Optimal growth path in an OLG economy without time-preference assumptions

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To the memory of professor Abd el Hamid Trad

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Abstract

The goal is to characterize optimal growth path in an OLG economy where capital accumulation is achieved through bequests, without using the assumptions of time preference theory on a social level.

First I investigated the Pareto optimality of a bequests plan, which is a sequence of bequests from one generation to the next one $(k_1, k_2...)$.

I found that, in the set $L_- \cup L_+$ which can be seen as the set of non-oscillating bequests plans, under conditions (regularity, concavity, monotony, interiority...), the condition:

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{D_2U(k_n, k_{n+1})}{D_1U(k_{n+1}, k_{n+2})} \right| \prec +\infty$$

is necessary and sufficient for Pareto-optimality, where $U(k_{n-1}, k_n)$ is the level of utility achieved by generation $g_n$with an heritage k_{n-1} and a bequest k_n, and D_1U and D_2U are respectively the first and the second partial derivative of U.

This condition implies that, for "most" generations we have:

$$-D_2U(k_{n-1}, k_n) \leq D_1U(k_n, k_{n+1})$$

which means that generation g_n wins less from a decrease of its bequest by one unit, than what is lost by generation g_{n+1}.

Since the concept of Pareto optimality is too wide to define the optimal bequests plan, I investigated consensual optimality. A consensually optimal bequests plan is solution to

$$\max_B \Psi \left[(U(b_{i-1}, b_i))_{i \geq 1} \right]$$

where Ψ is a consensual optimality criterion. I showed, under conditions on U and Ψ that if Ψ is sensitive to long run interest, a steady state consensual optimum converges necessarily to k^*, where $k^*$is solution of

$$D_1U(k^*, k^*) + D_2U(k^*, k^*) = 0$$

Moreover, there is a clash between Pareto-optimality and "egalitarianism" of such a criterion Ψ, where egalitarianism is the quality of a criterion which treats all generations the same way. However, global egalitarianism may lead to Pareto-optimality. I qualified such an egalitarian and Pareto-optimal bequests plan, as optimal growth path.

I then showed, under assumption of intra-life time neutrality, that the golden rule is asymptotically observed in an optimal growth path.

To make an optimal growth path consistent with spontaneous equilibrium, I used familial altruistic utility and determined what familial altruism intensity can lead to the optimal growth path.

1 Introduction

The aim of this work is to characterize optimal growth path in an OLG economy where capital accumulation is achieved through bequests, without using the assumptions of time preference theory on a social level, because such assumptions introduce necessarily inequality between the different generations of the society.

Section2 expounds the economic motivations of the problem and optimality concepts.

Section3 presents main formal definitions and assumptions.

To find out the first order conditions necessary for a bequests plan to be Pareto-optimal, I apply to it the Kuhn-Tucker theorem.[1]. This theorem uses the differentials of objective function and constraints function, and also the dual variables. That's why a study of the structure of l_∞^* is needed. It is the object of section4.

The study of l_∞^* in section4 leads to the calculus of the differential of a function on l_∞, which will be used to solve consensual optimality and extract optimal growth paths from the set of Pareto-optima.

Section5 sets up the first order necessary condition for Pareto-optima after computing the differentials of the objective function U_i and the constraints function H_i.

With additional assumptions on the life-utility function U, particularly concavity and differentiability on the border, section6 studies the sufficiency of first order condition to guarantee the Pareto-optimality of a bequests plan.

Section7 analyses examples of Pareto-optimal bequests plans, particularly steady states ones.

The concept of consensual optimality, properties of consensual optimality criterions and relation between consensual optima and Pareto optima are worked out in section8. Section9 shows that, under assumption of intra-life time neutrality, golden rule is an asymptotic property of optimal growth paths. Section10 determines to what extent optimal growth paths are consistent with spontaneous equilibrium.

To lighten the text, some proofs have been gathered in section11, while less important ones can be found in appendices.

Conclusion is written as an abstract for someone who read all the text, with hints on possible improvements. It emphasizes on economic interpretation and does not get back to mathematical aspects.

[1] See appendix4

2 Motivation

2.1 Infinite horizon sum

Optimization mathematics allowed to build economic models where decisive economic parameters like the rate of savings or the rate of technical change, were no longer data of the models, but results explained and computed by the models.

This increases the explanatory power of growth models, but make them reckon on the assumption of infinite horizon sum.

Indeed, if we denote u the instantaneous utility function and $c(t)$ the consumption at date t, the total utility of the society is supposed to be:

$$\int_0^{+\infty} u(c(t))e^{-\rho t}dt$$

Thus, to find the best *growth path* of economic growth, one maximizes the above criterion over $c(t)$ under an evolution constraint like:

$$\begin{aligned} \dot{k} + c(t) &= f(k(t)) - a \cdot k(t) \\ \text{with } c &\in [0,\ f(k)] \text{ and } k(0) = k_0 \end{aligned}$$

where a is the rate of capital depreciation, f the production function, k the capital and c the consumption.

The discrete version of this kind of optimality criterion is a discounted sum of utility levels, depending on time-periods or generations in OLG models:

$$\sum_{t=1}^{+\infty} \frac{u(c(t))}{(1+\rho)^t}$$

Thus, the economy would behave like an infinite-lived individual with a time discount rate equivalent to the intergenerational discount rate.

As shown by Barro (1974), with operative intergenerational transfers, this is acceptable in the context of the usual assumptions of time-preference theory and with a positive rate of own generation preference.

On the personal level, time-preference theory assumes that one always prefers present goods to future goods. Although this can be criticized in absence of uncertainty and irrationality, it is not the purpose of this work.

However, the concept of "*best growth path*" is mainly concerned about social optimality which relation to time-preference theory is somewhat questionable: is it natural for social optimality to prefer present time over future? or present generations over distant ones?

Therefore, we can relevantly ask if we should keep on using the same time-preference assumptions when looking for the *best growth path* and if we should keep on using an infinite horizon sum criterion which inevitably favours present generations to the detriment of distant ones.

If we should not favour present generations to the detriment of distant ones, as pointed out by Allais (1946) or Mankiw (2001), how can we then compute a social optimum and test its economic efficiency without using the infinite horizon sum?

This is the object of the present work.

First of all, we have to specify the concepts of Pareto optimality and consensual optimality that form social optimality and will be used to define the optimal growth path.

2.2 Pareto optimality

To understand optimality problems in OLG models, we need first to look for Pareto-optimal consumption allocations $c(t)$. This issue has been first studied by theorists like Cass(1972), Balasko-Shell(1980) and Wilson(1980), but with different mathematical tools and in a somewhat different context, probably more general because of the focus of this work on capital accumulation. However, this work permits (I hope) to give some interesting results in an issue as central as capital accumulation in growth theory.

Intergenerational Pareto-optimality of a consumption allocation means that it is not possible to find a better way to distribute the consumption so as to strictly enhance the utility of one generation without diminishing utilities of one or more other generations.

In the beginning of its economic life, a given generation receives a capital k_h as heritage. It consumes and invests during it's life and disappears bequeathing a capital k_l.

During its life, given k_h and k_l, the generation chooses the consumption $c(t)$ that maximizes its individual intra-life utility.

Let k_0 be the capital inherited by the first generation g_1, k_1 the capital bequeathed by $g_1$to g_2, k_2 the capital bequeathed by $g_2$to g_3 and so on...

Given the vector *bequests plan* $K = (k_1, k_2, \cdots)$, each generation g_i maximizes its individual intra-life utility and determines its consumption $c_i(t)$ and its life-utility U_i.

Thus, we can see that the allocation of consumption and the distribution of utility between generations depend only on the vector K. So, we can speak of intergenerational Pareto-optimality of the vector K: a bequests plan K is Pareto-optimal if there is no other bequests plan that strictly enhances the utility of one generation without diminishing the utilities of one or more other generations.

If we exclude technical changes, the utility level reached with a heritage k_{i-1} and a bequest k_i depends only on k_{i-1}, k_i. So, we can write:

$$U_i = U_i(K) = U(k_{i-1}, k_i)$$

We can immediately see that if K is Pareto-optimal, it is a solution to the program $P_i(K)$ for all $i \geq 1$:

$$\max_B U(b_{i-1}, b_i)$$
$$subject\ to \quad : \quad U(b_{j-1}, b_j) \geq U(k_{j-1}, k_j)\ \forall j \geq 1,\ j \neq i$$

2.3 Consensual optimality

The concept of Pareto-optimality corresponds to the idea of efficient use of resources, but it does not take into account the social **consensus** underlying social stability and durability. For example, a situation where a unique individual owns all the wealth can be Pareto-optimal, but it is clearly not a socially stable situation and a social optimum.

Consequently, **an optimal growth path has not only to be a Pareto-optimum, but it has also to respect a consensual criterion Ψ reflecting the social consensus**. Consensual optimality is then given by the program $S(\Psi)$:

$$\max_B \Psi\left[(U(b_{i-1}, b_i))_{i \geq 1}\right]$$

The form of Ψ depends on the political system, social values...It is just as if Ψ expresses the preferences of an "out of the society and time" planner who incarnates the values and has widely agreed moral authority. We can think about this criterion as a kind of intergenerational GDP.

3 Formalism

To concentrate on optimality problems, consider an OLG economy without demographic growth and where individuals of each generation are exactly similar. Moreover, exclude intra-generational exchanges to eliminate wealth-distribution and prices questions. Exclude also, as a first approach, technical change. Capital accumulation is achieved through bequests from one generation to the next one.

3.1 Definitions

Denote $B = (b_i)_{i \geq 1}$ the sequence of bequests from one generation to the next one and $U(b_{i-1}, b_i)$ the level of utility a generation g_i reaches with a heritage b_{i-1} and a bequest b_i, where U is a functional defined on a subset D_u strictly included in R_+^2.

$U(b_{i-1}, b_i)$ is then the life-utility. It is distinct from the instantaneous utility $u\left(c\left(t\right)\right)$ one achieves at the instant t with a consumption $c\left(t\right)$.

Name the sequence $(b_i)_{i \geq 1}$: a **bequests plan**.

Let $l_{\infty+}$ be the set of real positive and bounded sequences, k_0 a real positive number.

Denote:

$$D = \{K = (k_1, k_2, \cdots) \in l_{\infty+} \ / \ \forall i \geq 1 : (k_{i-1}, k_i) \in D_u\}$$

Suppose that D_u is closed, and that it is such as the interior of D is not empty. Let Ψ be a Frechet-differentiable functional on l_∞.

For $K \in D$ and i an integer ≥ 1, denote respectively $P_i\left(K\right)$ and $S\left(\Psi\right)$ the following programs:

$P_i\left(K\right) =$

$$\max_{B \in D} U(b_{i-1}, b_i)$$
$$\textit{subject to} \quad : \quad U(b_{j-1}, b_j) \geq U(k_{j-1}, k_j) \ \forall j \geq 1, \ j \neq i$$

where b_{i-1}, b_i are the i-1th and the ith components of B.

$S\left(\Psi\right) =$

$$\max_{B \in D} \Psi\left[\left(U(b_{i-1}, b_i)\right)_{i \geq 1}\right]$$

A bequests plan K is a Pareto-optimal bequests plan if and only if it is solution to $P_i\left(K\right)$ for all $i \geq 1$ and K is a consensual optimum if and only if it is solution to $S\left(\Psi\right)$.

The aim of this work is to characterize Pareto-optimal bequests plans and consensual optima. This amounts to characterize solutions of $P_i\left(K\right)$ and $S\left(\Psi\right)$.

3.2 Assumptions

The following assumptions will be adopted when necessary:

- D_u is strictly included in R^2, closed and with a non-empty interior
- The interior of D is not empty
- U is of class C_1 on the interior of D_u , and continuous on D_u
- $D_1U \succ 0$ (D_1U is the derivative of U with respect to its first variable)
- D_u convex, U concave. One can then show easily that D is also convex.

The condition that U is C_1 is a condition of preferences regularity.

The concavity of U means that every mixing between 2 bequests plans is preferred to the worst of them, which is an usual and acceptable assumption.

The condition $D_1U \succ 0$ means that life utility increases when one gets more heritage, which seems also quite reasonable.

