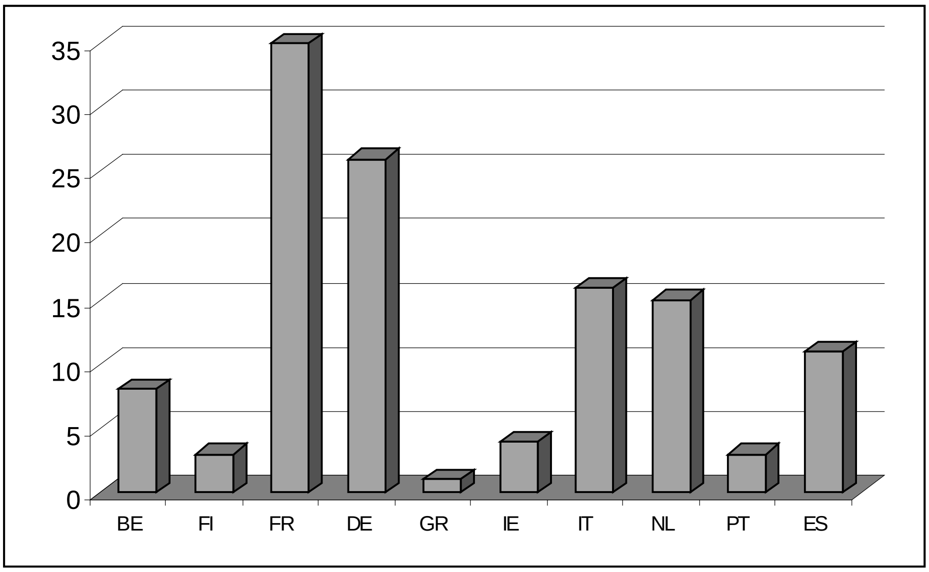

Figure 2: MSCI Euro index composition on October 16, 2003, (BE=Belgium, FI=Finland, FR=France, DE=Germany, GR=Greece, IE=Ireland, IT=Italy, NL=Netherlands, PT=Portugal, ES=Spain).

We assume a constant risk-free weekly interest rate for the liquidity component of the model, set to 0.03%, and constant transactions cost proportional to the traded value set to 0.2% both for buying and selling.

5.1 Scenario generation and simulation procedure

To generate scenario trees we use a historical simulation technique. At each step of the simulation period the scenario tree is generated using a non-parametric bootstrap technique from the past returns. To take into account the co-movements of the series we apply a simultaneous bootstrapping across the series. This approach allows to take into account the historical behavior of the returns on assets including extreme movements.

We assume no special predictive power and thus the tracking results can only be improved whenever we can apply more sophisticated forecasting tools.

To test the tracking model we apply a rolling simulation. For each date in the simulation period a 2-period scenario tree is generated, the corresponding optimization problem is solved and the portfolio uses the resulting first period optimal decision. The portfolio is held for a period and evaluated at the observed market prices, the value of the portfolio thus obtained represents the new starting value for the subsequent period. The sequence of portfolio returns is compared with the observed index returns.

To apply the simulation technique we consider two periods: the first, *bootstrap period*, is used for scenario generation while the second, *simulation period*, is used as testing period for the management of the tracking portfolio. There is a trade-off in the choice of width of the bootstrap period. A longer period can account for a more detailed description of the historical distribution while a shorter one can give a better representation of the current market condition.

For our tracking problem we consider 20-week simulation periods with weekly portfolio revision. At each step we generate a 2-period scenario tree corresponding to two weeks. For each simulation period we consider a bootstrap period which comprises all the observations in the dataset up to the beginning of the simulation period, i.e there is no overlapping between the two periods. In our experiments the bootstrap period is kept constant for all the steps in the simulation period even if it would be possible to update it in order to include, at each step, the new available information.

5.2 Computational results

In the following we present the tracking results for the portfolio management problem over the 20-week simulation period from April 27, 2000 to September 14, 2000. The corresponding bootstrap period ranges from October 15, 1998 to April 20, 2000.

We compare the tracking errors considering increasing scenarios and increasing number of assets using four different tracking measures: Mean Absolute Error (MAE), Root Mean Square Error (RMSE), Mean Percentage Absolute Error (MAPE) and Theil Index, given by

$$MAE = \frac{1}{K}\sum_{k=1}^{K}|\hat{x}_k - x_k| \qquad RMSE = \sqrt{\frac{1}{K}\sum_{k=1}^{K}(\hat{x}_k - x_k)^2}$$

$$MAPE = \frac{1}{K}\sum_{k=1}^{K}\left|\frac{\hat{x}_k - x_k}{x_k}\right| \qquad Theil = \frac{\sqrt{\frac{1}{K}\sum_{k=1}^{K}(\hat{x}_k - x_k)^2}}{\sqrt{\frac{1}{K}\sum_{k=1}^{K}\hat{x}_k^2} + \sqrt{\frac{1}{K}\sum_{k=1}^{K}x_k^2}}$$

where K denotes the number of steps in the simulation period, x_k denotes the value of the index and $\hat{x}_k$ the value of the tracking portfolio.

