Subextremal Functions and Lattice Programming[1]

by

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Abstract

Let M and N be the sets of minimizers of a function f over respective subsets K and L of a lattice with K being lower than L. Veinott (1965) and Topkis and Veinott (1968) showed that if f is subadditive, then M is lower than N. Veinott (1969) showed that if f is sub$*$ with $*$ being strictly (resp., properly) increasing, which is so, for example, when f is subadditive (resp., submeet or subjoin), then M is lower than (resp., weakly lower than) N. The present paper characterizes the class of functions f for which M is lower (resp., weakly lower, meet lower, join lower, chain lower) than N for all K lower than L. The resulting five classes of functions, called subextremal variants, have alternate characterizations by variants of the downcrossing-differences property, i.e., their first differences change sign at most once from plus to minus along complementary chains. In each case, the classes are closed under increasing transformations, and so may represent ordinal preferences. It is also shown that each subextremal variant class is closed under partial minimization. This paper subsumes earlier work of Li Calzi (1990) and Veinott (1991). Shannon (1990) and Milgrom and Shannon (1991) have done related overlapping work. Indeed, the last paper first showed M is lower than N for all K lower than L only if f has interval-downcrossing differences. The sub$*$ functions are lattice subextremal when $*$ is strictly increasing and subextremal when $*$ is properly increasing. Conditions are given on $*$ that assure sub$*$ functions have representations as strictly increasing functions of subadditive functions. Subadditive twice-differentiable strictly increasing transformations of twice-differentiable functions are characterized.

1. Introduction

Lattice programming is a qualitative theory of optimization that is concerned with the study of optimization problems for which one (global) minimizer of a function is increasing in the parameters of the problem. The first general sufficient condition for this to be so was developed by Veinott (1965) and improved by Topkis and Veinott (1968), viz., the function is subadditive[4] on a lattice. Subsequently, Veinott (1969) generalized the result to sub$*$ functions on a lattice where $*$ is a binary operation on a subset of the extended real numbers. A function is *sub$*$* if the $*$ product of the function values at two

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[4]The term subadditive was apparently suggested by K. Fan (1967), though the concept had been used by a number of authors during the preceding several decades. Among other names, subadditive functions are also called submodular by Edmonds (1970) and convex by Shapley (1971).

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points in the domain of the function majorizes the $*$ product of the function values at the meet and join of the two points. The sub$*$ functions are called *subadditive*, *submultiplicative*, *subjoin* or *submeet* according as $*$ is $+, \cdot, \vee$ or $\wedge$. These ideas—especially subadditivity—have proved useful in myriad applications by a wide variety of authors in the ensuing quarter of a century.

In the spring of 1990, Li Calzi (1990) and, independently, Shannon (1990a, b) (the latter inspired by a suggestion of Milgrom), discovered useful weaker sufficient conditions assuring that the optimal solution is increasing in the parameters of the problem. Shannon introduced the class of *quasisubmodular* functions. Li Calzi independently introduced the same class and four other variants thereof. In the language of the present paper, these five classes are defined by five variants of the *downcrossing differences*[5] property, i.e., the first differences of the function change sign from plus to minus at most once on "complementary" chains. In particular, the quasisubmodular functions are, in this terminology, the functions with interval-downcrossing differences. Subadditive functions have this property since their first differences are decreasing.

A function on a lattice is called *subextremal*[6] (resp., *strictly subextremal*) herein if it is either subjoin or submeet (resp., strictly subjoin or strictly submeet) on the sublattice hull of any pair of incomparable elements. In this terminology, Li Calzi (1990, pp. 13-14) showed implicitly that strictly subextremal functions have strictly downcrossing differences and established the converse explicitly. He also showed that functions with downcrossing differences are subextremal. Shannon (1990a, p. 18; 1990b, p. 23) independently established the weaker results that strictly subextremal functions have interval-downcrossing differences, and the latter are subextremal. Veinott (1991) introduced three additional subextremal variants called respectively meet-, join- and lattice-subextremal. Moreover, he showed that the five subextremal variants described above in fact characterize the five downcrossing-differences variants.

In the spring of 1991, Milgrom and Shannon (1991) showed that the functions with interval-downcrossing differences on a lattice characterize the functions f for which the set of minimizers of f over one subset of the lattice is lower than the set of minimizers of f over a second subset of the lattice whenever the first subset is lower than the second subset. In the summer of 1991, Veinott (1991) extended these results by developing comparative-minimizer characterizations for the other four downcrossing-differences variants in which the first instance of the lower-than relation given above is replaced by a variant thereof.

[5] A more precise phrase that we have eschewed for aesthetic reasons would be *single-down-zero-crossing differences*. The term *single crossing property* is used in [MS91] for a more restricted concept.

[6] The term subextremal is used in [Li90] to mean instead that a function is either subjoin or submeet (everywhere).

Section 2 introduces the notions of subextremal functions, functions with downcros sing differences, the lower-than relation, and the variants of each. In each case, the ap propriate comparative-minimizer characterization results are obtained. Section 3 show that sub$*$ functions are lattice subextremal if $*$ is strictly increasing, which is so, fo example, when $*$ is $+$, and are subextremal when $*$ is properly increasing, which is s when $*$ is $\wedge$ or $\vee$. For three classes of binary relations $*$, results of Aczél (1949, 1966 and Ling (1965) are used to represent sub$*$ functions as strictly increasing functions o subadditive functons. Twice-differentiable subadditive strictly increasing functions o twice differentiable functions are characterized.

2. Subextremal Functions and Comparative-Minimizer Characterization

In this section we consider when the set of minimizers of a function over one subse of a lattice "minorizes" the set of minimizers of the function over another subset in sense that is "compatible" with the partial order on the lattice. To make this compara tive-minimizer problem precise, let f be an extended real-valued function on a lattice partially ordered by $\leq$. For each $S \subseteq L$, denote by $f(S)$ the image of S under f, b argmin$f(S)$ the set of minimizers of f over S, by inf$f(S)$ the infimum of $f(S)$ and b min$f(S)$ the least element of $f(S)$ (if any). A comparative-minimizer characterizatio problem is to determine for $S, \Sigma \subseteq L$ the class of functions f for which argmin$f(S)$ "min orizes" argmin$f(\Sigma)$ when S "minorizes" Σ in another sense.