When the utility $U(b_{i-1}, b_i)$ of generation g_i comes from an optimization program like:

$$\begin{aligned} & \max_c \int_0^T u(c(t))\delta(t)dt \\ \text{subject to} \quad : \quad & \dot{k} + c(t) = f(k(t)) - a \cdot k(t); \\ k(0) \quad = \quad & k_h; k(T) = k_l \text{ and } c \in [0,\ f(k)] \end{aligned}$$

where u is the instantaneous utility function, δ is a function weighing instantaneous utilities of consumption during the life, a the rate of capital depreciation, f the production function and T the life period, we assume that u and f are **concave** and **increasing** and that f' **decreases below the parameter** a.[2] We can then check that the sequence of bequests is bounded.

We see also that for all $k_h \geq 0$ there is $k_{l\max}$ and $k_{l\min}$ such that:

$$k_l \succ k_{l\max} \Longrightarrow U(k_h, k_l) \text{ is not defined}$$

and

$$k_l \prec k_{l\min} \Longrightarrow U(k_h, k_l) \text{ is not defined}$$

So, D_u is indeed strictly included in R^2. This means that, with a heritage k_h, whatever be the consumption sacrifice consented, one cannot bequeath more than $k_{l\max}$, and whatever be the consumption abuse, one cannot bequeath less than $k_{l\min}$.

Hence, generally, it comes out that if a bequests plan K is at the frontier of D then: either the bequest of at least one generation is extreme, or there is a tendency, even episodically, to this behavior when time goes to infinity.

[2] Since the object of this work is not the critique of the decreasing returns hypothesis, but that of infinite horizon optimization, I choose to stay in the framework of neoclassical concav production function.

4 Preliminary : Study of l_∞^*

4.1 Notations and properties used

Let R be the real line and N the set of positive integers. Let c be the set of real converging sequences, c_0 the set of real sequences converging to 0, l_∞ the set of real bounded sequences and l_1 the set of real sequences verifying $\sum_{n\geq 1} |x_n| \prec +\infty$.

c, c_0 and l_∞ are Banach spaces for the norm $\|x\| = \sup\limits_{n\geq 1} |x_n|$ (where $x = (x_1, x_2...)$).

We have:

- $l_1 \subset c_0 \subset l_\infty$
- $l_1 \subset l_\infty^*$ [3]
- $l_1^* = l_\infty$ [4]
- $c_0^* = l_1$ [5]

4.2 Decomposition of an element $y \in l_\infty^*$

Denote δ_∞ the linear functional defined on c by $\delta_\infty(x) = \lim x_n$.

Lemma[6]*: Let $y \in l_\infty^*$. Then we can write in a unique manner:*

$$y = y_1 + y_2$$

where y_1 verifies:

$$\sum_{i=1}^{+\infty} |y_{1i}| \prec +\infty$$

and y_2 is such as its restriction to c is proportional to δ_∞.

[3] If E is a real normed vector space, we denote E* the dual of E.
[4] For proof, see Luenberger p108, quoted reference
[5] For the proof, see appendix2
[6] See 11-1 for proof

4.3 Application to the calculus of the differential of a function on l_∞

Let f be a function from l_∞ to R, Frechet-differentiable at $x_0 \in l_\infty$. Denote $\delta f(x_0)$ the Frechet-differential of f at x_0. By definition, $\delta f(x_0) \in l_\infty^*$. We then apply Lemma1:

$$\delta f(x_0) = \delta f_1(x_0) + \delta f_2(x_0)$$

where $\delta f_1(x_0) \in l_1$ and $\delta f_2(x_0) \in c_0^\perp$.

Denote the restriction of $\delta f_2(x_0)$ to c by $\delta f_2(x_0)\lfloor c$. Then, there is a real $\alpha\,(x_0)$ such that:

$$\delta f_2(x_0)\lfloor c = \alpha\,(x_0)\,\delta_\infty$$

We can consider $\delta f_1(x_0)$ as the finite part of the differential, and $\delta f_2(x_0)$ as the infinite part. Denote $\alpha\,(x_0)$ by $\frac{\partial f}{\partial \infty}\,(x_0)$. We prove[7] the following proposition:

Proposition1 *If f is a function from l_∞ to R, Frechet-differentiable at $x_0 \in l_\infty$, we have:*

$$\frac{\partial f}{\partial \infty}\,(x_0) = \lim_{\|h\| \longrightarrow 0, h \in c, \delta_\infty(h) \neq 0} \frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\delta_\infty(h)}$$

and

$$\frac{\partial f}{\partial \infty}\,(x_0) = \lim_{\|h\| \longrightarrow 0, h \in c, \delta_\infty(h) \neq 0} \frac{\liminf_n f(x_0 + r_n(h)) - f(x_0)}{\delta_\infty(h)}$$

where $r_n(h)$ is the sequence of c obtained by setting to 0 the n first terms of h.

[7] See appendix9

5 Necessary condition for the Pareto-optimality of a bequests plan

5.1 Introduction

In this section, we need the following assumptions on U:

- D_u is strictly included in R^2, closed and with a non-empty interior
- The interior of D is not empty
- U is of class C_1 on the interior of D_u , and continuous on D_u
- $D_1 U \succ 0$

Let $K \in \overset{\circ}{D}$, the interior of D. Let $i \geq 1$. If K is Pareto-optimal, it is solution of $P_i(K)$. We can write $P_i(K)$:

$$\max_{B \in D} U(b_{i-1}, b_i)$$
$$subject\ to \quad : \quad U_j(B) \geq U_j(K)\ \forall j \geq 1,\ j \neq i$$

where

$$U_j(B) = U(b_{j-1}, b_j)$$

For $B \in D$, define

$$G(B) = [U_j(B)]_{j \geq 1}$$

where $[U_j(B)]_{j \geq 1}$ is the sequence $(U_1(B), U_2(B), \ldots)$.

Define $H_i(B)$ as the sequence obtained by eliminating the ith term from the sequence $G(B) - G(K)$. So we have $H_i(B) = [h_{ij}(B)]_{j \geq 1}$ such as:

$$h_{ij}(B) = U_j(B) - U_j(K) \text{ if } 1 \leq j \prec i$$

and

$$h_{ij}(B) = U_{j+1}(B) - U_{j+1}(K) \text{ if } j \geq i$$

We can then write $P_i(K)$:

$$\max_{B \in D} U_i(B)$$
$$subject\ to \quad : \quad H_i(B) \geq 0$$

We prove[8], under the assumptions of paragraph5-1 on U, that G , U_i and H_i are Frechet-differentiable at a point K interior to D and such that $(k_0, k_1) \in \overset{\circ}{D}_u$. The respective Frechet-differentials are:

$$\delta G(K) \cdot \Delta K = [D_1 U(k_{j-1}, k_j) \Delta k_{j-1} + D_2 U(k_{j-1}, k_j) \Delta k_j]_{j \geq 1}$$
$$\delta U_i(B) = e_i \mid \delta G(B)$$
$$\delta H_i(B) = T_i(\delta G(B))$$

(See the definition of T_i in 11-2).

[8] See 11-2 for proof

5.2 Regularity of K for the inequality $H_i(B) \geq 0$

K is a regular point for the inequality $H_i(B) \geq 0$ if $H_i(K) \geq 0$ and if there is $x \in l_\infty$ such that $H_i(K) + \delta H_i(K) \cdot x \succ 0$[9].

The condition of regularity is required to apply the Kuhn-Tucker theorem[10].

First, observe that, if in a bequests plan there is i such that $D_2U(k_{i-1}, k_i) \succ 0$, it cannot be Pareto-optimal. Indeed, an increase of the bequest of g_i would increase the utility of g_i and also g_{i+1} since $D_1U(k_i, k_{i+1}) \succ 0$. Hence if K is Pareto-optimal, $K \in D_l = \{B \in D \diagup D_2U(b_{j-1}, b_j) \leq 0 \text{ for all } j \geq 1\}$.

Suppose then that $K \in D_l$. Moreover, suppose that $D_2U(k_{j-1}, k_j) \prec 0$ for all $j \geq 1$.

Let

$$L_- = \left\{ \begin{array}{l} B = (b_1, b_2, \cdots) \in D_l \diagup \exists \alpha \succ 0 \text{ and } n \geq 1 \\ \text{such that } \frac{-D_2U(b_{j-1},b_j)}{D_1U(b_{j-1},b_j)} \prec 1 - \alpha \text{ for } j \geq n \end{array} \right\}$$

and

$$L+ = \left\{ \begin{array}{l} B = (b_1, b_2, \cdots) \in D_l \diagup \exists \alpha \succ 0 \text{ and } n \geq 1 \\ \text{such that } \frac{-D_2U(b_{j-1},b_j)}{D_1U(b_{j-1},b_j)} \succ 1 + \alpha \text{ for } j \geq n \end{array} \right\}$$

Suppose that $K \in L_- \cup L_+$.

Under these assumptions on K, we show[11] that K is a regular point for the inequality $H_i(B) \geq 0$.

5.3 Application of the Kuhn-Tucker theorem

Under the assumptions of paragraph5-1 on U and the assumptions of paragraph5-2 on K , let $K \in \overset{\circ}{D}$ and such that $(k_0, k_1) \in \overset{\circ}{D}_u$.

As a consequence of the Kuhn-Tucker theorem, if K is a solution of $P_i(K)$ then there is $\lambda^* \in l_\infty^*$ such that, for all $\Delta K \in l_\infty$

$$\delta U_i(K) \cdot \Delta K + \langle \lambda^* \mid \delta H_i(K) \cdot \Delta K \rangle = 0 \tag{1}$$

with $\lambda^* \geq 0$.

Now apply lemma1 and write:

$$\lambda^* = \lambda + \beta$$

where $\lambda = (\lambda_1, \lambda_2, \cdots) \in l_1$ and $\beta \in c_0^\perp$. We see easily that if $\Delta K \in c_0$ then $\delta H_i(K) \cdot \Delta K \in c_0$ and $\delta U_i(K) \cdot \Delta K \in c_0$. So $\beta \mid \delta H_i(K) \cdot \Delta K = 0$. Replace λ^* by $\lambda + \beta$ in (1),we obtain

$$\delta U_i(K) \cdot \Delta K + \langle \lambda \mid \delta H_i(K) \cdot \Delta K \rangle = 0 \text{ for all } \Delta K \in c_0$$

[9] This condition means that $H_i(K) + \delta H_i(K) \cdot x$ is an interior point of $l_{\infty +}$ wich means that all components are strictly positive and that their limit inferior is also strictly positive

[10] See appendix4

[11] See 11-3 for proof

Henceforth, denote $u'_{hj} = D_1U(k_{j-1}, k_j)$ and $u'_{lj} = D_2U(k_{j-1}, k_j)$.

By development and identification, this comes to a system that gives

$$\begin{aligned}
\lambda_{i-p} &= (-1)^p \frac{u'_{hi} \cdots u'_{hi-p+1}}{u'_{li-1} \cdots u'_{li-p}} \text{ for } p \in [1, i-1] \\
\lambda_i &= -\frac{u'_{li}}{u'_{hi+1}} \\
\lambda_{i+p} &= (-1)^{p+1} \frac{u'_{li} \cdots u'_{li+p}}{u'_{hi+1} \cdots u'_{hi+p+1}} \text{ for } p \in [0, +\infty]
\end{aligned} \tag{2}$$

Since $\lambda \in l_1$, it verifies $\sum_{j=1}^{+\infty} |\lambda_j| \prec +\infty$. Replace λ_j by its value in (2), we obtain

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right| \prec +\infty \tag{3}$$

If K is a Pareto-optimal bequests plan, then the inequality (3) holds for all $i \geq 1$.

Now drop the assumption $D_2U(k_{j-1}, k_j) \prec 0$ for all $j \geq 1$, but keep $K \in D_l$.

Let $J = \{j / D_2U(k_{j-1}, k_j) = 0\}$. If J is up-bounded, let $q = \max J$.

If K is Pareto-optimal, the bequests plan extracted from K and beginning at the generation g_{q+1} : $(k_{q+1}, k_{q+2}, \cdots)$ is also necessarily Pareto-optimal when we take $(k_0, k_1, \cdots k_q)$ as fixed parameters. If K is in $\overset{\circ}{D} \cap (L_- \cup L_+)$, the extracted plan is also in $\overset{\circ}{D}_{k_q} \cap (L_- \cup L_+)$ and $(k_q, k_{q+1}) \in \overset{\circ}{D}_u$[12]. Since $\prod_{j=q+1}^{+\infty} D_2U(k_{j-1}, k_j) \neq 0$, it verifies necessarily the condition (3), but with g_{q+1} as first generation. This gives, for all $i \geq 1$:

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{lq+i+n}}{u'_{hq+i+1+n}} \right| \prec +\infty$$

Multiply the above inequality by

$$\prod_{n=0}^{q-1} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right|$$

we find again the inequality (3).