In our first experiment we consider all the 9 MSCI country indexes to track the benchmark. In figure 3 we compare the values of the MSCI Euro index with the values of optimized portfolio (portf9) and the equally weighted portfolio (ew9), along the simulation period considered. The equally weighted portfolio is built at the beginning of the simulation period and then the values of the portfolio are observed at the dates of interest without rebalancing. In table 1 we compare the error statistics.

In figure 4 we present the portfolio composition in percentage obtained with the 9 country indexes and the liquidity component. From the graph it is evident that the country indexes contributed in different way. In table 2 we describe the portfolio composition by country, in decreasing order of importance (i.e. from the country with

		MAE	RMSE	MAPE	Theil
400 scenarios	portf9	1.6522994	2.0215496	0.0016926	0.0010296
eq-weighted	ew9	6.2771974	7.4391145	0.0063862	0.0037972

Table 1: Error statistics - Comparison between optimized tracking portfolios with 9 assets (400 scenarios) and equally weighted portfolio with 9 components, over 20-week period from April 27, 2000 to September 14, 2000.

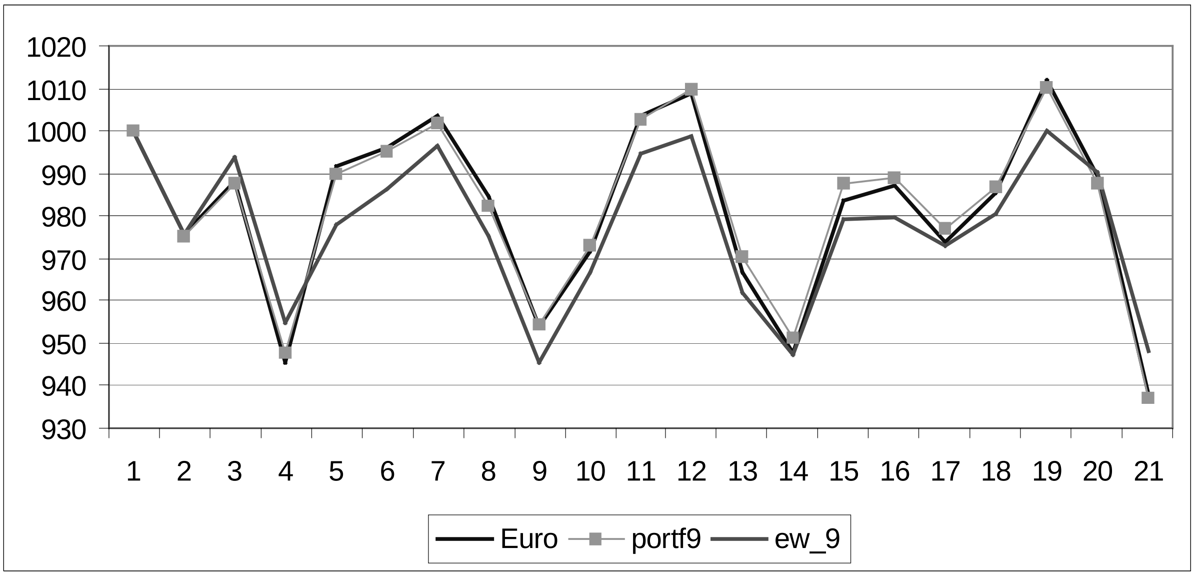

Figure 3: Portfolio values, comparing Euro index, optimal tracking portfolios with 9 components and the equally weighted portfolio with 9 assets. 20-week period from April 27, 2000 to September 14, 2000.

the biggest weight to the one with the smallest); with the parenthesis we denote that the corresponding weight in the portfolio is zero.

In table 3 we present the mean cumulative weight of the first n assets, in decreasing order of importance, in percentage over the whole portfolio; the mean values are obtained over the 20 weeks. It is interesting to observe that the first 4 assets represent, on average, more than the 80% of the portfolio while the first 6 assets represent more than the 95%. According to table 2 we can note that the first 4 assets (ordered by weight importance) are always the same (DE, FR, IT, NL), over the 20-week simulation, even if their relative importance changes over time. They are also the 4 more correlated with the benchmark, as we can see from table 4, where we present the correlation matrix computed on the log-returns, $r_t = ln\left(\frac{P_t}{P_{t-1}}\right)$, obtained from our dataset.