To give meaning to the term "minorize," it is necessary to introduce some binar set relations $\preceq$ on the set 2^L of nonempty subsets of L. It is natural to require these re ations $\preceq$ to be *compatible with* $\leq$ in the sense that if $\{s\}$ and $\{\sigma\}$ are singleton subset of L, then $\{s\} \preceq \{\sigma\}$ if and only if $s \leq \sigma$. The contained-in relation is a prominent exam ple of a set relation that is incompatible with $\leq$. Table 1 below defines seven set rela tions that are compatible with $\leq$ and are useful in the sequel. The relation $\leq_0$ (resp., $\leq$ s the weakest (resp., strongest) relation on 2^L that is compatible with $\leq$ on L. The re

Table 1. Seven Set Relations Compatible with $\leq$

Relation $\preceq$	*Definition of* $S \preceq \Sigma$
$\leq_0$	$s \in S$ and $\sigma \in \Sigma$ hold for some $s \leq \sigma$
$\leq_w$	$s \in S$ and $\sigma \in \Sigma$ imply either $s \wedge \sigma \in S$ or $s \vee \sigma \in \Sigma$
$\leq_\vee$	$s \in S$ and $\sigma \in \Sigma$ imply $s \vee \sigma \in \Sigma$
$\leq_\wedge$	$s \in S$ and $\sigma \in \Sigma$ imply $s \wedge \sigma \in S$
$\leq_l$	$s \in S$ and $\sigma \in \Sigma$ imply both $s \wedge \sigma \in S$ and $s \vee \sigma \in \Sigma$
$\leq_c$	$s \in S$ and $\sigma \in \Sigma$ imply s and σ are comparable, $s \wedge \sigma \in S$ and $s \vee \sigma \in \Sigma$
$\leq$	$s \in S$ and $\sigma \in \Sigma$ imply $s \leq \sigma$

ations $\leq_\vee$ and $\leq_\wedge$ are incomparable, but each is stronger than $\leq_w$ and weaker than $\leq_l$. Finally, the relation $\leq_l$ is weaker than $\leq_c$. The relations $\leq_0$ and $\leq$ are well known. The relation $\leq_l$ was introduced in [Ve65] and shown to partially order the nonempty sublattices of a lattice by Topkis and Veinott (1968). The relations $\leq_w$, $\leq_\vee$ and $\leq_\wedge$ were introduced in [Ve69]. The relation $\leq_c$ is new. A more detailed examination of all of these relations but $\leq_c$ appears in [Ve74] and [Ve87], and of all of them in [Vefo].

A simple comparative-minimizer characterization problem is to describe for a set relation $\preceq$ compatible with $\leq$, the class of functions f such that $\operatorname{argmin} f(\{s, s\wedge\sigma\}) \preceq \operatorname{argmin} f(\{s, s\vee\sigma\})$ for each $s,\sigma\in L$. Below we characterize these classes for the seven relations in Table 1. It turns out that the first (resp., last) two of these seven classes coincide. The middle five classes are distinct; consider one of these five classes and its associated set relation $\preceq$, say. The real interest of the class stems from the following additional characterizations and properties thereof.

- f is in the class if and only if $\operatorname{argmin} f(S)$ is increasing in S in the sense that $S \leq_l \Sigma$ implies $\operatorname{argmin} f(S) \preceq \operatorname{argmin} f(\Sigma)$.
- f is in the class if and only if that is so of the function $\min f\circ\mu^{-1}$ on $\mu(L)$ for each lattice morphism μ on L for which $\min f\circ\mu^{-1}$ exists on $\mu(L)$.[7]
- The class is invariant under strictly increasing transformations, so each function therein expresses ordinal as opposed to cardinal preferences among the elements of L.[8]
- The class includes families of functions that arise in practical optimization problems.

In the sequel we give two characterizations of each of the above five classes of functions. The first is in terms of "subextremal variants" and the second in terms of "downcrossing-differences variants." We now introduce these classes of functions.

Subextremal Variants. Let f be an extended real-valued function on a lattice L. Call f *subextremal* if for each (incomparable) $s,\sigma\in L$, either

$$f(s\vee\sigma)\vee f(s\wedge\sigma)\leq f(s)\vee f(\sigma) \tag{1}$$

or

$$f(s\vee\sigma)\wedge f(s\wedge\sigma)\leq f(s)\wedge f(\sigma) \tag{2}$$

holds. Observe that (1) (resp., (2)) is the statement that f is subjoin (resp., submeet) on the sublattice hull $\{s,\sigma\}^{\wedge\vee}$ of $\{s,\sigma\}$. Thus, to say that f is subextremal is to say that the function f is either subjoin or submeet on the sublattice hull of each pair of points in L. Keep in mind that whether f is subjoin or submeet depends on the pair of points

[7] Define $\min f\circ\mu^{-1}$ for each $t\in\mu(L)$ by $(\min f\circ\mu^{-1})(t)=\min f(\mu^{-1}(t))=\min_{\mu(s)=t} f(s)$. Define $\inf f\circ\mu^{-1}$ analogously.

[8] Invariance of each class follows from the invariance of the set of minimizers under such transformations.

Thus, a subjoin (resp., submeet) function is subextremal, but not every subextremal function is subjoin or submeet.

It turns out that there are four useful subextremal variants. These may be described as follows. Call f *join-* (resp., *meet-*, *lattice-*, *strictly*) *subextremal* if for each incomparable $s, \sigma \in L$, on adding $+\epsilon$ (resp., $+\epsilon$, $-\epsilon$, $-\epsilon$) to $f(s \wedge \sigma)$ (resp., $f(s \vee \sigma)$, $f(\sigma)$, $f(s)$ and $f(\sigma)$) in both (1) and (2), then either (1) or (2) holds for all small enough $\epsilon > 0$. In the sequel, we represent such inequalities by simply omitting the ϵ following the $+$ or $-$. Observe that f is *strictly subextremal* if and only if for each incomparable $s, \sigma \in L$, then either (1) or (2) holds with strict inequality.

Strictly subextremal functions are lattice-subextremal. In the sequel we show that the lattice-subextremal functions are precisely the functions that are both join- and meet-subextremal. Observe that f is subextremal on L if and only if f is subextremal on the dual of L. Similarly, f is join-subextremal on L if and only if f is meet-subextremal on the dual of L, and both are subextremal.

The five subextremal variants have characterizations in terms of the sign pattern of their differences on "complementary" chains. To develop this idea, it is necessary to formulate several variants of the notion of an extended real-valued function on a chain being *downcrossing*, i.e., changing sign at most once from $+$ to $-$ as one traverses the chain in its natural order.

Downcrossing Variants. A number x has *sign* $+, \overset{+}{0}, 0, \overset{-}{0}, -$ respectively according as $x > 0$, $x \geq 0$, $x = 0$, $x \leq 0$ or $x < 0$. Observe that positive (resp., negative) numbers have the first (resp., last) two signs and zero has the middle three signs. Let p (resp., n) be one of the first (resp., last) two of the above signs. Let g be an extended real-valued function on a chain C. Then g has the *sign pattern pn* on C if either g has sign p on C, or g has sign n on C, or there is an $r \in C$ such that $g(s)$ has sign p for $s < r$ and sign n for $s > r$ with $s \in C$. For example, if in the last event p is $+$ and n is $\overset{-}{0}$, then g is positive below r and nonpositive above r on C. Similarly, g has the *sign pattern* $+0-$ on C if either g has the $+-$ sign pattern on C or if there exists a subinterval I of C such that g is positive below I, zero on I and negative above I on C. Evidently, g has the sign pattern $+0-$ on C if and only if g has both the sign patterns $+\overset{-}{0}$ and $\overset{+}{0}-$ on C.