If J is not up-bounded, there is episodically a q such that $D_2U(k_{q-1}, k_q) = 0$. Consequently, in the sum $\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right|$ there is a finite number of nonzero terms. Thus, for all $i \geq 1$ the sum converges.

We can now state the following proposition:

[12] We denote D_{k_q} the set $\{B = (b_{q+1}, b_{q+2}, \cdots) \in l_{\infty+} \ / \ \forall i \geq 1 : (b_{q+i-1}, b_{q+i}) \in D_u\}$

Proposition2 *Let* $K \in \overset{\circ}{D} \cap (L_- \cup L_+)$ *and such that* $(k_0, k_1) \in \overset{\circ}{D}_u$*. Under the assumptions of paragraph5-1 on* U*, if* K *is a Pareto-optimal bequests plan then for all* $i \geq 1$ *we have:*

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right| \prec +\infty$$

6 Sufficiency

6.1 Introduction

Under the following assumptions, U is Frechet-differentiable on the border of D_u:[13]

- D_u is strictly included in R^2, closed, convex and with a non-empty interior
- U is of class C_1 on the interior of D_u , and continuous on D_u.
- D_1U and D_2U are extendable by continuity on $\widehat{D}_u = D_u - \overset{\circ}{D}_u$.

We add also to the previous assumptions:

- $D_1U \succ 0$
- The interior of D is not empty
- U concave

The assumption U *concave* implies that D is also closed and convex.

We then show, like in section5,[14] that G is Frechet-differentiable on the border of D.

As in section5, we conclude that U_i and H_i are Frechet-differentiable on the border of D too.

6.2 Stationarity of the Lagrangian

For $K \in D$ and $i \geq 1$ define the Lagrangian L_i from $D \times l_1$ to R such as $L_i(B, \mu) = U_i(B) + \langle \mu \mid H_i(B) \rangle$, and suppose, if $i \succ 1$, that $\prod_{j=1}^{i-1} u'_{lj} \neq 0$. The system (2) defines a sequence λ.

If

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right| \prec +\infty$$

then we can see that $\lambda \in l_1$. Like U_i and H_i , L_i is differentiable with respect to B on the interior and the border of D. Its differential, computed at $\mu = \lambda$ and $B = K$ is $\delta L_i(K, \lambda) = \delta U_i(K) + \langle \lambda \mid \delta H_i(K) \rangle$. Since $\lambda \in l_1$, for $\Delta K \in l_\infty$ we have

$$\begin{aligned} \delta L_i(K,\lambda) \cdot \Delta K &= \delta U_i(K) \mid \Delta K + \lambda \mid \langle \delta H_i(K) \cdot \Delta K \rangle \\ &= \delta U_i(K) \mid \Delta K + \lim_{n \longrightarrow +\infty} (\lambda_1, \lambda_2, \cdots, \lambda_n) \mid \langle \delta H_i(K) \cdot \Delta K \rangle \end{aligned}$$

[13] See appendix6 for the definition of Frechet-differentiability on the border and appendix7 for U Frechet-differentiability.

[14] See appendix8

Replace (λ_j) by their values in (2), we obtain

$$\delta U_i(K) \mid \Delta K + (\lambda_1, \lambda_2, \cdots, \lambda_n) \mid \langle \delta H_i(K) \cdot \Delta K \rangle = \lambda_n u'_{\ln+1} \Delta k_{n+1}$$

which tends to 0 when n tends to infinity.

Thus $\delta L_i(K, \lambda) = 0$. In other words, the Lagrangian L_i is stationary at (K, λ).

6.3 A sufficient condition to solve $P_i(K)$

Under the assumptions of paragraph6-1 on U, let $K \in D_l$ and i such that if $i \succ 1$ we have: $\prod_{j=1}^{i-1} u'_{lj} \neq 0$. We then deduce from the system (2) that $\lambda_j \geq 0$ for all $j \geq 1$. Since U is concave, we can easily see that U_i and H_i are concave too. Thus, L_i is concave with respect to B. As a result, for all $B \in D$ and $\alpha \in]0, 1[$ we have:

$$L_i((1-\alpha)K + \alpha B, \lambda) \geq (1-\alpha)L_i(K, \lambda) + \alpha L_i(B, \lambda)$$

then

$$\frac{L_i(K + \alpha(B - K), \lambda) - L_i(K, \lambda)}{\alpha} \geq L_i(B, \lambda) - L_i(K, \lambda)$$

since L_i is Frechet-differentiable, when α tends to 0 we obtain:

$$\delta L_i(K, \lambda) \mid (B - K) \geq L_i(B, \lambda) - L_i(K, \lambda) \tag{4}$$

We know that if

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right| \prec +\infty$$

we have $\delta L_i(K, \lambda) = 0$.

We then deduce from (4) that for all $B \in D$:

$$L_i(B, \lambda) - L_i(K, \lambda) \leq 0 \tag{5}$$

We now show that K solves $P_i(K)$. Suppose there is B in D such that $U_i(B) \succ U_i(K)$ and $H_i(B) \geq 0$. We have $H_i(K) = 0$ so $H_i(B) \geq H_i(K)$. Since $\lambda \geq 0$ we have $\lambda \mid H_i(B) \geq \lambda \mid H_i(K)$. Finally $U_i(B) + \lambda \mid H_i(B) \succ U_i(K) + \lambda \mid H_i(K)$. This contradicts (5). Hence, we can state:

Proposition3 *Under the assumptions of paragraph6-1 on U, let $K \in D_l$ and i such that if $i \succ 1$ we have: $\prod_{j=1}^{i-1} u'_{lj} \neq 0$.*

If

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right| \prec +\infty$$

then K is solution of $P_i(K)$.

If K is such that for all $i \geq 1$ we have $D_2U(k_{i-1}, k_i) \prec 0$, then we have:

$$\left(\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{l1+n}}{u'_{h2+n}} \right| \prec +\infty \right) \Longrightarrow \left(\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{li+n}}{u'_{hi+1+n}} \right| \prec +\infty \right) \text{ for all } i \geq 1$$

Now remember that if K is a solution of $P_i(K)$ for all $i \geq 1$, then K is a Pareto-optimal bequests plan. Thus, we can state:

Proposition4 *Under the assumptions of paragraph6-1 on U, let $K \in D$ such that for all $i \geq 1$ we have $D_2U(k_{i-1}, k_i) \prec 0$.*
If

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{l1+n}}{u'_{h2+n}} \right| \prec +\infty$$

then K is a Pareto-optimal bequests plan.

From proposition2 and proposition4, we deduce the following theorem:

Theorem1 *Under the assumptions of paragraph6-1 on U, let $K \in \overset{\circ}{D} \cap (L_- \cup L_+)$ such that $(k_0, k_1) \in \overset{\circ}{D}_u$ and such that for all $i \geq 1$ we have $D_2U(k_{i-1}, k_i) \prec 0$. Then K is a Pareto-optimal bequests plan if and only if*

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{l1+n}}{u'_{h2+n}} \right| \prec +\infty$$

7 Examples of Pareto optimal plans

7.1 A region of Pareto-optima

We adopt in all this section the same assumptions than in last section.

Define the open region (supposed not empty) l_- between the two lines : $L = \left\{(h,l) \in D_u \diagup \frac{-D_2U(h,l)}{D_1U(h,l)} = 1\right\}$ and $L_0 = \{(h,l) \in D_u \diagup D_2U(h,l) = 0\}$ as represented in the following figure.

If K is in

$$L_- = \left\{ \begin{array}{l} B = (b_1, b_2, \cdots) \in D_l \diagup \exists \alpha \succ 0 \text{ and } n \geq 1 \\ \text{such that } \frac{-D_2U(b_{j-1},b_j)}{D_1U(b_{j-1},b_j)} \prec 1-\alpha \text{ for } j \geq n \end{array} \right\}$$

then

$$\begin{aligned}
\sum_{p=0}^{+\infty}\prod_{j=0}^{p}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right| &= \sum_{p=0}^{n}\prod_{j=0}^{p}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right| + \sum_{p=n+1}^{+\infty}\prod_{j=0}^{p}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right| \\
&= \sum_{p=0}^{n}\prod_{j=0}^{p}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right| + \left[\prod_{j=0}^{n}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right|\right]\sum_{p=n+1}^{+\infty}\prod_{j=n+1}^{p}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right| \\
&\prec \sum_{p=0}^{n}\prod_{j=0}^{p}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right| + \left[\prod_{j=0}^{n}\left|\frac{u'_{l1+j}}{u'_{h2+j}}\right|\right]\sum_{p=n+1}^{+\infty}(1-\alpha)^{p-n} \prec +\infty
\end{aligned}$$

Moreover, if $u'_{lj} \prec 0$ for all $j \geq 1$, we can then apply theorem1 and conclude that K is Pareto-optimal.

$K \in L_-$ means graphically that from a given index n, K's components are under the line L without approaching it.

Thus :

Proposition5 *Under the assumptions of paragraph6-1 on U, all bequests plans which components are, from a given index in l_- without approaching the line L, are Pareto-optima.*

This condition implies that, for "most" generations we have:

$$-D_2U(k_{n-1}, k_n) \prec D_1U(k_n, k_{n+1})$$

which means that **if generation g_n decreases its bequest by one unit, it wins less than what is lost by generation g_{n+1}**. The condition

$$D_2U(k_{n-1}, k_n) \prec 0$$

can be taken as an **interiority** condition with respect to the set of relevant bequests plans.

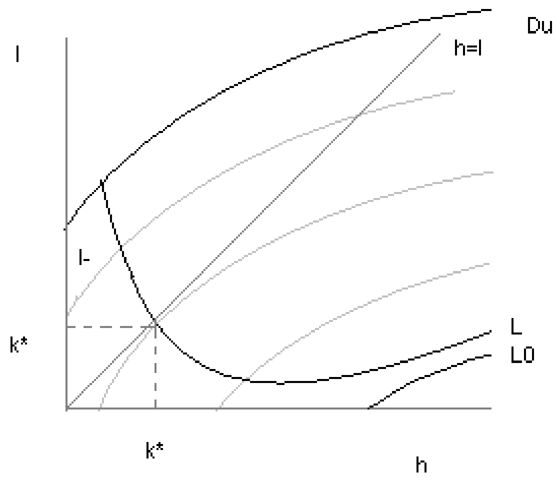

Figure 1:

7.2 Steady state Pareto-optima

At the point (k^*, k^*) , the line $h = l$ is tangent to a line $U(h, l) = U^*$, so we have

$$D_1U(k^*, k^*) = -D_2U(k^*, k^*)$$

Such a point does not necessarily exist. But if it does[15], a steady state bequests plan tending to k^* is an optimal steady state bequests plan with respect to the criterion

$$\max_K \lim U(k_{j-1}, k_j)$$

We will suppose henceforth that k^* exists and that $D_2U(k^*, k^*) \neq 0$.

None of the steady state bequests plans which limits are over k^* is Pareto-optimum. All steady state bequests plans under $k^*$are Pareto-optima.

Let B^* be a steady state bequests plan which components are equal to k^* from a given index. B^* is not a regular point of the inequality $H_i(B) \geq 0$. Indeed, from a given index, $H_i(B^*) + \delta H_i(B^*) \cdot x$ is proportional to $[(x_{j-1} - x_j)]_{j\geq1}$with a positive coefficient of proportionality. To obtain $H_i(B^*) + \delta H_i(B^*) \cdot x \succ 0$ we have to take x decreasing. Since x is in l_∞, the sequence converges and

$$x_{j-1} - x_j \longrightarrow 0 \text{ when } j \longrightarrow +\infty$$

Consequently, $H_i(B^*) + \delta H_i(B^*) \cdot x$ cannot be in the interior of $l_{\infty+}$.

[15] The question of existence of $k^*$is not a difficult problem. To lighten the text, it has been avoided. Nevertheless, it is addressed in a forthcoming work.