Moving from these observations we test the tracking performance of the model using a smaller number of assets including up to four of the more correlated indexes since they represent a consistent part of the portfolio in the case with 9 indexes. Choosing

week1	week2	week3	week4	week5	week6	week7	week8	week9	week10
FR	FR	FR	DE	FR	FR	FR	FR	FR	FR
DE	DE	DE	FR	DE	DE	DE	DE	DE	DE
IT	IT	IT	IT	IT	IT	IT	IT	IT	IT
NL	NL	NL	NL	NL	NL	NL	NL	NL	NL
FI	FI	FI	ES	FI	FI	FI	FI	FI	FI
ES	ES	ES	FI	ES	ES	ES	ES	ES	ES
cash	BE	BE	PT	BE	BE	BE	BE	BE	BE
PT	PT	IE	BE	PT	PT	PT	PT	PT	PT
IE	IE	PT	IE	(IE)	cash	(IE)	cash	(IE)	IE
(BE)	(cash)	(cash)	cash	(cash)	(IE)	(cash)	(IE)	(cash)	(cash)
week11	week12	week13	week14	week15	week16	week17	week18	week19	week20
DE	FR	DE	DE	DE	DE	DE	DE	DE	DE
FR	DE	FR	FR	FR	FR	FR	FR	FR	FR
IT	IT	IT	IT	IT	IT	IT	IT	IT	IT
NL	NL	NL	NL	NL	NL	NL	NL	NL	NL
FI	ES	FI	ES	ES	FI	FI	FI	FI	FI
ES	FI	BE	BE	BE	ES	ES	ES	ES	ES
BE	BE	ES	FI	FI	BE	BE	BE	BE	BE
PT	PT	PT	PT	PT	PT	PT	PT	PT	PT
IE	cash	IE	(IE)	cash	IE	IE	IE	IE	IE
(cash)	IE	(cash)	(cash)	(IE)	(cash)	(cash)	(cash)	(cash)	(cash)

Table 2: Portfolio composition in decreasing order of importance, the parenthesis denote that the corresponding weight is exactly zero. 20-week period from April 27, 2000 to September 14, 2000.

n. assets included	cumulative %
1^{st}	27.36871
1^{st}:2^{nd}	52.42472
1^{st}:3^{rd}	67.61705
1^{st}:4^{th}	80.92942
1^{st}:5^{th}	88.83516
1^{st}:6^{th}	95.32503
1^{st}:7^{th}	98.24130
1^{st}:8^{th}	99.56298
1^{st}:9^{th}	99.98713
1^{st}:10^{th}	100.00000

Table 3: Cumulative portfolio composition, in percentage, with increasing number of assets - mean value over 20 weeks.

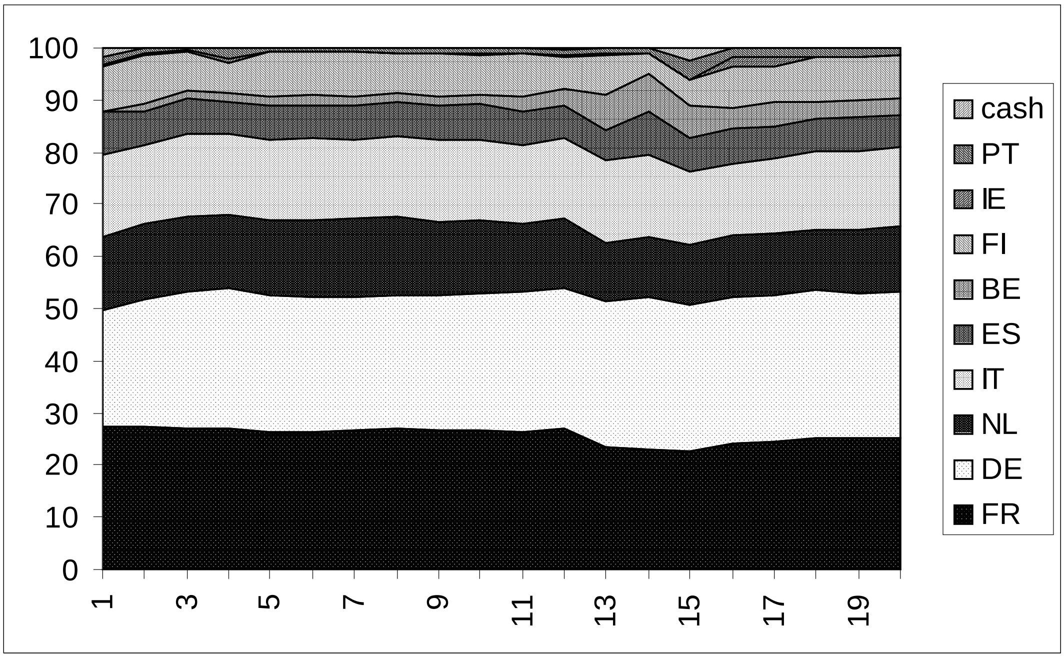

Figure 4: Percentage portfolio composition (9 components and cash, 20-week period from April 27, 2000 to September 14, 2000).

the more correlated indexes induce us to expect a good tracking performance, thus we test also the opposite experiment choosing the less correlated ones.
We denote with case 1, respectively case 2, the tests carried using the more correlated, respectively the less correlated, indexes with respect to the benchmark.