The sign patterns of an extended real-valued function g on a chain C that are useful for characterizing subextremal variants turn out to be the ones that can be characterized by collections of implications of the form: $g(r)\,\gamma\, 0 \Leftarrow g(s)\,\delta\, 0$ for all $r < s$ in C where γ is $>$ or $\geq$ and δ is $>$ or $\geq$, i.e., $g(r)$ stands in the relation γ to 0 if $g(s)$ stands in the relation δ to 0. The implied-by column of Table 2 below characterizes five sign patterns by suitable collections of implications of the above type. For example, the fact that $g(r)$

Table 2. Three Characterizations of Sign Patterns

Sign Pattern	*Implied-by*	*Implies*	*Either-or*
	For each $r < s$ in C	For each $r < s$ in C	For each $r < s$ in C
$\overset{+}{0}\overset{-}{0}$	$g(r) \geq 0 \Leftarrow g(s) > 0$	$g(r) < 0 \Rightarrow g(s) \leq 0$	$g(r) \geq 0$ or $g(s) \leq 0$
$+\overset{-}{0}$	$g(r) > 0 \Leftarrow g(s) > 0$	$g(r) \leq 0 \Rightarrow g(s) \leq 0$	$g(r) > 0$ or $g(s) \leq 0$
$\overset{+}{0}-$	$g(r) \geq 0 \Leftarrow g(s) \geq 0$	$g(r) < 0 \Rightarrow g(s) < 0$	$g(r) \geq 0$ or $g(s) < 0$
$+0-$	$\left\{\begin{matrix} g(r) > 0 \Leftarrow g(s) > 0 \\ g(r) \geq 0 \Leftarrow g(s) \geq 0 \end{matrix}\right\}$	$\left\{\begin{matrix} g(r) \leq 0 \Rightarrow g(s) \leq 0 \\ g(r) < 0 \Rightarrow g(s) < 0 \end{matrix}\right\}$	$\left\{\begin{matrix} g(r) > 0 \text{ or } g(s) \leq 0 \\ g(r) \geq 0 \text{ or } g(s) < 0 \end{matrix}\right\}$
$+-$	$g(r) > 0 \Leftarrow g(s) \geq 0$	$g(r) \leq 0 \Rightarrow g(s) < 0$	$g(r) > 0$ or $g(s) < 0$

> 0 if $g(s) \geq 0$ for all $r < s$ in C characterizes the sign pattern $+-$. Notice that the implications in the implies column of Table 2 are contrapositives of the corresponding ones in the implied-by column. Also, each statement in the either-or column is equivalent to the corresponding implications in the implied-by and implies columns. Thus, Table 2 gives three equivalent characterizations of each sign pattern.

In the sequel, call an extended real-valued function on a chain *downcrossing, join-downcrossing, meet-downcrossing, interval-downcrossing or strictly downcrossing* respectively according as the function has the sign pattern $\overset{+}{0}\overset{-}{0}$, $+\overset{-}{0}$, $\overset{+}{0}-$, $+0-$ or $+-$ on the chain. Call functions of these five types *downcrossing variants*. The downcrossing variants are illustrated in Figure 1. Notice that a function on a chain is a downcrossing variant if and only if the function is that downcrossing variant on each doubleton subset of the chain.

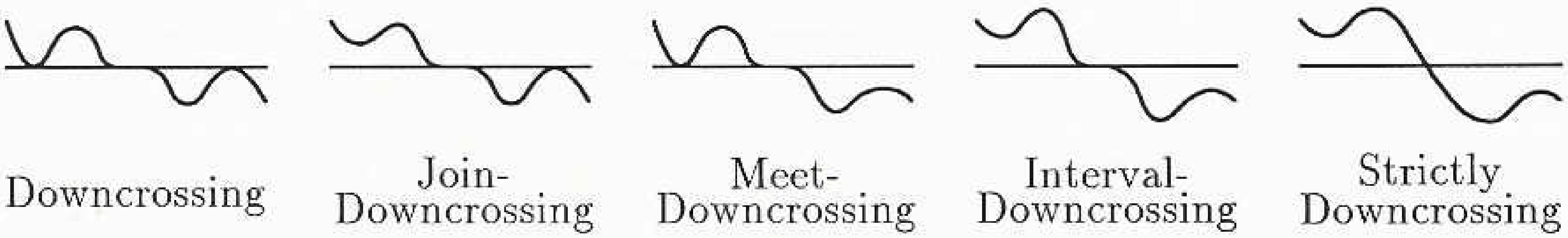

Figure 1. Downcrossing Variants

Are there any other sign patterns that can be characterized by collections of implications of the type described above? The answer is no. To see why, observe that since there are exactly four different individual statements of the form $g(r)\,\gamma\,0 \Leftarrow g(s)\,\delta\,0$ for all $r < s$ in C where γ and δ can each be either $>$ or $\geq$. With one exception, for each pair of such statements, one implies the other. The exceptional pair of statements is $g(r) > 0 \Leftarrow g(s) > 0$ and $g(r) \geq 0 \Leftarrow g(s) \geq 0$; neither implies the other. This pair of statements implies one and is implied by the other of the two remaining individual statements. Thus, the above exceptional pair is the desired fifth collection of statements. All other collections of statements of the above type are equivalent to one of these five. To see this, observe that it suffices to consider collections of statements of the above form that do not contain duplicates. Each such collection consists of at most four distinct statements. If the collection is not one of the five described above, then the collection contains a pair of statements other than the exceptional pair. But then, one of the statements in the pair is implied by the other and so can be eliminated. Repeating this procedure at most twice more produces the desired result.

Downcrossing-Differences Variants. Suppose f is an extended real-valued function on a lattice L. For each σ in L, call $\Delta_\sigma f(s) \equiv f(s \vee \sigma) - f(s)$ the *σ-difference* of f at $s \in L$. Interpret both $\infty - \infty$ and $-\infty - -\infty$ as zero to remove ambiguity in this definition. Call C a *complementary σ-chain* if C is a subchain of L with least element $r < \sigma$ and $s \wedge \sigma = r$ for each $s \in C$. Figure 2 illustrates one such subchain in $\Re^3$.