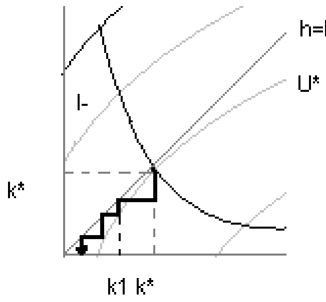

Figure 2:

For B^*, from a given index, we have $\left|\frac{u'_{l1+n}}{u'_{h2+n}}\right| = 1$. So

$$\sum_{p=0}^{+\infty}\prod_{n=0}^{p}\left|\frac{u'_{l1+n}}{u'_{h2+n}}\right| = +\infty$$

But it does not mean that B^* is not Pareto-optimal since B^* is not a regular point of $H_i(B) \geq 0$.

On the contrary, we can show that:

Proposition6 *B^* is Pareto-optimal.*

Proof: We first observe that, for $K = (k_1, k_{2,} \cdots) \in D_l$, if an extracted plan $(k_n, k_{n+1,} \cdots)$ is Pareto-optimal for generations $(g_n, g_{n+1,} \cdots)$, then K is Pareto-optimal[16]. It is therefore enough to show Pareto-optimality for the plan K^* which all components are equal to $k^*$and with $k_0 = k^*$. We show this graphically. Consider the first generation to try to enhance its utility level. It will bequeath a smaller capital than k^*, say k_1. All the following generations will decrease their bequest to keep at least a utility level of $U^* = U(k^*, k^*)$. We see in the following figure that there will be a generation that will fall short of the level U^* even if it bequeaths 0. Hence, it is not possible to enhance utility level of one generation without decreasing utility levels of some other generations. Thus K^* is Pareto-optimal, and so is B^*.

■

The criterion $max_K \lim U(k_{j-1}, k_j)$ gives best plans for remote generations. Thus B^* is the best steady state plan for remote generations.

If the economy begins with $k_0 \succ k^*$, generations have to bequeath less than they inherit until they reach k^*. If they do not, not only their

[16] We can show that by supposing that K is not Pareto-optimal. This will leads us to find a plan that dominates $(k_n, k_{n+1,} \cdots)$, which contradicts its Pareto-optimality.

bequests plan will not be good for remote generations but also it will not even be Pareto-optimal.

If the economy begins with $k_0 \leq k^*$, either it can tend to a steady state limit $k_\infty \prec k^*$with a Pareto-optimal bequests plan, or it can tend to the limit k^* which is better for remote generations but require from immediate generations to bequeath more. Consequently, **if $k_0 \prec k^*$, one cannot enhance utility level of remote generations without decreasing immediate generations ones.**

The condition $k_\infty \leq k^*$implies that

$$-D_2U(k_\infty, k_\infty) \leq D_1U(k_\infty, k_\infty)$$

which means that, asymptotically, one unit decrease of bequest adds less utility than what is lost by the simultaneous and equal decrease of heritage.

If we consider long run interest, each generation should increase bequest until its marginal utility for next generation comes to heritage marginal utility. Then we get to the value $k^*$where

$$-D_2U(k^*, k^*) = D_1U(k^*, k^*)$$

8 Consensual optimum

8.1 First order condition

We adopt in all this section the same assumptions than in last section.

As defined in section2, a consensual optimum is a bequests plan maximizing an inter-generations criterion $\Psi(G(B))$, where $\Psi(x)$ is a Frechet-differentiable function from l_∞ to R. Suppose that $G(\overset{\circ}{D}) \neq \varnothing$. Suppose also that Ψ is nondecreasing and concave[17].This means that an increase in the utility of generation g_i without change for other generations, increases the value of Ψ. For example, we can take a linear criterion $\Psi(G(B)) = y \mid G(B)$ where $y \in l^*_{\infty +}$.

Suppose that K is an interior solution of

$$\max_{B \in D} \Psi(G(B)) \tag{6}$$

The necessary first order condition is

$$\delta\left[\Psi G\right](K) \cdot \Delta B = 0 \text{ for all } \Delta B \in l_\infty$$

and this condition is sufficient even if K is not interior since ΨG is concave.

Using proposition1, we write:

$$\delta\Psi(x) = \delta\Psi_1(x) + \delta\Psi_2(x) \text{ where } \delta\Psi_1(x) \in l_1 \text{ and } \delta\Psi_2(x) \in c_0^\perp$$

For $x \in c$ we have[18]

$$\delta\Psi_2(G(K)) \cdot x = \left[\frac{\partial\Psi}{\partial\infty}(G(K))\right] \lim x_n$$

Denote the vector $\delta\Psi_1(G(K)) = (\frac{\partial\Psi}{\partial x_1}(G(K)), \frac{\partial\Psi}{\partial x_2}(G(K)), \cdots)$ by $(\Psi'_1, \Psi'_2, \cdots)$ and $\frac{\partial\Psi}{\partial\infty}(G(K))$ by Ψ'_∞.

We have[19]:

$$\Psi'_{i+1} = \Psi'_1 \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}}$$

for all $i \geq 1$.

and[20]:

Proposition7 *Under the assumptions of paragraph6-1, all interior maximizers of the criterion $\Psi(G(B))$ verifying $\frac{\partial\Psi}{\partial x_1}(G(K)) \neq 0$ and $D_2U(k_{i-1}, k_i) \prec 0$ for all $i \geq 1$, where Ψ is a Frechet-differentiable function on l_∞, are Pareto-optima.*

[17] For the same reason than the concavity of U

[18] See paragraph4-3

[19] See 11-4-1 for proof

[20] See 11-4-2 for proof

8.2 Egalitarianism

First, we define some interesting properties for Ψ.

Definition1 Ψ *is non-saturable at infinity if and only if* $\Psi'_\infty(G(B)) \succ 0$ *for all* $B \in D$.

This property means that the consensual criterion always increases strictly when utility of remote generations increases strictly. Thus this property warrants that the criterion is sensitive to long run interest.

Definition2 Ψ *is "locally egalitarian" at a point* G *if and only if* $\Psi'_i(G) = 0$ *for all* $i \geq 1$.

As seen in section2, one of the essential reasons to work in an OLG context, is the preoccupation about equality between generations. Even if equality is not a demanded condition, we need at least to compare it to the analyzed situation[21].

The following proposition[22] clarifies definition2:

Proposition8 *Let* s *be a one-to-one mapping on* N^* *and define* $\widehat{s}$ *the transformation on* l_∞ *such that, for* $G = (g_i) \in l_\infty$, $\widehat{s}(G) = (g_{s(i)})$. *Then,* Ψ *is locally egalitarian at a point* G *if and only if:*

$$\delta\Psi(G) \cdot \Delta G = \delta\Psi(G) \cdot \widehat{s}(\Delta G) \tag{7}$$

for all s *and for all* $\Delta G \in c$ *in a given neighborhood of* G.

The condition (7) means that if we change components order in ΔG, it does not change the consensual value. Thus, it expresses the idea of an equal importance of wealth increase for all generations in the eyes of what we called *social consensus* or *"out of society and time" planner* in section2.

[21] For much of the ideas developed here, I am indebted to M. Allais's "Economie et interet". I found also a more accessible exposé of some important issues in *Macroeconomics* of G.Mankiw. For example, about intergeneration equality we can read p116: "We then see that optimal capital accumulation is essentially function of the importance that we give to present and future generations. If we put them (generations) on the same level, (optimal path) will have to reach the golden rule's capital level." This is exactly what is shown in present and next section.

Notice that original G.Mankiw text is certainly somewhat different from what I quoted because I translated back to English the French translation available to me. Nevertheless, I hope that the meaning is preserved.

[22] See 11-5

8.3 Egalitarianism versus efficiency in steady states case

Suppose now that the solution K of (6) is in c (it is equivalent to say that K is a steady state plan) and take $\Delta B \in c$. Then $\delta G(K) \cdot \Delta B \in c$. Denote $u'_l = \lim D_2U(k_{n-1}, k_n)$ and $u'_h = \lim D_1U(k_{n-1}, k_n)$ and $\Delta b = \lim \Delta b_n$. We then have

$$\delta\Psi_2(G(K)) \cdot [(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i)_{i\geq 1}] = \Psi'_\infty(u'_l + u'_h)\Delta b$$

If Ψ is non-saturable at infinity, we have necessarily

$$u'_l + u'_h = 0 \tag{8}$$

Then K converges to k^*.

With the help of proposition7 we deduce:

Theorem2 *If the criterion defining a consensual optimum is non-saturable at infinity, all steady state bequests plans that are interior consensual optima converge necessarily to k^*. Let K be such an interior consensual optimum such that $D_2U(k_{i-1}, k_i) \prec 0$ for all $i \geq 1$. If Ψ is not locally egalitarian at $G(K)$, then K is Pareto-optimal.*

Hence, as long as non-saturability at infinity, which means sensitivity at infinity, is respected, changing the criterion Ψ can only change the speed of convergence to k^* (number of generations necessary to get close enough to k^*) but not the "destination" k^*.

The second assertion of theorem2 is somewhat amazing : **local egalitarianism does not warrant efficiency** (Pareto-optimality), **while its opposite** (favouritism) **does**.

To try to find more "optimistic" properties for egalitarianism, let's define global egalitarianism:

Definition3 Ψ *is "globally egalitarian" if and only if*

$$\Psi'_i(G) = 0$$

for all $i \geq 1$ and for all $G \in c$.

The following proposition[23] clarifies definition3:

Proposition9 *Let s be a one-to-one mapping on N^* and define $\widehat{s}$ as in proposition8. Then, if Ψ is globally egalitarian, we have:*

$$\Psi(\widehat{s}(G)) = \Psi(G)$$

for all s and for all $G \in c$[24].

[23] See proof in 11-6

[24] Although the reciprocal implication seems true, I have failed yet to prove it.

Remark *If we suppress the condition $G \in c$ in the definition of global egalitarianism and $\Delta G \in c$ in the definition of local egalitarianism, these concepts would be tighter but much more difficult to characterize. So, I kept $G \in c$.*

Let K be an interior consensual optimum for the non-saturable-at-infinity criterion Ψ such that Ψ is **not locally egalitarian** at $G(K)$ and $D_2U(k_{i-1}, k_i) \prec 0$ for all $i \geq 1$. We then have (see in the above paragraph) $\Psi'_{i+1} = \Psi'_1 \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}}$. Ψ non locally egalitarian implies that $\Psi'_1 \neq 0$. Since $\lim_n \Psi'_n = 0$, we then have $\lim_i \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}} = 0$. This implies that (k_{n-1}, k_n) never reaches endlessly (k^*, k^*) which means that there is, at least episodically, a **deviant generation** that gets away from k^*. However, K is Pareto-optimal.

Although, rigorously, condition (8) is only necessary to interior consensual optimality, I wrongly call a plan checking (8): a consensual optimum. If Ψ is **globally egalitarian**, all consensual optima are not necessarily Pareto-optima, but some of them are. For example, the plan where first generations bequeath their maximum bequest until they reach k^* and then the remaining generations stay at k^*, is a consensual optimum and a Pareto optimum. This plan is the **faster** way to reach k^*.

Observing that this plan is the **best state for long run interest**, we can assert:

Proposition10 *A globally egalitarian non-saturable-at-infinity consensual criterion enables the fastest consensus-optimal and Pareto-optimal attainment of the best state for long run interest k^*. Whereas with a non locally egalitarian, non-saturable-at-infinity consensual criterion, every consensual optimum is Pareto optimal but there is endlessly a deviant generation from k^*.*

8.4 Optimal growth path

As said in section2, an optimal growth path has to be Pareto-optimal and consensus-optimal. We limit henceforth our interest to steady state optimal growth paths.

Denote k_∞ the limit of a steady state bequests plan. $k_\infty = k^*_=$ means that from a given index n we have $k_n = k^*$, $k_\infty = k^*_-$ means that $k_n \longrightarrow k^*$ with $k_n \prec k^*$, $k_\infty = k^*_+$ means that $k_n \longrightarrow k^*$ with $k_n \succ k^*$, $k_\infty = k^*_{-+}$ means that $k_n \longrightarrow k^*$ with oscillations round k^*.

With a globally egalitarian criterion (non saturable at infinity), the following table characterizes steady state plans to show up which ones are optimal growth paths.

k_∞	$\prec k^*$	k^*_-	$k^*_=$	k^*_{-+}	k^*_+	$\succ k^*$
Pareto optimality	yes	?	yes	?	?	no
consensual optimality	no	yes	yes	yes	yes	no

For the cases k^*_+ , k^*_- and k^*_{-+}, $K \notin \overset{\circ}{D} \cap (L_- \cup L_+)$. We cannot apply theorem1.