In each experiment we consider 3 different tracking portfolios composed with 1, 2 and 4 indexes respectively, according to table 5.
The end-of-week prices of the MSCI equity indexes used in the experiments are presented in figures 5 and 6.

In tables 6 and 7 we present the tracking error statistics of the optimized tracking portfolios, considering an increasing number of scenarios, and the equally-weighted

	Euro	Belgium	Finland	France	Germany	Ireland	Italy	Netherland	Portugal	Spain
Euro	1.0000	0.7202	0.7041	0.9666	0.9507	0.6071	0.8974	0.9099	0.5728	0.8393
Belgium	0.7202	1.0000	0.3672	0.6944	0.6969	0.5370	0.6032	0.7478	0.4059	0.5989
Finland	0.7041	0.3672	1.0000	0.6382	0.5968	0.3634	0.5374	0.5515	0.4022	0.5199
France	0.9666	0.6944	0.6382	1.0000	0.8916	0.5678	0.8578	0.8638	0.5549	0.8052
Germany	0.9507	0.6969	0.5968	0.8916	1.0000	0.6035	0.8322	0.8513	0.5376	0.7649
Ireland	0.6071	0.5370	0.3634	0.5678	0.6035	1.0000	0.5175	0.5735	0.3864	0.5132
Italy	0.8974	0.6032	0.5374	0.8578	0.8322	0.5175	1.0000	0.8022	0.5212	0.7720
Netherland	0.9099	0.7478	0.5515	0.8638	0.8513	0.5735	0.8022	1.0000	0.4367	0.7187
Portugal	0.5728	0.4059	0.4022	0.5549	0.5376	0.3864	0.5212	0.4367	1.0000	0.5952
Spain	0.8393	0.5989	0.5199	0.8052	0.7649	0.5132	0.7720	0.7187	0.5952	1.0000

Table 4: Correlations for MSCI indexes (October 15, 1998 to October 16, 2003).

	Case 1	Case 2
portf1	FR, cash	PT, cash
portf2	FR, DE, cash	PT, IE, cash
portf4	FR, DE, NL, IT, cash	PT, IE, FI, BE, cash

Table 5: Indexes used in the tracking portfolios in the two cases.

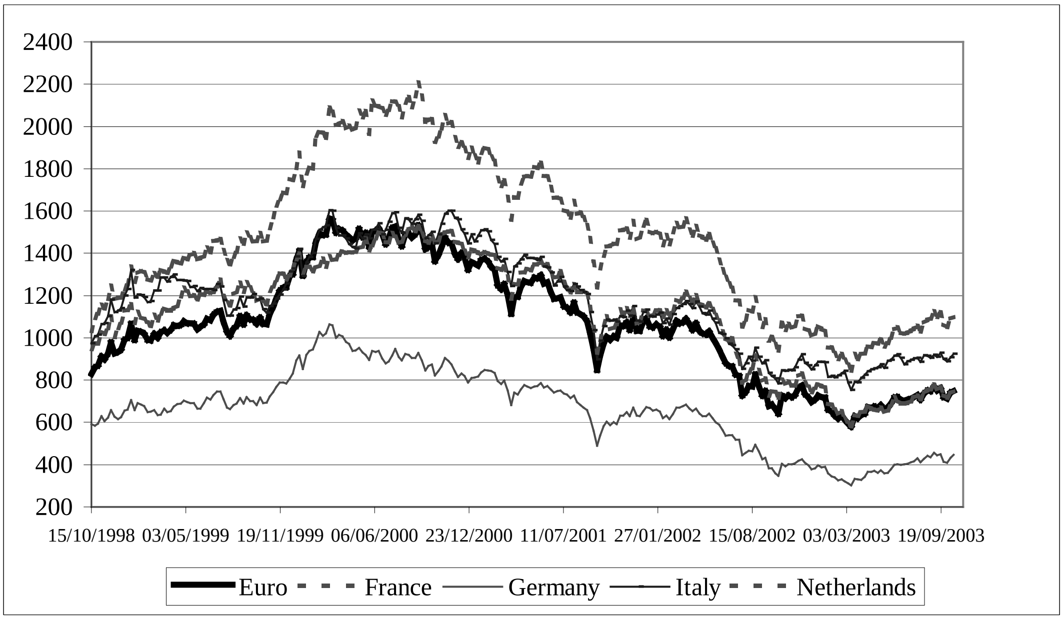

Figure 5: MSCI indexes (Euro, France, Germany, Italy, Netherlands) end-of-week prices, from October 15, 1998 to October 16, 2003.

portfolios.

As we may expect the tracking performances are better in case 1. In both cases the tracking performances improve considering an increasing number of scenarios and the optimized portfolios generally perform better than the corresponding equally-weighted portfolios. The obtained results confirm the effectiveness of the dynamically optimized tracking strategy.

In figures 7 and 10 we compare the values of the MSCI Euro index with the values of optimized portfolios (1, 2 and 4 indexes), along the simulation period considered.