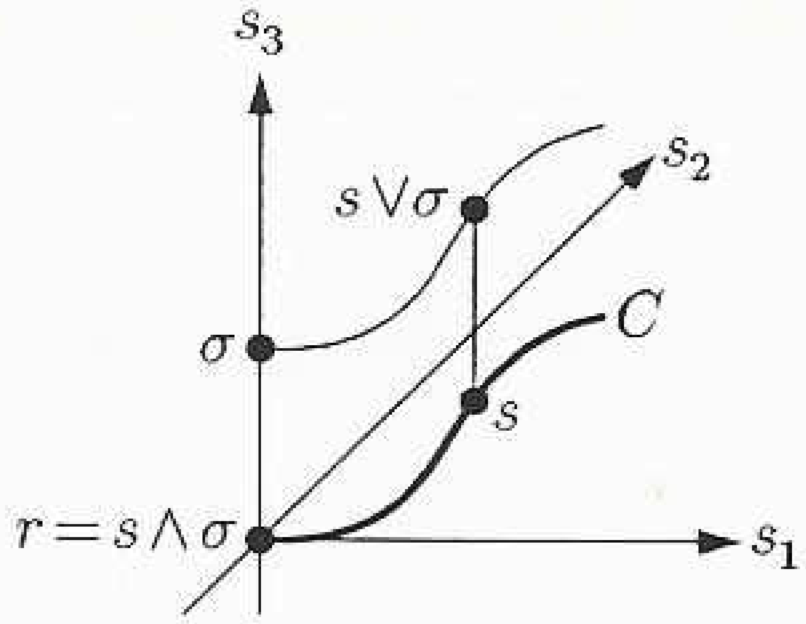

Figure 2. Complementary σ-Chain in $\Re^3$

In the sequel when we speak of properties of the differences of f we mean properties of the σ-differences of f on complementary σ-chains for each $\sigma \in L$. Thus we can and do speak of f as having *downcrossing* (resp., *join-downcrossing, meet-downcrossing, interval-downcrossing, strictly downcrossing*) *differences*. It follows from the characterizations in Table 2 that f has this property on L if and only if f has the property on $\{s, \sigma\}^{\wedge\vee}$ for each pair $\{s, \sigma\}$ of incomparable elements of L. In this event, the unique complementary σ-chain with cardinality two or more is $\{s \wedge \sigma, s\}$. Using this observation and setting $g \equiv \Delta_\sigma f$ on $\{s \wedge \sigma, s\}$, Table 3 below summarizes the characterizations of the five downcrossing-differences variants obtained from the "implied-by" column of Table 2. Similar tables can be constructed for corresponding characterizations obtained from the "implies" and "either-or" columns of Table 2. We leave these to the reader, though we use them frequently in the sequel.

Table 3. Characterizations of Downcrossing-Differences Variants

Differences of f	*Implied-by Characterizations*
	For each incomparable $s,\sigma \in L$
Downcrossing	$f(s\wedge\sigma) \le f(\sigma) \Leftarrow f(s) < f(s\vee\sigma)$
Join-Downcrossing	$f(s\wedge\sigma) < f(\sigma) \Leftarrow f(s) < f(s\vee\sigma)$
Meet-Downcrossing	$f(s\wedge\sigma) \le f(\sigma) \Leftarrow f(s) \le f(s\vee\sigma)$
Interval-Downcrossing	$\left\{\begin{matrix} f(s\wedge\sigma) < f(\sigma) \Leftarrow f(s) < f(s\vee\sigma) \\ f(s\wedge\sigma) \le f(\sigma) \Leftarrow f(s) \le f(s\vee\sigma)\end{matrix}\right\}$
Strictly Downcrossing	$f(s\wedge\sigma) < f(\sigma) \Leftarrow f(s) \le f(s\vee\sigma)$

Denote by $\mathfrak{L}_L$ the set of subsets S of L for which $\operatorname{argmin} f(S) \neq \emptyset$.

Theorem 1. Characterization of Subextremal Functions. *The following properties of an extended real-valued function f on a lattice L are equivalent.*

1° *f is subextremal.*

2° *f has downcrossing differences.*

3° $\operatorname{argmin} f(\{s, s\wedge\sigma\}) \le_0 \operatorname{argmin} f(\{\sigma, s\vee\sigma\})$ *for all $s,\sigma \in L$.*

$3'$ $\operatorname{argmin} f(\{s, s\wedge\sigma\}) \le_w \operatorname{argmin} f(\{\sigma, s\vee\sigma\})$ *for all $s,\sigma \in L$.*

4° $\operatorname{argmin} f(S) \le_0 \operatorname{argmin} f(\Sigma)$ *for all $S \le_l \Sigma$ in $\mathfrak{L}_L$.*

$4'$ $\operatorname{argmin} f(S) \le_w \operatorname{argmin} f(\Sigma)$ *for all $S \le_l \Sigma$ in $\mathfrak{L}_L$.*

5° $\inf f\circ\mu^{-1}$ *is subextremal on $\mu(L)$ for each lattice morphism μ on L.*

Proof. $1^\circ \Leftrightarrow 2^\circ$. Consider incomparable $s,\sigma \in L$. Then by the either-or version of Table 3, 2° holds on $\{s,\sigma\}^{\wedge\vee}$ if and only if $\gamma: f(s\vee\sigma) \le f(s)$ or $\delta: f(s\wedge\sigma) \le f(\sigma)$ holds and $\Gamma: f(s\vee\sigma) \le f(\sigma)$ or $\Delta: f(s\wedge\sigma) \le f(s)$ holds. This is so if and only if either (a) γ and Γ hold or δ and Δ hold, or (b) γ and Δ hold or δ and Γ hold. Now (a) and (b) are respectively equivalent to (1) and (2). Thus (a) or (b) holds if and only if 1° holds on $\{s,\sigma\}^{\wedge\vee}$.

$2^\circ \Leftrightarrow 3^\circ$. From Table 3, 2° is false if and only if $f(s\wedge\sigma) > f(s)$ and $f(s\vee\sigma) > f(\sigma)$ for some incomparable $s,\sigma \in L$, or equivalently 3° is false.

$3^\circ \Leftrightarrow 3' \Leftrightarrow 4^\circ \Leftrightarrow 4'$. Observe $4'$ implies $3'$ and 4° implies 3° as one sees by setting $S = \{s, s\wedge\sigma\}$ and $\Sigma = \{\sigma, s\vee\sigma\}$. Also, $3'$ implies 3° since $\le_w$ is stronger that $\le_0$. Thus, suppose 3° holds and that $S \le_l \Sigma$ are in $\mathfrak{L}_L$. Choose $s \in \operatorname{argmin} f(S)$ and $\sigma \in \operatorname{argmin} f(\Sigma)$. Then $s\wedge\sigma \in S$ and $s\vee\sigma \in \Sigma$ since $S \le_l \Sigma$. Therefore $s \in \operatorname{argmin} f(\{s, s\wedge\sigma\})$ and $\sigma \in \operatorname{argmin} f(\{\sigma, s\vee\sigma\})$. Thus by 3°, either $s\wedge\sigma \in \operatorname{argmin} f(\{s, s\wedge\sigma\}) \subseteq \operatorname{argmin} f(S)$ or $s\vee\sigma \in \operatorname{argmin} f(\{\sigma, s\vee\sigma\}) \subseteq \operatorname{argmin} f(\Sigma)$, establishing $4'$.

$5^\circ \Rightarrow 1^\circ$. Since the identity map μ is a lattice morphism from L into itself, it follows that $\inf f \circ \mu^{-1} = f$ is subextremal on $\mu(L) = L$.