However, for the case k^*_- , we can apply proposition4 if $\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{l1+n}}{u'_{h2+n}} \right|$converges. Since we cannot clearly see that, we may use Rabee-Duhamel criterion which says that if we can write $\left| \frac{u'_{l1+n}}{u'_{h2+n}} \right| = 1 - \frac{a}{n+1} + r_{n+1}$, with $a \succ 1$ and $\sum r_n \prec +\infty$, then $\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{u'_{l1+n}}{u'_{h2+n}} \right| \prec +\infty$. The interpretation is that the bequests plan must not tend "too rapidly" to k^*.

Consequently, only $k^*_=$ is doubtlessly an optimal growth path. On top of that, the plan $k^*_=$ is the fastest to reach k^*.

9 Golden rule

The golden rule : *capital marginal productivity = rate of capital depreciation,* characterizes the best steady state in an economy governed by a Solow model, which is one of the first, the simplest, but more insightful models in growth theory. In this model, the sharing between consumption and saving is not the result of an optimization decision, but a fixed parameter as supposed in keynesian theory. Indeed, Solow's goal was to criticize Harrod-Domar model by accepting all its keynesian assumptions except that of fixed proportions production . Solow's economy converges to a steady state depending on the savings rate. The golden rule steady state is obtained by imposing the golden savings rate equal to the *quotient of marginal productivity on average productivity.*

Here, under intra-life time neutrality assumption, we find **similar results**, but with a dynamic of **savings behaviors** governed by the needs of **consensual optimality** and **Pareto optimality**.

Intra-life time neutrality means that a generation does not care when it consumes, as long as it is during its lifetime. Thus, the discount rate ρ in the expression of life utility $\int_0^T u(c(t))e^{-\rho t}dt$ is taken 0[25].

Proposition11 *Suppose that $U(h,l)$ comes from the resolution of the following program:*

$$\begin{aligned} U(h,l) &= \max_c \int_0^T u(c(t))dt \\ \text{subject to} &: \dot{k} + c(t) = f(k(t)) - a \cdot k(t) \ ; \\ k(0) &= h \ ; k(T) = l \text{ and } c \in [0,\ f(k)] \end{aligned}$$

where u is a concave function, C_1 on $]0,+\infty[$, such that all section6 conditions on U are verified. Then any optimal growth path verifies the golden rule, and the asymptotic savings rate is the golden savings rate. Moreover, this is true for any bequests plan tending to k^*.

Proof: Write the Hamiltonian:

$$H = u(c) + \lambda(f(k) - a.k - c)$$

We have $c^* = \arg\max H$. If $\lambda \in [u'f(k), u'(0)]$ then $u'(c) = \lambda$ and $\frac{d\lambda}{dt} = -\lambda(f'(k) - a)$. Denote k_g the capital verifying the golden rule $f'(k) = a$. In the phase plan (k, λ), the line $\frac{d\lambda}{dt} = 0$ is then the straight line$\{k = k_g\}$. The line $\frac{dk}{dt} = 0$ is $\{\lambda = u'(f(k) - a.k)\}$. k_g is a minimizer of this line. Below the line $\{\lambda = u'f(k)\}$, the condition $c \leq f(k)$ becomes active and the equation $\frac{d\lambda}{dt} = -\lambda(f'(k) - a)$ is no longer true. We can now draw the phase plane of this program.

[25] This assumption may look as naive but $\rho \neq 0$ would mean that consumption in the beginning of life weighs more than in other periods of life, which would not reflect life utility, but utility at a given time.

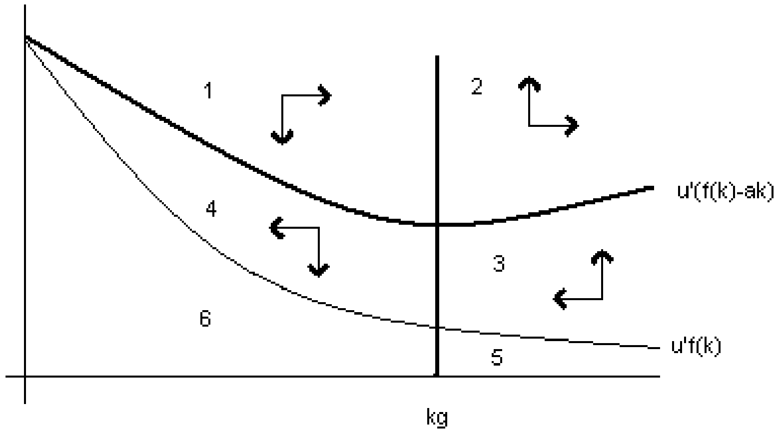

Figure 3:

We see in the above figure that if one begins with a capital $h = k_g$ and then changes his capital stock by his consumption choice, he cannot return on the line $\{k = k_g\}$. We deduce that if $h = k_g$ and $l = k_g$, the optimal $c(t)$ must hold capital stock exactly at the level k_g. We then have $\lambda_g = u'(f(k_g) - a.k_g)$ and $c(t) = f(k_g) - a.k_g$ for all $t \in [0, T]$.

Define $V(h, l, r, s)$ as follows:

$$\begin{aligned} V(h,l,r,s) &= \max_c \int_r^s u(c(t))dt \\ \text{subject to} &: \dot{k} + c(t) = f(k(t)) - a \cdot k(t) \; ; \\ k(0) &= h \; ; \; k(T) = l \text{ and } c \in [0, \; f(k)] \end{aligned}$$

We have $V(h, l, 0, T) = U(h, l)$, $V_h'(h, l, 0, T) = U_h'(h, l)$ and $V_l'(h, l, 0, T) = U_l'(h, l)$. We know that if there is a point k^* such that $U_h(k^*, k^*) + U_l(k^*, k^*) = 0$, k^* is unique (see previous section). As a result of the equations of Hamilton-Jacobi-Bellman[26] applied on V at the point $(k_g, k_g, 0, T)$, we have:

$$\begin{aligned} c(0) &= \arg\max \{u(c) + V_h'(f(k_g) - ak_g - c)\} \\ c(T) &= \arg\max \{u(c) - V_l'(f(k_g) - ak_g - c)\} \end{aligned}$$

[26] Applied to the present case, these equations are:

$$\begin{aligned} V_r' + Max_c \{u(c) + V_h'(f(k_g) - ak_g - c)\} &= 0 \\ -V_s' + Max_c \{u(c) - V_l'(f(k_g) - ak_g - c)\} &= 0 \end{aligned}$$

See Marti, quoted reference.

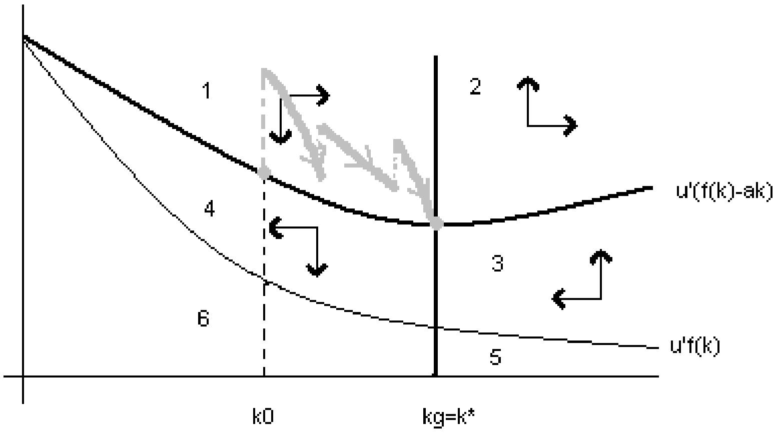

Figure 4:

since $u'(f(k_g)) \leq \lambda_g \leq u'(0)$ (see phase plane) and $\lambda(r) = V_h'$ and $\lambda(s) = -V_l'$[27], this gives:

$$\begin{aligned} u'(c(0)) - V_h' &= 0 \\ u'(c(T)) + V_l' &= 0 \end{aligned}$$

We have seen above that c is constant if $h = l = k_g$. Then $u'(c(0)) = u'(c(T))$ and $V_h' + V_l' = 0$.

k_g is then the unique solution k^* of $U_h'(k,k) + U_l'(k,k) = 0$, which implies that:

$$k_g = k^*$$

■

We verify easily that the savings rate tends to

$$f'(k^*)\frac{k^*}{f(k^*)}$$

We look now to the savings behaviors necessary to reach k^*.

If $k_0 \prec k^*$: Suppose that before generation1, economy was stationary at $k = k_0$. A generation who will enhance bequest will have to enhance its savings rate above $\frac{ak}{f(k)}$. So it will necessarily sacrifice utility as we have seen in section7. In the following figure, dotted lines in region1 represent jumps from a generation to the next one. Continuous lines represent trajectories of “sacrificed ”generations.

[27] See Marti, quoted reference.

As we get closer to k^*, the savings rate will decrease back to $\frac{ak}{f(k)}$ and the life utility $U(h, l)$ will increase. Hence, **the efforts accepted by immediate generations will benefit to the infinity of remote generations and there is not another way to reach** k^*.

If $k_0 \succ k^*$: As said in section7, staying in this situation is not Pareto-optimal. **Decreasing savings rate is good for immediate generations as for remote ones**. Economy will depart from region3 and end to $k^*$and savings rate will decrease to its golden rule level.

If $k_0 = k^*$: **Generations will just have to keep a constant consumption** $c^* = f(k^*) - a.k^*$which warrants the golden savings rate.

10 Welfare analysis

10.1 Selfish and altruistic utility

If everyone behaves only according to what his pure selfishness dictates, the optimal growth path would not be stable. The first generation that is free to deviate would do it and bequeath the minimum capital. This capital $l_{\min}$ is determined by

$$U_l'(k^*, l_{\min}) = 0$$

If the generation does not bequeath $l_{\min}$, it must be because of one of the three following reasons:

First, it can be compelled by the planner. But it is hard to imagine that. Planner can fix prices or production, but not bequests.

Second, it can be so altruistic that it maximizes the consensual criterion Ψ instead of its utility. This supposition is not completely utopian because some people determine their behavior according to collective interests. For example, some Europeans don't buy Japanese cars although they may be competitive, because it would lead to European GDP decrease. We can also combine individual utility U an consensual criterion Ψ and use something like $\lambda U + (1 - \lambda)\Psi$.

Thirdly, we can suppose that individual utility depends on heirs' utility. This assumption appears as the more appropriate to our problem. We can name it : familial altruism or intergenerational altruism. This concept have been used by Brenheim-Ray(1987) or Lakshmi(2002). The point is then to see if this limited altruism could lead to the optimal growth path. In other words, **we try to find out what familial altruism intensity can lead to the optimal growth path**.

We suppose that we can decompose the individual utility V as follows:

$$V = U + A$$

where U is the classical selfish utility depending on personal consumption and A is the altruistic utility depending on the capital bequeathed to next generation. Thus

$$V(h, l) = U(h, l) + A(l)$$

where U is the utility function used in previous sections (with assumptions of section6) and A is the altruistic utility.

A has to be nondecreasing with l since altruistic feelings are satisfied when bequest increases. We also suppose that A is C_1 and strictly concave[28] on R.

[28] Concavity requirement on A means decreasing marginal altruistic utility.

10.2 Honour your heir, but not more than yourself!

Name **spontaneous equilibrium** a bequests plan where each generation g_n chooses k_n solution to:

$$\begin{aligned} & \max_l U(h,l) + A(l) \\ \text{where } h &= k_{n-1} \end{aligned} \tag{9}$$

Denote $l_{\min}(h)$ and $l_{\max}(h)$ respectively the lower bound and the upper bound of $\{l / (h,l) \in D_u\}$. If[29]

$$U_l'(h, l_{\max}(h)) + A'(l_{\max}(h)) \leq 0 \text{ and } U_l'(h, l_{\min}(h)) + A'(l_{\min}(h)) \geq 0$$

then the solution $\varphi(h)$ of (9) verifies

$$U_l'(h, \varphi(h)) + A'(\varphi(h)) = 0.$$

We will suppose that.

In a spontaneous equilibrium we have $k_n = \varphi(k_{n-1})$ for $n \geq 1$. If a spontaneous equilibrium is an optimal growth path, then $k^* = \varphi(k^*)$. But $U_h'(k^*, k^*) = -U_l'(k^*, k^*)$ then, we can assert

Proposition12 *If a spontaneous equilibrium is an optimal growth path, then*

$$U_h'(k^*, k^*) = A'(k^*) \tag{10}$$

The left hand side of equation (10) is the increase of selfish utility resulting from an increase of heritage by one unit. The right hand side is the increase of altruistic utility resulting from an increase of bequest by one unit. The interpretation is that **spontaneous equilibrium cannot be optimal growth path unless generations feel** (asymptotically) **about their heirs as they feel about themselves**.