In figures 8 and 9 we present the comparison between the MSCI Euro index, the optimized tracking portfolios and the corresponding equally-weighted portfolios, along the simulation period, in case 1. The same in figures 11 and 12 for case 2.
To analyze the role of the penalty term, introduced to reduce the portfolio turnover, we compare the compositions of the optimized portfolio along a testing period under different values of the parameter γ which accounts for penalties on transaction costs. In figure 13 we present the results related to a 10-week simulation period, from

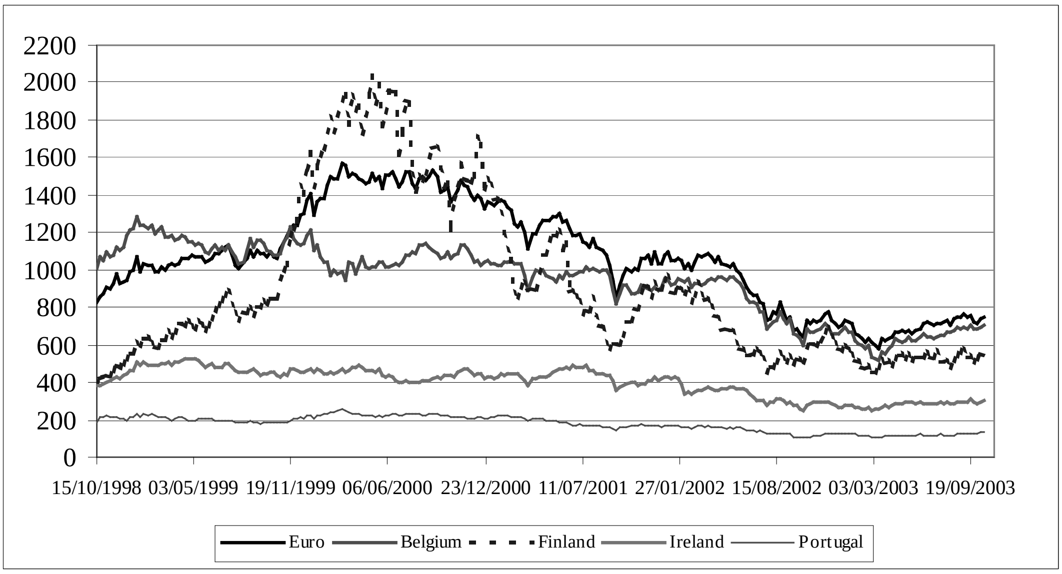

Figure 6: MSCI indexes (Euro, Portugal, Ireland, Finland, Belgium) end-of-week prices, from October 15, 1998 to October 16, 2003.

		MAE	RMSE	MAPE	Theil
	portf1	15.2409633	18.4441628	0.0156294	0.0093284
100 scenarios	portf2	7.1329005	8.6309208	0.0072752	0.0044052
	portf4	7.5959154	9.6077199	0.0078239	0.0048803
	portf1	13.8424462	16.8031368	0.0142496	0.0036651
400 scenarios	portf2	6.1604654	7.1973992	0.0062943	0.0085053
	portf4	6.6230482	9.0864636	0.0068367	0.0046140
eq-weighted	ew2	6.6182411	7.9195683	0.0067625	0.0040302
	ew4	24.3481804	26.6740613	0.0249141	0.0134222

Table 6: Error statistics - Comparison between optimized tracking portfolios with 1, 2, 4 assets and equally weighted portfolio with 2 and 4 components, over 20-week period from April 27, 2000 to September 14, 2000 - Case 1 (more correlated).

		MAE	RMSE	MAPE	Theil
	portf1	50.4548902	59.1034617	0.0518457	0.0293533
100 scenarios	portf2	25.1082042	30.1882847	0.0256653	0.0153064
	portf4	18.5033082	24.4694372	0.0189764	0.0124795
	portf1	42.8315089	51.8853848	0.0440924	0.0258699
400 scenarios	portf2	20.3095160	25.6375723	0.0209126	0.0129419
	portf4	13.3671423	17.0109510	0.0137131	0.0086830
eq-weighted	ew2	31.7867019	38.7660474	0.0325101	0.0200461
	ew4	29.4195896	33.5099816	0.0299315	0.0173169

Table 7: Error statistics - Comparison between optimized tracking portfolio with 1, 2, 4 assets and equally weighted portfolio with 2 and 4 components, over 20-week period from April 27, 2000 to September 14, 2000 - Case 2 (less correlated).