$2^\circ \Rightarrow 5^\circ$. Suppose μ is a lattice morphism on L. Put $F = \inf f \circ \mu^{-1}$. Suppose t, τ are incomparable and $F(t) < F(t \vee \tau)$. Choose $\epsilon > 0$ so $(F(t)+\epsilon) \vee (-\epsilon^{-1}) < F(t \vee \tau$ there exist incomparable $s \in \mu^{-1}(t)$ and $\sigma \in \mu^{-1}(\tau)$ with $f(s) \leq (F(t)+\epsilon) \vee (-\epsilon^{-1})$ an $\leq (F(\tau)+\epsilon) \vee (-\epsilon^{-1})$. Since $\mu(s \vee \sigma) = t \vee \tau$, $f(s) \leq (F(t)+\epsilon) \vee (-\epsilon^{-1}) < F(t \vee \tau) \leq f$ Hence, by $\mu(s \wedge \sigma) = t \wedge \tau$, 2° and Table 3, $F(t \wedge \tau) \leq f(s \wedge \sigma) \leq f(\sigma) \leq (F(\tau)+\epsilon) \vee$ Now $\epsilon > 0$ is arbitrary, so $F(t \wedge \tau) \leq F(\tau)$. ▮

Theorem 2. Characterization of Join-Subextremal Functions. *The following p ties of an extended real-valued function f on a lattice L are equivalent.*

1° *f is join-subextremal.*

2° *f has join-downcrossing differences.*

3° $\operatorname{argmin} f(\{s, s \wedge \sigma\}) \leq_\vee \operatorname{argmin} f(\{\sigma, s \vee \sigma\})$ *for all $s, \sigma \in L$.*

4° $\operatorname{argmin} f(S) \leq_\vee \operatorname{argmin} f(\Sigma)$ *for each $S \leq_l \Sigma$ in $\mathfrak{L}_L$.*

5° $\min f \circ \mu^{-1}$ *is join-subextremal on $\mu(L)$ for each lattice morphism μ on which* $\min f \circ \mu^{-1}$ *exists on $\mu(L)$.*

6° *f is subextremal and $f(s) = f(s \wedge \sigma)$ implies $f(s \vee \sigma) \leq f(\sigma)$ for all $s, \sigma \in L$.*

Proof. $1^\circ \Leftrightarrow 2^\circ$. Same as the proof that $1^\circ \Leftrightarrow 2^\circ$ in Theorem 1 on replacing $f(s$ $f(s \wedge \sigma)+$.

$2^\circ \Leftrightarrow 3^\circ$. From Table 3, 2° is false if and only if $f(s \wedge \sigma) \geq f(s)$ and $f(s \vee \sigma)$ for some incomparable $s, \sigma \in L$, or equivalently 3° is false.

$3^\circ \Leftrightarrow 4^\circ$. To see that 4° implies 3°, set $S = \{s, s \wedge \sigma\}$ and $\Sigma = \{\sigma, s \vee \sigma\}$. Conv suppose 3° holds and $S \leq_l \Sigma$ are in $\mathfrak{L}_L$. Choose $s \in \operatorname{argmin} f(S)$ and $\sigma \in \operatorname{argmin} f(\Sigma$ $s \wedge \sigma \in S$ and $s \vee \sigma \in \Sigma$ since $S \leq_l \Sigma$. Thus $s \in \operatorname{argmin} f(\{s, s \wedge \sigma\})$ and $\sigma \in$ argmi $s \vee \sigma\})$, whence by 3°, $s \vee \sigma \in \operatorname{argmin} f(\{\sigma, s \vee \sigma\}) \subseteq \operatorname{argmin} f(\Sigma)$, establishing 4°.

$5^\circ \Rightarrow 1^\circ$. Since the identity map μ is a lattice morphism from L into itself, it that $\min f \circ \mu^{-1} = f$ is join-subextremal on $\mu(L) = L$.

$2^\circ \Rightarrow 5^\circ$. Let μ be a lattice morphism on L for which $F \equiv \min f \circ \mu^{-1}$ exists o Suppose $t, \tau \in \mu(L)$ are incomparable and $F(t) < F(t \vee \tau)$. Now there exist incomp $s \in \mu^{-1}(t)$ and $\sigma \in \mu^{-1}(\tau)$ with $f(s) = F(t)$ and $f(\sigma) = F(\tau)$. Thus since $\mu(s \vee \sigma)$ $f(s) = F(t) < F(t \vee \tau) \leq f(s \vee \sigma)$. Hence, by $\mu(s \wedge \sigma) = t \wedge \tau$, 2° and Table 3, $F(t$ $f(s \wedge \sigma) < f(\sigma) = F(\tau)$.

$6^\circ \Leftrightarrow 2^\circ$. Immediate from Table 3 and $1^\circ \Leftrightarrow 2^\circ$ of Theorem 1. ▮

The next result follows from Theorem 2 by duality.

Theorem 3. Characterization of Meet-Subextremal Functions. *The following prope of an extended real-valued function f on a lattice L are equivalent.*

1° *f is meet-subextremal.*

2° *f has meet-downcrossing differences.*

3° $\operatorname{argmin} f(\{s, s \wedge \sigma\}) \leq_\wedge \operatorname{argmin} f(\{\sigma, s \vee \sigma\})$ *for all $s, \sigma \in L$.*

4° $\operatorname{argmin} f(S) \leq_\wedge \operatorname{argmin} f(\Sigma)$ *for each $S \leq_l \Sigma$ in $\mathfrak{L}_L$.*

5° $\min f \circ \mu^{-1}$ *is meet-subextremal on $\mu(L)$ for each lattice morphism μ on L f which* $\min f \circ \mu^{-1}$ *exists on $\mu(L)$.*

6° *f is subextremal and $f(s) = f(s \vee \sigma)$ implies $f(s \wedge \sigma) \leq f(\sigma)$ for all $s, \sigma \in L$.*

Theorem 4. Characterization of Lattice-Subextremal Functions. *The following prop of an extended real-valued function f on a lattice L are equivalent.*

1° *f is lattice-subextremal.*

$1'$ *f is join- and meet-subextremal.*

2° *f has interval-downcrossing differences.*

3° $\operatorname{argmin} f(\{s, s \wedge \sigma\}) \leq_l \operatorname{argmin} f(\{\sigma, s \vee \sigma\})$ *for all $s, \sigma \in L$.*

4° $\operatorname{argmin} f(S) \leq_l \operatorname{argmin} f(\Sigma)$ *for each $S \leq_l \Sigma$ in $\mathfrak{L}_L$.*

5° $\min f \circ \mu^{-1}$ *is lattice-subextremal on $\mu(L)$ for each lattice morphism μ on L f which* $\min f \circ \mu^{-1}$ *exists on $\mu(L)$.*

6° *f is subextremal, $f(s) = f(s \wedge \sigma)$ implies $f(s \vee \sigma) \leq f(\sigma)$, and $f(s) = f(s \vee \sigma)$ i plies $f(s \wedge \sigma) \leq f(\sigma)$ for all $s, \sigma \in L$.*

Proof. Condition 1° holds if and only if 2° of both Theorems 2 and 3 hold. This fo exactly as in the proof that $1^\circ \Leftrightarrow 2^\circ$ in Theorem 1 on replacing $f(\sigma)$ by $f(\sigma)-$ an $\{s, \sigma\}^{\wedge\vee}$" by "at s, σ." Also, $1'$ (resp., 2°, 3°, 4°, 5°, 6°) holds if and only if 1° (resp., 2 °, 5°, 6°) of Theorems 2 and 3 both hold. ∎

Theorem 5. Characterization of Strictly Subextremal Functions. *The following prop s of an extended real-valued function f on a lattice L are equivalent.*