If $A'(k_\infty) \prec U_h'(k_\infty, k_\infty)$, then $U_h'(k_\infty, k_\infty) + U_l'(k_\infty, k_\infty) \succ 0$ which implies $k_\infty \prec k^*$. So **if feelings toward heirs are deficient, economy will stay in underaccumulation**.

Similarly, if $A'(k_\infty) \succ U_h'(k_\infty, k_\infty)$, then $k_\infty \succ k^*$. **If feelings toward heirs are excessive, economy will go in overaccumulation**.

10.3 Transitory state

The condition (10) is necessary for a spontaneous equilibrium to be an optimal growth path, but it is not sufficient. We have to make sure that the sequence $k_n = \varphi(k_{n-1})$ converges.

[29] The condition $U_l'(h, l_{\min}(h)) + A'(l_{\min}(h)) \geq 0$ means that it is always better to bequest more than $l_{\min}(h)$. Without this condition, adding A to U would not change anything to spontaneous equilibrium and economy would go in underaccumulation. The condition $U_l'(h, l_{\max}(h)) + A'(l_{\max}(h)) \leq 0$ means that it is always better to bequest less than $l_{\max}(h)$. Without this condition, U would be useless and economy would go in overaccumulation.

We work out this under assumptions that U and A are C_2 on their definition sets.

We then have

$$\varphi'(h) = -\frac{U''_{lh}(h,\varphi(h))}{U''_{ll}(h,\varphi(h)) + A''(\varphi(h))}$$

We can impose a minimum concavity condition on A to make sure the spontaneous equilibrium converges.

Indeed, if there is $\alpha \in]0,1[$ such that

$$A''(\varphi(h)) \prec -U''_{ll}(h,\varphi(h)) - \frac{\left|U''_{lh}(h,\varphi(h))\right|}{\alpha} \tag{11}$$

for all h for which the last expression is defined, then $|\varphi'(h)| \prec \alpha$. This implies $|k_{n+1} - k_n| \preceq \alpha\,|k_n - k_{n-1}|$ and sequence (k_n) converges to the unique solution of $k = \varphi(k)$, k^* .

If we want only to have convergence for k_0 close enough to k^*, it suffices to have

$$A''(k^*) \prec -U''_{ll}(k^*,k^*) - \left|U''_{lh}(k^*,k^*)\right| \tag{12}$$

Under conditions (10) and (11) (or (12) if k_0 is close enough to k^*), consensual optimality is warranted but still not Pareto optimality.

Indeed, if $\varphi'(k^*) \neq 0$, we will be in the cases k^*_- , k^*_+ or k^*_{-+} which are not necessarily Pareto-optima according to paragraph8-4. But if $\varphi'(k) = 0$ in a neighborhood of k^*, which is equivalent to

$$U''_{lh}(k,k) = 0$$

and if k_0 is in this neighborhood, we will be in the case $k^*_=$ which is an optimal growth path.

11 Proofs

11.1 Proof of lemma1

11.1.1 Projection from l_∞^* on l_1

For $i \geq 1$, let e_i be the element of l_∞ such that all its components are zero except the i^{th} which is 1.

Let $y \in l_\infty^*$. Consider the sequence: $(y \mid e_i)_{i \geq 1}$. This sequence is in l_1.[30] Denote Φ the mapping from l_∞^* to l_1 which associates to y the sequence $(y \mid e_i)_{i \geq 1}$. Φ is a projection from l_∞^* to l_1. Indeed, it is a linear transformation and, considering $l_1$as a subset of l_∞^*, if $y \in l_1$then $\Phi(y) = y$.

11.1.2 Decomposition of an element $y \in l_\infty^*$ by Φ

Consider the mapping Identity I from c_0 to l_∞:

$$I : c_0 \underset{x \longrightarrow x}{\longrightarrow} l_\infty$$

We can verify easily that Φ is the adjoint operator of I, what we write:

$$\Phi = I^*$$

I being linear and continuous, we deduce that Φ is a continuous[31] linear operator.

Furthermore, we have:[32]

$$R(I)^\perp = N(I^*)$$

where

$$R(I) = \{y \in l_\infty / \exists x \in c_0 : I(x) = y\} = c_0$$

and

$$N(I^*) = \{x \in c_0 / I^*(x) = 0\}$$

which means:

$$N(\Phi) = c_0^\perp$$

For $y \in l_\infty^*$, define $k = \Phi(y) - y$. We can write:

$$y = \Phi(y) + k.$$

with $\Phi(y) \in l_1$ and $k \in c_0^\perp$.

We have decomposed an element y of l_∞^* as a sum of an element of l_1 and an element of $c_0^\perp$. We easily show that this decomposition is unique.

[30] We show this like we have shown that $(f(e_n))_{n \geq 1}$is in l_1. See appendix2.

[31] The adjoint of a continuous linear operator is continuous too.

[32] See Luenberger p155 quoted reference.

11.1.3 Study of $c_0^\perp$

We have:

$$\|\delta_\infty\| = \sup_{x\in c}\frac{|\lim x_n|}{\|x\|} = \sup_{x\in c}\frac{|\lim x_n|}{\sup|x_n|} = 1$$

and

$$\forall\alpha \in R : \|\alpha\delta_\infty\| = |\alpha|\,\|\delta_\infty\| = |\alpha|$$

so we can apply Hahn-Banach theorem[33], and extend $\alpha\delta_\infty$ with an element of l_∞^*, say β.

Denote B the set of such linear functionals. We now show that $c_0^\perp = B$. We see easily that B is a vector subspace of l_∞^* included in $c_0^\perp$. Reciprocally, let $\beta \in c_0^\perp$ and $x \in c$. Denote $e = (1, 1\cdots)$. We have $x - (\delta_\infty \mid x)e \in c_0$, so $\langle\beta \mid (x - (\delta_\infty \mid x)e)\rangle = 0$. Thus $\beta \mid x = (\beta \mid e)(\delta_\infty \mid x)$. This proves that the restriction of β to c is proportional to δ_∞. Then $\beta \in B$ and $c_0^\perp \subset B$.■

11.2 Differentiability of G, U_i and H_i

11.2.1 Differentiability of U_i and H_i

Let T_i be the transformation which suppresses the ith component of an element of l_∞, replaces it by the next one, and shifts all the following components backward. We have

$$H_i(B) = T_i(G(B) - G(K)) \tag{13}$$

We have also

$$U_i(B) = e_i \mid G(B) \tag{14}$$

Suppose that G is Frechet-differentiable at B. The symbol δ preceding a transformation means its Frechet-differential. From equations (13) and (14) we can easily show that

$$\delta H_i(B) = T_i(\delta G(B))$$

and

$$\delta U_i(B) = e_i \mid \delta G(B)$$

It remains to prove Frechet-differentiability of G.

11.2.2 Differentiability of G

First, we build a compact subset $\overline{E(K,r)}$ included in $\overset{\circ}{D}_u$ such that the open sphere $S(K,r)$ of center K and radius r is included in $\overset{\circ}{D}$ and such that all its points have their successive pairs of components in $\overline{E(K,r)}$[34]. r is a strictly positive real. Then we define the function o :

$$o(u, v, x, y) = \frac{U(x + u, y + v) - U(x, y) - D_1U(x, y)u - D_2U(x, y)v}{\sqrt{u^2 + v^2}}$$

[33] See appendix3

[34] See appendice5

if $\sqrt{u^2+v^2} \neq 0$ and

$$o(u,v,x,y) = 0$$

if $\sqrt{u^2+v^2} = 0$.

o is continuous on $\left[0, \frac{r}{2}\right]^2 \times \overline{E(K,r)}$. Since this set is compact, o is uniformly continuous on it. Hence, for $\varepsilon \succ 0$ there is $\alpha \succ 0$ such that

$$\sqrt{u^2+v^2} \prec \alpha \Longrightarrow |o(u,v,x,y)| \prec \frac{\varepsilon}{\sqrt{2}} \text{ for all } (x,y) \text{ in } \overline{E(K,r)}. \tag{15}$$

For $j=1$, take $\Delta k_{j-1} = \Delta k_0 = 0$.

If $K + \Delta K \in S(K,r)$, then $(k_{j-1} + \Delta k_{j-1}, k_j + \Delta k_j) \in \overline{E(K,r)}$. Moreover, if $\|\Delta K\| \prec \frac{\alpha}{\sqrt{2}}$, then for all $j \geq 1$ $\sqrt{\Delta k_{j-1}^2 + \Delta k_j^2} \prec \alpha$. We can then apply (15) an write $|o(\Delta k_{j-1}, \Delta k_j, k_{j-1}, k_j)| \prec \frac{\varepsilon}{\sqrt{2}}$.

If we define $\delta G(K)$ by

$$\delta G(K) \cdot \Delta K = \left[D_1 U(k_{j-1}, k_j)\Delta k_{j-1} + D_2 U(k_{j-1}, k_j)\Delta k_j\right]_{j \geq 1}$$

we have

$$\begin{aligned}
& \frac{\|G(K+\Delta K) - G(K) - \delta G(K)\cdot \Delta K\|}{\|\Delta K\|} \\
= & \frac{1}{\sup_{j\geq 1}|\Delta k_j|} \sup_{j\geq 1} \left| \begin{array}{c} U(k_{j-1}+\Delta k_{j-1}, k_j+\Delta k_j) - U(k_{j-1},k_j) \\ -D_1U(k_{j-1},k_j)\Delta k_{j-1} + D_2U(k_{j-1},k_j)\Delta k_j \end{array} \right| \\
= & \frac{\sup_{j\geq 1}\left| o(\Delta k_{j-1}, \Delta k_j, k_{j-1}, k_j)\sqrt{\Delta k_{j-1}^2 + \Delta k_j^2}\right|}{\sup_{j\geq 1}|\Delta k_j|} \\
\prec & \frac{\varepsilon}{\sqrt{2}} \frac{\sup_{j\geq 1}\left|\sqrt{\Delta k_{j-1}^2 + \Delta k_j^2}\right|}{\sup_{j\geq 1}|\Delta k_j|} \leq \varepsilon
\end{aligned}$$

This shows that

$$\frac{\|G(K+\Delta K) - G(K) - \delta G(K)\cdot \Delta K\|}{\|\Delta K\|} \longrightarrow 0 \text{ when } \|\Delta K\| \longrightarrow 0$$

This proves that $\delta G(K)$ is the Frechet-differential of G at K.■

11.3 Regularity of K

Denote $u'_{hj} = D_1U(k_{j-1}, k_j)$ and $u'_{lj} = D_2U(k_{j-1}, k_j)$.

We have $H_i(K) = 0$. Denote $\theta_j = u'_{hj}x_{j-1} + u'_{lj}x_j$, where $x_0 = 0$ and $j \geq 1$. Then $H_i(K) + \delta H_i(K) \cdot x = (\theta_j)_{j\geq 1, j\neq i}$.

For $j \geq i+1$, if $K \in L_-$, take x_i a strictly positive real. Take x_j such that $0 \prec x_j \prec \inf(\frac{u'_{hj}}{-u'_{lj}}x_{j-1}, x_{j-1})$. Then $\theta_j = u'_{hj}x_{j-1} + u'_{lj}x_j \succ 0$. For $m \geq n$ we have $\frac{u'_{hm}}{-u'_{lm}} \succ \frac{1}{1-\alpha} \succ 1$. Then $\inf(\frac{u'_{hm}}{-u'_{lm}}x_{m-1}, x_{m-1}) = x_{m-1}$. When $j \longrightarrow +\infty$

, we have $\theta_j = (u'_{hj} + u'_{lj})x_{n-1} \geq u'_{hj}x_{n-1}\alpha$. We know that k_j is in $[0, \|K\|]$ so $\liminf u'_{hj} \succ 0$ and then $\liminf\ \theta_j \succ 0$. If $K \in L_+$ take x_i a strictly negative real. Take x_j such that $x_j \prec \inf(\frac{u'_{hj}}{-u'_{lj}}x_{j-1}, x_{j-1})$. Then $\theta_j = u'_{hj}x_{j-1} + u'_{lj}x_j \succ 0$. For $m \geq n$ we have $\frac{u'_{hm}}{-u'_{lm}} \prec \frac{1}{1+\alpha} \prec 1$. Then $\inf(\frac{u'_{hm}}{-u'_{lm}}x_{m-1}, x_{m-1}) = x_{m-1}$. When $j \longrightarrow +\infty$, we have $\theta_j = (u'_{hj} + u'_{lj})x_{n-1} \geq u'_{hj}(-x_{n-1})\alpha$. Similarly, I deduce that $\liminf\ \theta_j \succ 0$.