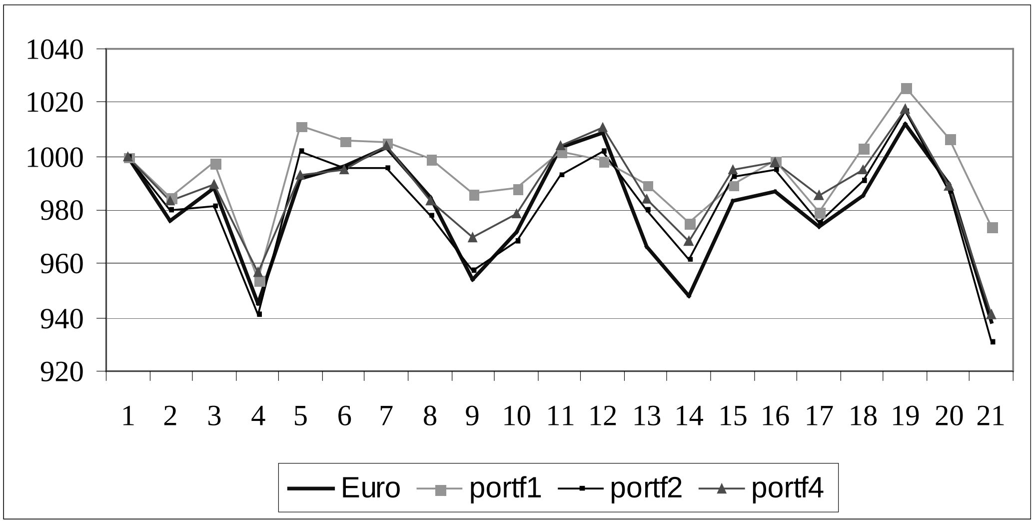

Figure 7: Portfolio values, comparing Euro index, optimal tracking portfolios with 1, 2 and 4 components respectively. 20-week period from April 27, 2000 to September 14, 2000 - Case 1 (more correlated).

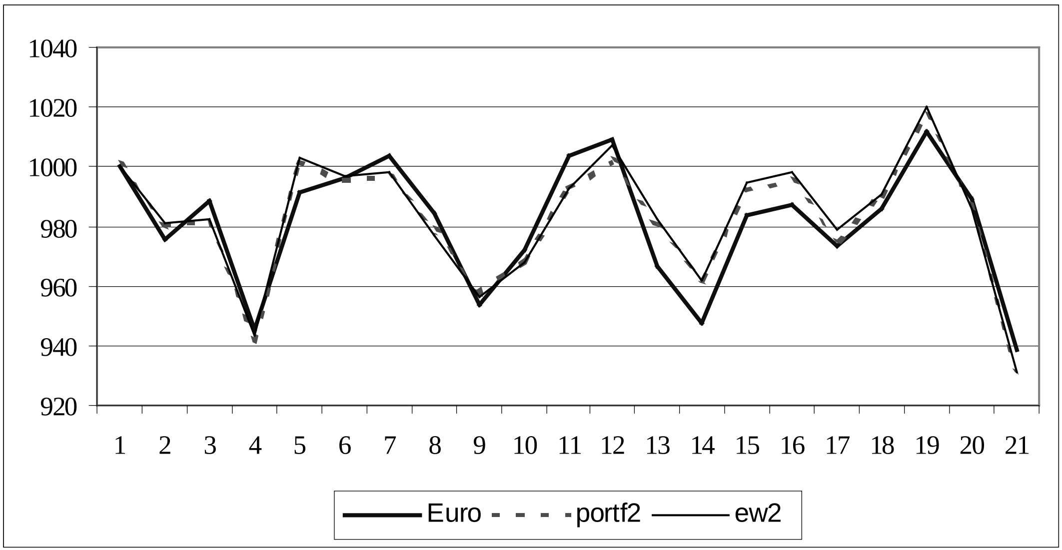

Figure 8: Comparison between the Euro index, the optimized portfolio and the equally weighted portfolio with 2 assets, 20-week period from April 27, 2000 to September 14, 2000 - Case 1 (more correlated).

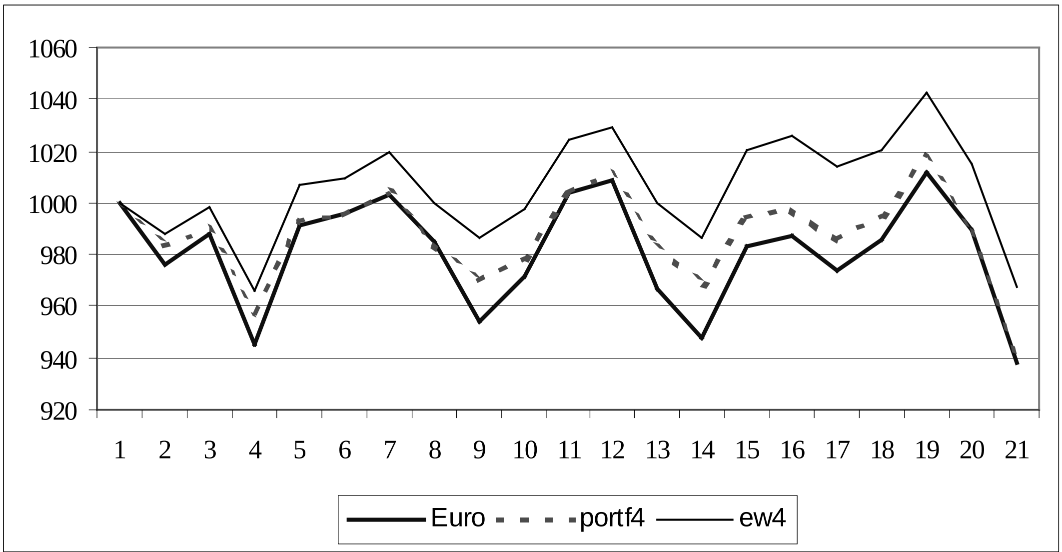

Figure 9: Comparison between the Euro index, the optimized portfolio and the equally weighted portfolio with 4 assets, 20-week period from April 27, 2000 to September 14, 2000 - Case 1 (more correlated).