1° *f is strictly subextremal.*

2° *f has strictly downcrossing differences.*

3° $\operatorname{argmin} f(\{s, s \wedge \sigma\}) \leq_c \operatorname{argmin} f(\{\sigma, s \vee \sigma\})$ *for all (resp., all incomparable) $s, \sigma \in L$*

$3'$ $\operatorname{argmin} f(\{s, s \wedge \sigma\}) \leq \operatorname{argmin} f(\{\sigma, s \vee \sigma\})$ *for all incomparable $s, \sigma \in L$.*

4° $\operatorname{argmin} f(S) \leq_c \operatorname{argmin} f(\Sigma)$ *for each $S \leq_l \Sigma$ in $\mathfrak{L}_L$.*

5° $\min f \circ \mu^{-1}$ *is strictly subextremal on $\mu(L)$ for each lattice morphism μ on L fo which* $\min f \circ \mu^{-1}$ *exists on $\mu(L)$.*

6° *f is subextremal, $f(s) = f(s \wedge \sigma)$ implies $f(s \vee \sigma) < f(\sigma)$, and $f(s) = f(s \vee \sigma)$ im plies $f(s \wedge \sigma) < f(\sigma)$ for all incomparable $s, \sigma \in L$.*

Proof. $1^\circ \Leftrightarrow 2^\circ$. Same as the proof that $1^\circ \Leftrightarrow 2^\circ$ in Theorem 1 on replacing $f(s)$ and $f(\sigma)$ by $f(s)-$ and $f(\sigma)-$ respectively.

3°. The statements of 3° reading with and without parentheses are equivalent since the claimed inequality is trivially true when $s, \sigma \in L$ are comparable.

$2^\circ \Leftrightarrow 3^\circ$. From Table 3, 2° is false if and only if $f(s \wedge \sigma) \geq f(s)$ and $f(s \vee \sigma) \geq f(\sigma)$ for some incomparable $s, \sigma \in L$, or equivalently 3° is false.

$3^\circ \Leftrightarrow 3' \Leftrightarrow 4^\circ$. Suppose 4° holds and $s, \sigma \in L$ are incomparable. Set $S = \{s, s \wedge \sigma\}$ and $\Sigma = \{\sigma, s \vee \sigma\}$. Then from 4°, either $s \notin \operatorname{argmin} f(\{s, s \wedge \sigma\})$ or $\sigma \notin \operatorname{argmin} f(\{\sigma, s \vee \sigma\})$, whence $3'$ holds. Also, $3'$ implies 3° because $\leq$ is stronger than $\leq_c$. Finally, suppose 3° holds, $S \leq_l \Sigma$ in $\mathfrak{L}_L$, $s \in \operatorname{argmin} f(S)$ and $\sigma \in \operatorname{argmin} f(\Sigma)$. Then since 3° implies 3° and hence 4° of Theorem 4, $s \wedge \sigma \in \operatorname{argmin} f(S)$ and $s \vee \sigma \in \operatorname{argmin} f(\Sigma)$. Now s and σ are comparable, for if not, it follows from 3°, $s \in \operatorname{argmin} f(\{s, s \wedge \sigma\})$ and $\sigma \in \operatorname{argmin} f(\{\sigma, s \vee \sigma\})$ that $s \leq \sigma$, which is a contradiction.

$5^\circ \Rightarrow 1^\circ$. Same as the proof that $5^\circ \Rightarrow 1^\circ$ in Theorem 2 on replacing "join" by "strictly."

$2^\circ \Rightarrow 5^\circ$. Same as the proof that $2^\circ \Rightarrow 5^\circ$ in Theorem 2 on replacing the inequality $F(t) < F(t \vee \tau)$ by $F(t) \leq F(t \vee \tau)$.

$6^\circ \Leftrightarrow 2^\circ$. Immediate from Table 3 and $6^\circ \Leftrightarrow 2^\circ$ of Theorem 4. ∎

A *subextremal-variant class* of functions on a lattice is one of the following five classes: the subextremal, join-subextremal, meet-subextremal, lattice-subextremal or strictly subextremal functions on the lattice. The next result asserts that each member of a subextremal-variant class of functions on a lattice expresses ordinal rather than cardinal preferences among the elements of L.

Corollary 6. *Each subextremal-variant class of functions on a lattice is closed under strictly increasing transformations.*

Proof. Immediate from Table 3 and 2° of Theorems 1-5 since an inequality between the values of a function f at two points in its domain holds if and only if that is so of $g \circ f$ whenever f is in one of the subextremal-variant classes on a lattice and g is a strictly increasing function from the extended reals to itself. ∎

Corollary 7. *The subextremal class of functions on a lattice is closed under increasing transformations and pointwise limits.*

Proof. For the first part, suppose f is subextremal on a lattice L and g is an increasing function from the extended reals to itself. Suppose $s, \sigma \in L$ and $g \circ f(s) < g \circ f(s \vee \sigma)$.

Since g is increasing $f(s) < f(s \vee \sigma)$. Hence by Table 3 and 2° of Theorem 1, $f(s \wedge \sigma) \leq f(\sigma)$, so because g is increasing, $g \circ f(s \wedge \sigma) \leq g \circ f(\sigma)$. This implies $g \circ f$ is subextremal by Table 3 and 2° of Theorem 1.

For the second part, suppose $\{f_n\}$ is a sequence of subextremal functions on L and $f_n \to f$ pointwise. Suppose $s, \sigma \in L$ and $f(s) < f(s \vee \sigma)$. Then $f_n(s) < f_n(s \vee \sigma)$ for all large enough n, so by Table 3 and 2° of Theorem 1, $f_n(s \wedge \sigma) \leq f_n(\sigma)$ for all large enough n. Letting $n \to \infty$ yields $f(s \wedge \sigma) \leq f(\sigma)$, so f is subextremal by Table 3 and 2° of Theorem 1. ∎

The assertions of Corollary 7 are false for the other four subextremal-variant classes of functions. This accounts for the fact the infimum in 5° of Theorem 1 must be replaced by a minimum in 5° of the other four Theorems.

Corollary 8. *The restrictions of the five subextremal-variant classes to one-to-one functions coincide.*

Proof. It follows from the 6° of Theorems 2-5 that if f is one-to-one, then f is subextremal if and only if it is strictly (resp., meet-, join-, lattice-) subextremal. ∎

Call $S \subseteq L$ a *quasisublattice*[9] of L if $s, \sigma \in S$ imply either $s \wedge \sigma \in S$ or $s \vee \sigma \in S$. Table 4 below summarizes the largest subset of 2^L on which each of the seven set relations $\preceq$ in Table 1 is reflexive by characterizing the $S \in 2^L$ for which $S \preceq S$. The next result follows from Table 4 on taking $S = \Sigma = L$ in 4° of each of Theorems 1-5.