For j such that $1 \prec j \leq i-1$ take $x_1 \prec 0$ and x_j such that $x_j \prec \frac{u'_{hj}}{-u'_{lj}}x_{j-1}$. Then $\theta_j = u'_{hj}x_{j-1} + u'_{lj}x_j \succ 0$ and $\theta_1 = u'_{l1}x_1 \succ 0$.

We have then found $x = (x_j)_{j\geq 1} \in l_\infty$such that $(\theta_j)_{j\geq 1, j\neq i} = H_i(K) + \delta H_i(K) \cdot x \succ 0$. Then K is a regular point for the inequality $H_i(B) \geq 0$. For $i = 1$, we can show similarly that if $K \in L_- \cup L_+$, K is a regular point for the inequality $H_1(B) \geq 0$.■

11.4 First order condition

11.4.1 Proof of : $\Psi'_{i+1} = \Psi'_1 \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}}$

We have:

$$\delta\left[\Psi G\right](K) = \delta\Psi(G(K)) \cdot \delta G(K)$$

and

$$\delta G(K) \cdot \Delta B = (u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i)_{i\geq 1}$$

then

$$\begin{aligned} & \delta\left[\Psi G\right](K) \cdot \Delta B \\ = \; & \delta\Psi_1(G(K)) \cdot [(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i)_{i\geq 1}] \\ & + \delta\Psi_2(G(K)) \cdot [(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i)_{i\geq 1}] \end{aligned}$$

Replace $\delta\Psi_1(G(K))$ with $(\Psi'_1, \Psi'_2, \cdots)$, then

$$\begin{aligned} \delta\left[\Psi G\right](K) \cdot \Delta B &= \sum_{i=1}^{+\infty} \Psi'_i(u'_{hi}\Delta b_{i-1} + u'_{li}\Delta b_i) + \Psi'_\infty(u'_l + u'_h)\Delta b \\ &= \sum_{i=1}^{+\infty} (\Psi'_i u'_{li} + \Psi'_{i+1}u'_{hi+1})\Delta b_i + \Psi'_\infty(u'_l + u'_h)\Delta b \end{aligned}$$

This has to be identically zero for all ΔB in c. It implies that $\Psi'_i u'_{li} + \Psi'_{i+1}u'_{hi+1} = 0$ for all $i \geq 1$, which gives $\Psi'_{i+1} = \Psi'_i \frac{-u'_{li}}{u'_{hi+1}}$. Thus

$$\Psi'_{i+1} = \Psi'_1 \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}}$$

■

11.4.2 Proposition7

Since $(\Psi'_1, \Psi'_2, \cdots) \in l_1$, we have:

$$|\Psi'_1| \sum_{i=2}^{+\infty} \prod_{j=1}^{i} \left| \frac{-u'_{lj}}{u'_{hj+1}} \right| \prec +\infty$$

If $\Psi'_1 \neq 0$ we deduce that

$$\sum_{i=2}^{+\infty} \prod_{j=1}^{i} \left| \frac{-u'_{lj}}{u'_{hj+1}} \right| \prec +\infty$$

If K is such that $D_2U(k_{i-1}, k_i) \prec 0$ for all $i \geq 1$, according to proposition4, K is Pareto-optimal.■

11.5 Proposition8

Take all components of ΔG zero except i^{th} equal to Δg, and s the one-to-one mapping on N^* inverting i with n without changing the other indexes. (7) implies that $\Psi'_i(G)\Delta g = \Psi'_n(G)\Delta g$. But $\lim_n \Psi'_n(G) = 0$ so $\Psi'_i(G) = 0$ for all $i \geq 1$.Reciprocally, if $\Psi'_i(G) = 0$ for all $i \geq 1$, $\delta\Psi(G) \cdot \Delta G = \Psi'_\infty(G)\delta_\infty(\Delta G)$ where $\Delta G \in c$. Let s be a one-to-one mapping on N^*. Suppose $\liminf s^{-1}(n) \in N$ then there would be episodically p such that $s^{-1}(p) = \liminf s^{-1}(n)$ which is impossible since s^{-1} is a one-to-one mapping. Then $\liminf s^{-1}(n) = +\infty$ which implies $\lim s^{-1}(n) = +\infty$. Denote the n^{th} term of $\widehat{s}(\Delta G)$ by $s\Delta g_n$. We then have $\delta_\infty(\widehat{s}(\Delta G)) = \lim s\Delta g_n = \lim \Delta g_{s^{-1}(n)} = \lim \Delta g_n = \delta_\infty(\Delta G)$. Then $\delta\Psi(G) \cdot \Delta G = \Psi'_\infty(G)\delta_\infty(\Delta G) = \Psi'_\infty(G)\delta_\infty(\widehat{s}(\Delta G)) = \delta\Psi(G) \cdot \widehat{s}(\Delta G)$■.

11.6 Proposition9

First, for an integer $n \geq 1$, define s_n such that if i or $s(i) \in \{1, \cdots, n\}$ then $s_n(i) = i$, else : $s_n(i) = s(i)$. It is easy to verify that s_n is a one-to-one mapping on N^*. For $x \in R^n$ and $G \in c$ denote $x\widehat{s}G$ the sequence obtained by replacing the n first terms of $\widehat{s}(G)$ by x. Notice that, as shown in the proof of proposition8, if $G \in c$ then $\widehat{s}(G) \in c$. $\Psi'_i(G) = 0$ for all $i \geq 1$ and $G \in c$ implies that the function $f(x) = \Psi(x\widehat{s}G)$ is constant on R^n. The 2 sequences $\widehat{s}(G)$ and $\widehat{s}_n(G)$ are equal from the $n+1^{th}$ index, we can then write $\widehat{s}_n(G) = x\widehat{s}G$, which implies $\Psi(\widehat{s}(G)) = \Psi(\widehat{s}_n(G))$. Now denote h_p the p^{th} term of $\widehat{s}_n(G)$. If $p \succ n$, there is $m \succ n$ such that $h_p = g_m$. Then $\|\widehat{s}_n(G) - G\| = \sup_{p \succ n} |h_p - g_p| \leq \sup_{m,p \succ n} |g_m - g_p|$ which tends to 0 when n tends to infinity since $G \in c$. Then $\Psi(\widehat{s}_n(G)) \longrightarrow \Psi(G)$ which implies $\Psi(\widehat{s}(G)) = \Psi(G)$■

12 Conclusion

The need to specify the concept of *optimal growth path* without infinite horizon optimization, has lead to try to mathematically characterize optimality between generations in an OLG economy.

For Pareto-optimality, we establish, under some conditions, that for "most" generations we should have:

$$-D_2U(k_{n-1}, k_n) \leq D_1U(k_n, k_{n+1})$$

which means that if generation g_n decreases its bequest by one unit, it wins less than what is lost by generation g_{n+1}.

To select from Pareto-optima, this paper introduces consensual optimality with egalitarian and non-saturable-at-infinity criterion. These concepts are used for the definition of optimal growth paths, which are shown to converge necessarily to the capital k^* defined by

$$-D_2U(k^*, k^*) = D_1U(k^*, k^*)$$

Moreover, with intra-life time neutrality, k^*observes the golden rule.

Then, with the use of familial altruistic utility, we have shown that if marginal altruistic utility of bequest is equal to marginal selfish utility of heritage, spontaneous equilibrium is consistent with optimal growth path.

However, bequests plans which are not in $L_- \cup L_+$ have not been examined here, particularly Pareto optima that cross cyclically the line L. They don't meet regularity requirement, but their study should be interesting. For example, it could help know to what extent, changes in bequeathing behavior affects long period economic cycles.

It should also be interesting to drop intra-life time neutrality assumption and see consequences on golden rule observance.

Appendices

1 Norm and continuity of a linear functional

- By definition, a linear functional on a normed space X is bounded if and only if here is M such as for all $x \in X : |f(x)| \leq M \|x\|$.

$\|f\|$ is the smallest of such M.

- A linear functional is continuous if and only if it is bounded[35].

2 Proof of $c_0^* = l_1$

For $i \geq 1$, let e_i be the element of l_∞ such that all its components are zero except the i^{th} which is 1. Let $x \in c_0$ and $f \in c_0^*$. We have $\sum_1^n x_i e_i \longrightarrow x$, so $f\left(\sum_1^n x_i e_i\right) \longrightarrow f(x)$, then $\sum_1^{+\infty} x_i f(e_i) = f(x)$. One the other hand, f *continuous* $\Leftrightarrow \frac{|f(x)|}{\|x\|} \leq \|f\|$ for all $x \in c_0$ (see appendix1). $\|e_i\| = 1$ gives $|f(e_i)| \leq \|f\|$ for all $i \geq 1$. Let $\alpha \in \,]0,1[$. Take $x_n = sign(f(e_n)) \cdot \frac{1}{n^\alpha}$ then $x = (x_n)_{n\geq 1} \in c_0$. We have

$$\sum_1^{+\infty} \frac{|f(e_n)|}{n^\alpha} = |f(x)| \leq \|x\| \,.\, \|f\| = \|f\|$$

Now, let $\varphi(\alpha) = \sum_1^{+\infty} \frac{|f(e_n)|}{n^\alpha}$. Then φ is bounded and decreasing on $]0,1[$. Hence, it has a finite limit as $\alpha \longrightarrow 0$. We can show easily that this limit is $\sum_1^{+\infty} |f(e_n)|$. Thus the sequence $(f(e_n))_{n\geq 1}$ is in l_1. Owing to the equality $\sum_1^{+\infty} x_i f(e_i) = f(x)$, we can identify it to f.

So $c_0^* \subset l_1$. The inverse inclusion is evident.

3 Hahn-Banach theorem

If f is a bounded functional defined on a subspace M of a real linear normed space X, then there is a bounded functional F defined on X which is an extension of f and which norm is equal to $\|f\|$[36].

4 Kuhn-Tucker theorem.[37]

Let X be a vector space and Z a normed vector space having positive cone P. Assume that P contains an interior point. Let Ω be a subset of X with a non empty interior.

Let f be a real valued functional on Ω and G a mapping from Ω into Z. Let $x_0 \in \overset{\circ}{\Omega}$ such that f and G are Gateaux differentiable at x_0. Assume that the

[35] See Luenberger p104, quoted reference

[36] The proof of this theorem is given in Luenberger p111, but only in the case where X is separable. l_∞ is not separable, but, according to the same reference, with Zorn lemma we can extend the theorem to a non separable space .

[37] For proof, see Luenberger p249, quoted reference

Gateaux differentials are linear in their increments[38]. Suppose x_0 maximizes f subject to $G(x) \geq 0$ and that x_0 is a regular point of the inequality $G(x) \geq 0$. Then there is $z_0^* \in Z^*$, $z_0^* \geq 0$ such as

$$\delta f(x_0) \mid y + \langle z_0^* \mid (\delta G(x_0) \cdot y) \rangle = 0 \text{ for all } y \in X$$

and $z_0^* \mid G(x_0) = 0$.

5 Construction of $\overline{E(K,r)}$

Let $K \in \overset{\circ}{D}$ and such that $(k_0, k_1) \in \overset{\circ}{D}_u$. Denote $S(K,r)$ the open sphere of center K and radius r. There is $r_0 \succ 0$ such that $S(K,r_0) \subset \overset{\circ}{D}$[39]. For $r \in]0, r_0]$ define $E(K,r) =$

$\{(k_h, k_l) \in R^2 \diagup \exists B \in S(K,r) \text{ and } i \geq 1 \text{ such that } b_{i-1} = k_h \text{ and } b_i = k_l \}$. We have $E(K,r) \subset D_u$. Denote $\overline{E(K,r)}$ the closure of $E(K,r)$. Since D_u is closed, we have also $\overline{E(K,r)} \subset D_u$. Moreover, if $(k_h, k_l) \in E(K,r)$, there is $B \in S(K,r)$ and $i \geq 1$ such that $b_{i-1} = k_h$ and $b_i = k_l$. Thus $|k_h| = |b_{i-1}|$ and $|k_l| = |b_i|$, so $|k_h| \leq \sup(\|B\|, k_0)$ and the same thing for k_l. But $B \in S(K,r)$ implies $\|B\| \leq \|K\| + r$, so $E(K,r) \subset [0, \sup(\|K\|, k_0)]^2$, and so is $\overline{E(K,r)}$. Consequently $\overline{E(K,r)}$ is a closed and bounded subset of R^2,it is then compact.