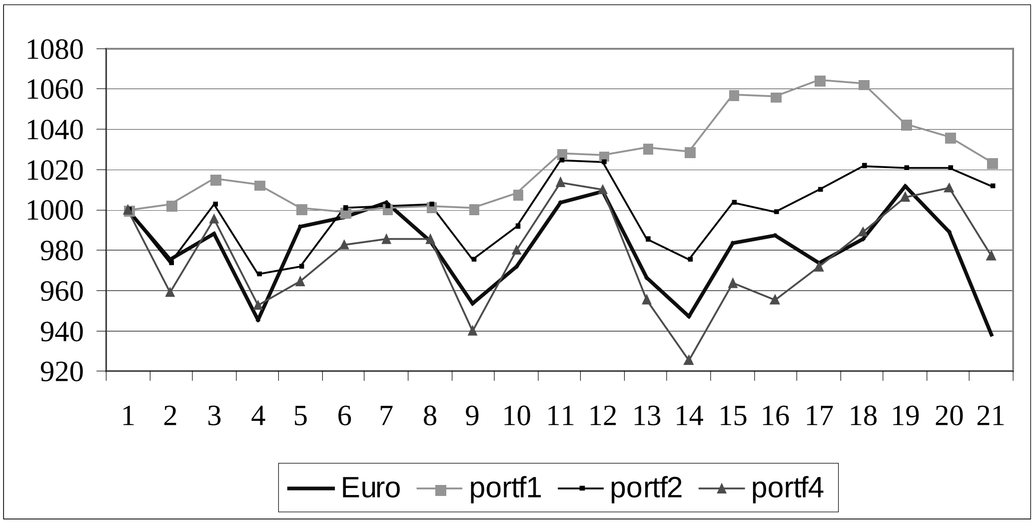

Figure 10: Portfolio values, comparing Euro index, optimal tracking portfolios with 1, 2 and 4 components respectively. 20-week period from April 27, 2000 to September 14, 2000 - Case 2 (less correlated).

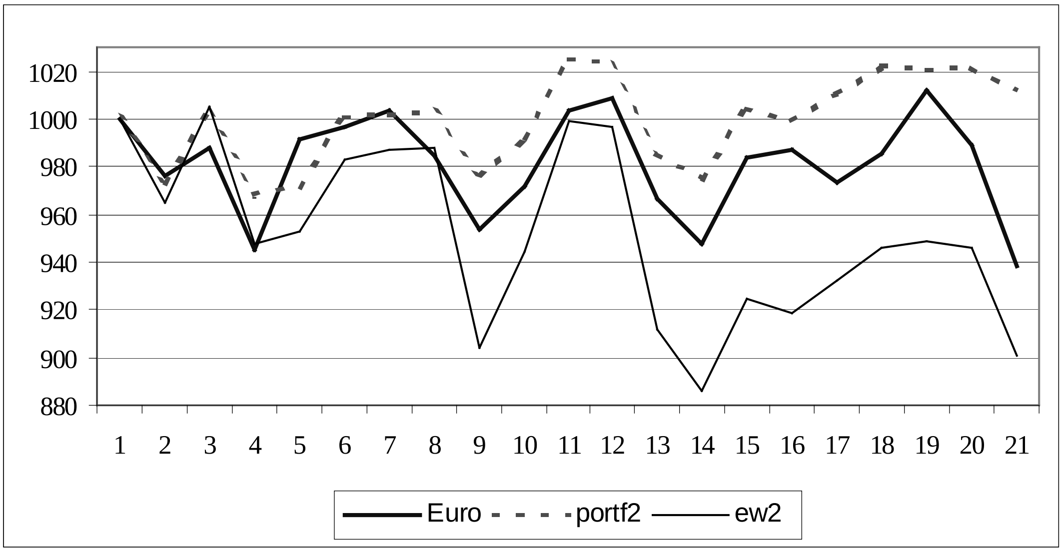

Figure 11: Comparison between the Euro index, the optimized portfolio and the equally weighted portfolio with 2 assets, 20-week period from April 27, 2000 to September 14, 2000 - Case 2 (less correlated).