Table 4. Characterization of Where Seven Set Relations are Reflexive

Relation $\preceq$	*Characterization of* $S \preceq S$ *with* $S \neq \emptyset$
$\leq_0$	S is a subset of L
$\leq_w$	S is a quasisublattice of L
$\leq_\vee$	S is a join-sublattice of L
$\leq_\wedge$	S is a meet-sublattice of L
$\leq_l$	S is a sublattice of L
$\leq_c$	S is a subchain of L
$\leq$	S is a singleton subset of L

Corollary 9. Characterization of Set of Minimizers. *The set of minimizers of a subextremal, join-subextremal, meet-subextremal, lattice-subextremal or strictly subextremal function on a lattice is respectively a quasisublattice, join-sublattice, meet-sublattice, sublattice or subchain.*

[9] Quasisublattices and quasisublattice functions were introduced and studied by the second author in 1976, c.f., [Ve87].

Let $\underline{\Re}$ be the set of extended real numbers.

Lemma 10. Subextremal are Quasisublattice Functions. *A subextremal function on a lattice is a quasisublattice function, i.e., has quasisublattice level sets.*[10]

Proof. Suppose f is subextremal on a lattice L, $t \in \underline{\Re}$, $\text{epi}_t f \equiv \{s \in L : f(s) \le t\}$, $s, \sigma \in \text{epi}_t f$. If (1) holds, then $f(s \vee \sigma) \vee f(s \wedge \sigma) \le f(s) \vee f(\sigma) \le t$, so $s \vee \sigma, s \wedge \sigma \in \text{epi}_t f$. If (2) holds, then $f(s \vee \sigma) \wedge f(s \wedge \sigma) \le f(s) \wedge f(\sigma) \le t$, so either $s \vee \sigma$ or $s \wedge \sigma$ is in $\text{epi}_t f$. ∎

Theorems 1-5 are coordinate-free formulations of problems in which decisions are not distinguished from parameters. In practice, there are often separate decision variables and parameters. We now explore the implications of Theorems 1-5 in this setting.

Subextremal Variants with Decisions and Parameters. Suppose D is a poset of *decisions*, P is a poset of exogenous *parameters* and $L \subseteq D \times P$ is the set of *feasible decision-parameter pairs*. Let $\pi_P L \equiv \{p \in P : (d,p) \in L \text{ for some } d \in D\}$ be the *projection* of L on P, i.e., the set of *feasible parameters*, and $L_p \equiv \{d \in D : (d,p) \in L\}$ be the *p-section* of L, i.e., the set of *feasible decisions* when the parameter is $p \in \pi_P L$. Let f be an extended real-valued *cost function* on L. Finally, let $F(p) \equiv \inf_{d \in L_p} f(d,p)$ be the *minimum-cost* and $L_p^0 \equiv \{d \in L_p : F(p) = f(d,p)\}$ be the *optimal decision set* when the parameter is $p \in \pi_P L$. The next result is a consequence of Theorems 1-5.

Corollary 11. Minimization of Subextremal Variants.[11] *If L is a sublattice of $D \times P$ and f is subextremal (resp., join-subextremal, meet-subextremal, lattice-subextremal, strictly subextremal) on L, then $L_p^0 \preceq L_q^0$ for each $p \le q$ in $\pi_P L$ for which $L_p^0 \ne \emptyset$ and $L_q^0 \ne \emptyset$ where $\preceq$ is $\le_w$ (resp., $\le_\vee$, $\le_\wedge$, $\le_l$, $\le_c$). Moreover, F is subextremal (resp., join-subextremal, meet-subextremal, lattice-subextremal, strictly subextremal) on $\pi_P L$ provided, except for subextremal functions, $L_p^0 \ne \emptyset$ for each $p \in \pi_P L$.*

Proof. For the first assertion, observe that since L is a sublattice of $D \times P$, then $S \equiv L_p \times \{p\} \le_l L_q \times \{q\} \equiv \Sigma$. Then by Theorem 1 (resp., 2, 3, 4, 5), $L_p^0 \times \{p\} = \text{argmin} f(S) \preceq \text{argmin} f(\Sigma) = L_q^0 \times \{q\}$, so $L_p^0 \preceq L_q^0$. For the second assertion, use the lattice morphism $\mu(d,p) = p$ from L onto $\pi_P L$, so $F = \inf f \circ \mu^{-1}$, and Theorem 1 (resp., 2, 3, 4, 5). ∎

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[10] As Christina Shannon has pointed out to us, not every quasisublattice function has downcrossing differences.

[11] A slightly stronger result can be obtained by using coordinate versions of Theorems 1-5. We leave the details to the reader.

Sub* Functions. In this section we study an important subclass of subextremal functions. The subclass of functions is called sub* and is defined in terms of a binary operation $*$ on a set T. The *identity* for $*$ is an element e of T such that $t*e=e*t=t$ for all $t\in T$. The *null element* for $*$ is an element o of T such that $t*o=o*t=o$ for all $t\in T$. If an identity (resp., null element) exists, it is unique.

Suppose $*$ is a binary operation on a chain T. Call $*$ *increasing* (resp., *strictly increasing, properly increasing*) if for each t,u,τ,v in T for which $(t,u)\le(\tau,v)$ (resp., $(t,u)<(\tau,v)$, $t<\tau$ and $u<v$), one has $t*u\le\tau*v$ (resp., $t*u<\tau*v$, $t*u<\tau*v$). Similarly, call $*$ *weakly increasing* if $t*u<\tau*u$ and $u*t<u*\tau$ for each $t<\tau$ in T for which u is not a null element of T. For example, $+$ is strictly increasing on $\Re$; $+$ is weakly, but not strictly, increasing on $\bar{\Re}\equiv\Re\cup\{\infty\}$; $\vee$ and $\wedge$ are properly, but not weakly, increasing on $\Re$; and a trivial (constant) binary operation like $t*\tau=0$ for all $t,\tau\in\Re$ is increasing, but not properly so. Nevertheless, a strictly increasing binary operation is weakly increasing; a weakly increasing one is properly increasing by Lemma 13 below; and a properly increasing one is increasing. The above examples show that these implications are not reversible.

Lemma 12. *If a weakly increasing binary operation on a chain has a null element, then it is the least or greatest element of the chain.*

Proof. Suppose $*$ is a weakly increasing binary operation on a chain T. If the assertion is false, there exist $t,\tau\in T$ such that $t<o<\tau$. Since $*$ is weakly increasing, $t*\tau<o*\tau=o=t*o<t*\tau$, which is a contradiction. ∎

Lemma 13. *A weakly increasing binary operation on a chain is properly increasing.*

Proof. Suppose $*$ is a weakly increasing binary operation on a chain T. Then for each $t<\tau$ and $u<v$ in T, $t*u\le\tau*v$. If t or v is the null element o, say, then $u<o<\tau$, which contradicts Lemma 12. Thus, neither t nor v is the null element. Hence $t*u<t*v<\tau*v$. ∎

Let L be a lattice, $T\subseteq\bar{\Re}$, $f:L\to T$ and $*$ be an increasing binary operation on T. Call f *sub** if

$$f(s\vee\sigma)*f(s\wedge\sigma)\le f(s)*f(\sigma)$$

for all $s, \sigma \in L$. If the inequality holds strictly for all incomparable $s, \sigma \in L$, call f *strictly sub**. Sub* functions were introduced by Veinott in 1969 and studied in [Ve69, Ve74, Ve87]. In particular, he noted that the sub* functions include the subadditive ($* = +$), submultiplicative ($* = \cdot$), subjoin ($* = \vee$) and submeet ($* = \wedge$) functions.