We show now that there is $r \in]0, r_0]$ such that $\overline{E(K,r)} \subset \overset{\circ}{D}_u$. Denote $\widehat{D}_u = \left\{ x \in D_u \diagup x \notin \overset{\circ}{D}_u \right\}$.

Suppose that for all $r \in]0, r_0]$, $\overline{E(K,r)}$is not included in $\overset{\circ}{D}_u$. Since $\overline{E(K,r)} \subset D_u$, it implies $\overline{E(K,r)} \cap \widehat{D}_u \neq \emptyset$. For $r \prec r'$, we have $\overline{E(K,r)} \subset \overline{E(K,r')}$. Let r_n be a sequence of $]0, r_0]$ decreasing to 0. The sets $(\overline{E(K,r_n)} \cap \widehat{D}_u)_{n \in N}$ are embedded together. They are compact. Thus, their intersection $\cap_{n \in N} (\overline{E(K,r_n)} \cap \widehat{D}_u)$ is not empty[40]. But $\cap_{n \in N} (\overline{E(K,r_n)} \cap \widehat{D}_u) = \left[\cap_{n \in N} \overline{E(K,r_n)} \right] \cap \widehat{D}_u$ so $\left[\cap_{n \in N} \overline{E(K,r_n)} \right] \cap \widehat{D}_u \neq \emptyset$.

Let $x \in \left[\cap_{n \in N} \overline{E(K,r_n)} \right] \cap \widehat{D}_u$.

$$\begin{aligned} x &\in \left[\cap_{n \in N} \overline{E(K,r_n)} \right] \Longrightarrow \\ \text{for all } \mu &\succ 0 \text{ there is } i \geq 1 \text{ such that } \|(x_h, x_l) - (k_{i-1}, k_i)\| \prec \mu \\ x &\in \widehat{D}_u \Longrightarrow \\ \text{for all } \mu &\succ 0,\ S(x,\mu) \text{ contains an element that doesn't belong to } D_u. \end{aligned}$$

[38] Theese hypothesises concerning Gateaux differentials are all verified when the functions are Frechet differentiable.

[39] *Proof:*

Suppose that for all $r \succ 0$, $S(K,r)$ cuts the border of D. Let M be a point of (border of D) $\cap$ $S(K,r)$. Since $S(K,r)$ is open, it contains an open sphere of center M. Since M is a border point, the last sphere must contain an element outside of D_k. This contradicts the fact that K is in an interior of D.

[40] *Proof:*

Let (C_n) be a sequence of embedded compact subsets. We extract from every C_n an element x_n. The sequence (x_n) have an accumulation point x. We then see that $x \in \cap_n C_n$.

Let $\alpha \succ 0$. I take $\mu = \frac{\alpha}{2}$. Let $i \geq 1$ such that $\|(x_h, x_l) - (k_{i-1}, k_i)\| \prec \frac{\alpha}{2}$ and $y \notin D_u$ such that $\|(x_h, x_l) - (y_h, y_l)\| \prec \frac{\alpha}{2}$. Then

$$\|(k_{i-1}, k_i) - (y_h, y_l)\| \leq \|(x_h, x_l) - (k_{i-1}, k_i)\| + \|(x_h, x_l) - (y_h, y_l)\| \prec \alpha$$

Take B such that $b_j = k_j$ for all $j \neq i$ and $j \neq i-1$, and $b_i = y_l$ and $b_{i-1} = y_h$.

If $i \geq 2$, $y \notin D_u$ implies that $B \notin D$. If $i = 1$ then $y \in S((k_0, k_1), \alpha)$.

Thus, for all $\alpha \succ 0$ we have either found B such as $\|B - K\| \prec \alpha$ but $B \notin D$ or $y \in S((k_0, k_1), \alpha)$ but $y \notin D_u$. This contradicts $K \in \overset{\circ}{D}$ and $(k_0, k_1) \in \overset{\circ}{D}_u$■

6 Frechet-differentiability on the border

While this property doesn't add anything in the general case, it permits to characterize border optima in the concave case.

Definition *Let D be a convex subset of a real normed space X and T a transformation from $\overline{D}$[41] to a real normed space Y. Denote $\widehat{D}$ the border of D. $\widehat{D} = \overline{D} - \overset{\circ}{D}$. T is said to be Frechet-differentiable on the border of D if there is for all x of $\widehat{D}$ a continuous linear transformation $\delta T(x)$ such that*

$$\lim_{x' \in D,\ x' \longrightarrow x} \frac{\|T(x') - T(x) - \delta T(x) \cdot (x' - x)\|}{\|x' - x\|} = 0$$

Remark *In this case, the Frechet-differential in not necessarily unique as it is when T is defined in a neighborhood of x. But we verify easily that, if we define $\overset{\circ}{D} - x = \left\{ x' - x / x' \in \overset{\circ}{D} \right\}$, its restriction to $\overset{\circ}{D} - x$ is unique.*

7 U is Frechet-differentiable on the border of D_u

Under 5-1 conditions on U and D_u and for $A \in \widehat{D}_u$ we show that:

$$\lim_{M \longrightarrow A\ ,\ M \in D_u} \frac{\left\| U(M) - U(A) - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{AM} \right\|}{\left\| \overrightarrow{AM} \right\|} = 0$$

Let $\alpha \in [0, 1]$, $M \in \overset{\circ}{D}_u$ and $\varphi(\alpha) = U((1-\alpha)A + \alpha M)$. Since $]A, M] \subset \overset{\circ}{D}_u$(accessibility lemma), φ is continue on $[0, 1]$, derivable on $]0, 1]$ and $\varphi'(\alpha) = \overrightarrow{\nabla U}((1-\alpha)A + \alpha M) \cdot \overrightarrow{AM}$. Owing to the mean value theorem, there is $\beta \in$

[41] We denote $\overline{D}$ the closure of D.

$]0,\alpha[$such that $\varphi'(\beta) = \frac{\varphi(\alpha)-\varphi(0)}{\alpha}$. Take $\alpha = 1$. Then $\varphi'(\beta) = U(M) - U(A)$. This gives $\overrightarrow{\nabla U}((1-\beta)A + \beta M) \cdot \overrightarrow{AM} = U(M) - U(A)$. So:

$$\begin{aligned} & \frac{\left\| U(M) - U(A) - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{AM} \right\|}{\left\| \overrightarrow{AM} \right\|} \\ = & \frac{\left\| \overrightarrow{\nabla U}((1-\beta)A + \beta M) \cdot \overrightarrow{AM} - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{AM} \right\|}{\left\| \overrightarrow{AM} \right\|} \\ \leq & \left\| \overrightarrow{\nabla U}((1-\beta)A + \beta M) - \overrightarrow{\nabla U}(A) \right\| \longrightarrow 0 \text{ when } M \longrightarrow A \end{aligned}$$

If $M \in \widehat{D}_u$, I take $M' \in \overset{\circ}{D}_u$ such that $MM' \leq AM^2$.

$$\frac{\left\| U(M) - U(A) - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{AM} \right\|}{\left\| \overrightarrow{AM} \right\|} \leq \frac{\left\| U(M') - U(A) - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{AM'} \right\|}{\left\| \overrightarrow{AM'} \right\|} \frac{\left\| \overrightarrow{AM'} \right\|}{\left\| \overrightarrow{AM} \right\|} + \frac{\left\| U(M) - U(M') - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{M'M} \right\|}{\left\| \overrightarrow{AM} \right\|}$$

we have

$$\frac{\left\| U(M') - U(A) - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{AM'} \right\|}{\left\| \overrightarrow{AM'} \right\|} \longrightarrow 0 \text{ when } M' \longrightarrow A$$

and

$$\frac{\left\| \overrightarrow{AM'} \right\|}{\left\| \overrightarrow{AM} \right\|} \longrightarrow 1 \text{ when } M \longrightarrow A$$

Similarly, there is $M" \in]M, M'[$ such that $U(M') - U(M) = \overrightarrow{\nabla U}(M") \cdot \overrightarrow{MM'}$. Then

$$\frac{\left\| U(M) - U(M') - \overrightarrow{\nabla U}(A) \cdot \overrightarrow{M'M} \right\|}{\left\| \overrightarrow{AM} \right\|} = \frac{\left\| (\overrightarrow{\nabla U}(M") - \overrightarrow{\nabla U}(A)) \cdot \overrightarrow{M'M} \right\|}{\left\| \overrightarrow{M'M} \right\|} \frac{\left\| \overrightarrow{M'M} \right\|}{\left\| \overrightarrow{AM} \right\|}$$

$$\leq \left[\left\| \overrightarrow{\nabla U}(M") \right\| + \left\| \overrightarrow{\nabla U}(A) \right\| \right] \frac{\left\| \overrightarrow{M'M} \right\|}{\left\| \overrightarrow{AM} \right\|} \longrightarrow 0 \text{ when } M \longrightarrow A$$

■

8 G is Frechet-differentiable on the border of D

Let $K \in \widehat{D}$, $r \succ 0$ and $S_f(K,r)$ the closed sphere of center K and radius r. Denote $F(K,r) =$
$\left\{(k_h, k_l) \in R^2 \diagup \exists B \in S_f(K,r) \cap D \text{ and } i \geq 1 \text{ such that } b_{i-1} = k_h \text{ and } b_i = k_l \right\}$.
$F(K,r) \subset D_u$ and $\overline{F(K,r)} \subset D_u$ since D_u is closed.

I show like I have done in appendix5 for $\overline{E(K,r)}$, that $\overline{F(K,r)}$ is bounded. It is then a compact.

As in appendix5, although it is easier here because U is Frechet-differentiable on the border of D_u and not only in the interior of D_u, I define o and use its uniform continuity on $\left[0, \frac{r}{2}\right]^2 \times \overline{F(K,r)}$ to end to

$$\lim_{B \in D,\ B \longrightarrow K} \frac{\|G(B) - G(K) - \delta G(K) \cdot (B-K)\|}{\|B-K\|} = 0$$

■

9 Calculus of $\frac{\partial f}{\partial \infty}(x_0)$

Let $h \in c$ and let $r_n(h)$ be the sequence of c obtained by setting to 0 the n first terms of h. Since f is a function from l_∞ to R , Frechet-differentiable at $x_0 \in l_\infty$, for all $\varepsilon \succ 0$ there is $\alpha \succ 0$ such that:

$$\|h\| \prec \alpha \Longrightarrow \frac{|f(x_0+h) - f(x_0) - \delta f(x_0).h|}{\|h\|} \prec \varepsilon$$

But $\|h\| \prec \alpha \Longrightarrow \|r_n(h)\| \prec \alpha$ for all $n \geq 1$, then

$$|f(x_0 + r_n(h)) - f(x_0) - \delta f(x_0).r_n(h)| \prec \varepsilon \|r_n(h)\|$$

Thus

$$\left| f(x_0 + r_n(h)) - f(x_0) - \sum_{i=n+1}^{+\infty} \frac{\partial f}{\partial x_i}(x_0).h_i - \frac{\partial f}{\partial \infty}(x_0).\delta_\infty(h) \right| \prec \varepsilon \|r_n(h)\|$$

Moreover, $\|r_n(h)\| = \sup_{i \succ n} |h_i|$. It is a positive and decreasing sequence converging to $|\delta_\infty(h)|$. We have also $\sum_{i=n+1}^{+\infty} \frac{\partial f}{\partial x_i}(x_0).h_i \longrightarrow 0$ when $n \longrightarrow +\infty$. Then

$$\left| \limsup_n f(x_0 + r_n(h)) - f(x_0) - \frac{\partial f}{\partial \infty}(x_0).\delta_\infty(h) \right| \leq \varepsilon |\delta_\infty(h)|$$

which gives

$$\left| \frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\delta_\infty(h)} - \frac{\partial f}{\partial \infty}(x_0) \right| \leq \varepsilon$$

This proves that

$$\frac{\partial f}{\partial \infty}(x_0) = \lim_{\|h\| \longrightarrow 0, h \in c, \delta_\infty(h) \neq 0} \frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\delta_\infty(h)}$$

We prove similarly the formula for $\liminf$.■

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