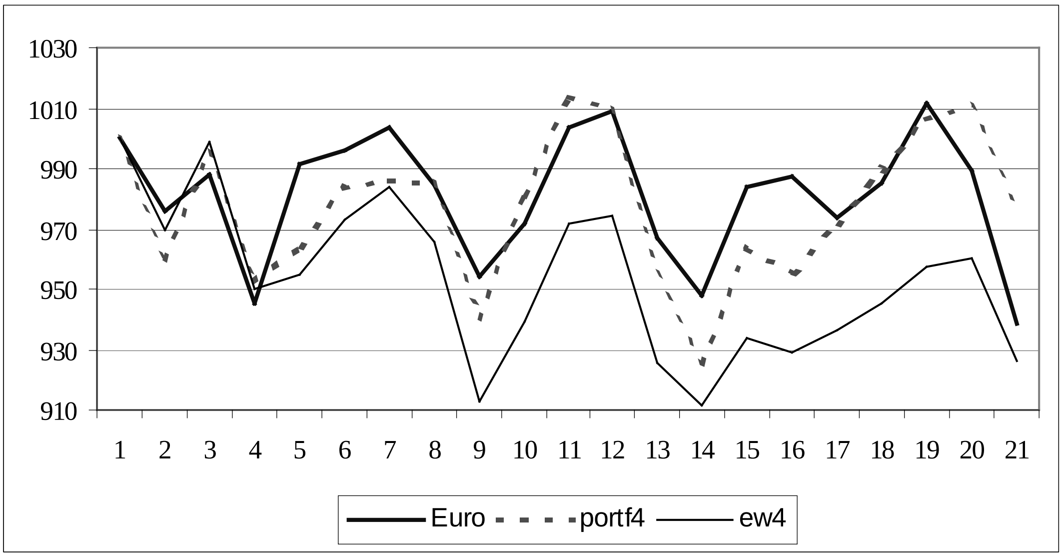

Figure 12: Comparison between the Euro index, the optimized portfolio and the equally weighted portfolio with 4 assets, 20-week period from April 27, 2000 to September 14, 2000- Case 2 (less correlated).

September 07, 2000 to November 16, 2000, considering two different values of the penalty parameter γ, respectively $\gamma = 1$ and $\gamma = 10$. The tracking portfolio used considers the first 4 more correlated indexes, i.e. DE, FR, IT, NL.

The results show that the optimal portfolio composition is quite stable along the simulation period and moreover it is barely sensitive to changes in the parameter γ.

6 Concluding remarks

We have considered the problem of tracking an index in a multistage framework. We focused on partial replication with transaction costs and liquidity component in the portfolio. The resulting stochastic dynamic problem can be approached through stochastic programming techniques using a double decomposition approach which allow to obtain smaller and easier to solve subproblems.

We test the model using data considering the problem of dynamically tracking an index over different periods. We test the model with increasing number of scenarios and increasing number of assets in the tracking portfolio. The tracking portfolio performance is compared with the performance of equally weighted portfolios. The results are positive and the optimized portfolios overperform the equally weighted ones even if the scenario generation technique is very simple.

The tracking portfolio performance is related to the scenarios used in the optimization process. In particular the portfolio will perform well if scenarios reflect the real market conditions and future evolutions. To generate scenario trees we use

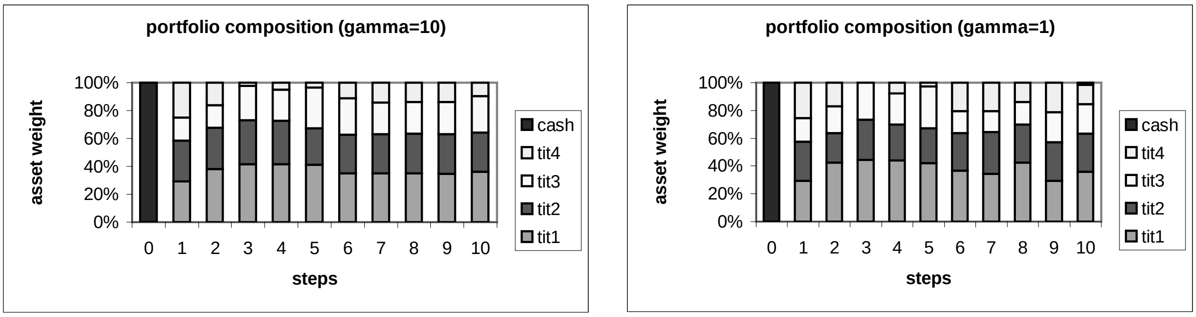

Figure 13: Optimal portfolio compositions with $\gamma = 10$ and $\gamma = 1$, 10-week simulation period from September 07, 2000 to November 16, 2000 - portf4 - Case 1.

a historical simulation assuming no special predictive power thus any enhancement in the scenario generation technique might improve the results.

A further step would be the extension of the model to include asymmetric tracking error measures more suitable in the case of actively managed portfolios. Moreover the scenario generation technique can be improved using a filtered historical simulation, see [2], or some parametric models to be estimated on historical data.

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