Sub* Functions that are Subextremal Variants. Sub* functions need not be subextremal. For example, if $t * \tau = 0$ on $\Re$, every real-valued function on a lattice is sub*, but not every such function is subextremal. On the other hand, mild strict monotonicity conditions on $*$ assure that sub* functions are indeed subextremal.

Lemma 14. Sub* Functions that are Subextremal Variants. *Suppose f is a function from a lattice to $T \subseteq \bar{\Re}$ and $*$ is a binary operation on T. If f is sub* and $*$ is properly (resp., strictly) increasing, then f is subextremal (resp., lattice subextremal). If f is strictly sub*, then f is strictly subextremal.*

Proof. If f is not subextremal, then there exist incomparable s, σ such that $f(s) < f(s \vee \sigma)$ and $f(\sigma) < f(s \wedge \sigma)$. If also $*$ is properly increasing, $f(s) * f(\sigma) < f(s \vee \sigma) * f(s \wedge \sigma)$, so f is not sub*. If f is not lattice-subextremal, then there exist incomparable s, σ such that $f(s) < f(s \vee \sigma)$ and $f(\sigma) \leq f(s \wedge \sigma)$. If $*$ is strictly increasing, then $f(s) * f(\sigma) < f(s \vee \sigma) * f(s \wedge \sigma)$, so f is not sub*. Finally, if f is not strictly lattice-subextremal, then there exist incomparable s, σ such that $f(s) \leq f(s \vee \sigma)$ and $f(\sigma) \leq f(s \wedge \sigma)$. If $*$ is increasing, then $f(s) * f(\sigma) \leq f(s \vee \sigma) * f(s \wedge \sigma)$, so f is not strictly sub*. ∎

These results are sharp as the following examples show. Let $f_1(s) = s_1 s_2$ and $f_2 = 1$ on $L = \{0,1\}^2$. Now multiplication is (vacuously) strictly increasing on $\{1\}$, but only properly so on $\{0,1\}$. Also, f_1 and f_2 are submultiplicative, but not strictly so, on L and its dual. But f_1 is not join-subextremal on L and not meet-subextremal on its dual, and f_2 is not strictly subextremal on L.

Lemma 14 together with Theorem 1 (resp., 4) give an alternate proof of the following two results of Veinott [Ve69, Ve74, Ve87]. If f is sub* on a lattice L and $*$ is properly (resp., strictly) increasing, then $\operatorname{argmin} f(S) \leq_w$ (resp., $\leq_l$) $\operatorname{argmin} f(\Sigma)$ for all $S \leq_l \Sigma$ in $\mathfrak{L}_L$. The special case of this result reading with parentheses and $*$ being $+$ on $\Re$ (whence $+$ is strictly increasing) is due to Topkis and Veinott [Ve65], [TV68].

Subadditivizable Functions. The most important sub* functions are the subadditive ones. This raises the question when a function is subadditivizable. Call a real-valued tion f on a lattice *subadditivizable* (resp., *strictly subadditivizable*) ... continuous strictly increasing extended real-valued function g on an interval in $\bar{\Re}$ containing the image of f such that the composite function $g \circ f$ is subadditive (resp., strictly subadditive). In this event, we say f is *subadditivizable* (resp., *strictly subadditivizable*) *by* g. The next result is immediate from Corollary 6 and Lemma 14.

Lemma 15. Subadditivizable Functions that are Subextremal Variants. *A real-valued subadditivizable (resp., strictly subadditivizable) function on a lattice is lattice-subextremal (resp., strictly subextremal).* ∎

By an example of Shannon [Sh90a, b], this result is easily seen to be sharp.

Sub∗ Functions that are Subadditivizable. In this section we explore when sub∗ functions are subadditivizable, and conversely. The main tools for so doing are some known representation theorems for binary operations on $\Re$. First, however, we note that if f is a function from a lattice to an interval T in $\bar{\Re}$ and if f is subadditivizable (resp., strictly subadditivizable) by g on T with $g(T)$ being closed under addition, then f is sub∗ with $t * \tau = g(t) + g(\tau)$ for all $t, \tau \in T$.

To develop more detailed results, we require a few definitions. Let $*$ be a binary operation on a set T. Recall $*$ is *associative* if $(t*u)*v = t*(u*v)$ for all $t,u,v \in T$; $*$ is *commutative* if $t*u = u*t$ for all $t,u \in T$; and $*$ is *medial* if $(t*u)*(v*w) = (t*v)*(u*w)$ for all $t,u,v,w \in T$. The last property is also known as *bisymmetry*. Finally, $*$ is *idempotent* if $t*t = t$ for all $t \in T$.

Call an interval in $\bar{\Re}$ *proper* if it has positive length. Observe that a proper interval T in $\Re$ is closed under addition if and only if T is one of the intervals $\Re$, (a,∞), $[a,\infty)$, $(-\infty,-a)$ or $(-\infty,-a]$ for some $a \geq 0$. If g is a continuous extended real-valued function on an interval T in $\bar{\Re}$, then $g(T)$ is an interval. If also g is strictly increasing, then T is proper if and only if $g(T)$ is proper.

If $*$ is a binary operation on a compact interval $[a,b]$ in $\Re$, call $*$ *triangular* if it is associative, commutative and weakly increasing, and if either a or b is an identity. In that event, it is known and easy to show that if b (resp., a) is the identity of $*$, then a (resp., b) is its null element. For example, multiplication is a triangular operation on $[0,1]$ with zero being its null element and one being its identity.

Recall now three representation theorems for binary operations due to Aczél and Ling. The first two results apply to associative and the third to medial operations. The ... particular in [Ac49; Ac87, p. 107], [Li65] and [Ac66, pp. 254-256].

Theorem 16. Aczél (1949). *Suppose T is a proper interval in $\Re$. Then $*$ is an associative, continuous and strictly increasing binary operation on T if and only if there exists a continuous, strictly increasing real-valued function g on T such that $g(T)$ is closed under addition and $t * \tau = g^{-1}(g(t) + g(\tau))$ for each $t, \tau \in T$.*

Theorem 17. Ling (1965). *Suppose T is a proper compact interval in $\Re$. Then $*$ is a triangular, continuous binary operation on T if and only if there exists a continuous, strictly increasing extended real-valued function g on T such that $g(T)$ is $[-\infty, 0]$ or $[0, \infty]$, and $t * \tau = g^{-1}(g(t) + g(\tau))$ for each $t, \tau \in T$.*

Theorem 18. Aczél (1966). *Suppose T is a proper interval in $\Re$. Then $*$ is a medial, commutative, idempotent, continuous and strictly increasing binary operation on T if and only if there exists a continuous, strictly increasing real-valued function g on T such that $t * \tau = g^{-1}(\frac{1}{2}g(t) + \frac{1}{2}g(\tau))$ for each $t, \tau \in T$.*

We now apply Theorems 16-18 to represent sub$*$ functions for the three classes of binary operations $*$ described therein as subadditivizable. We prove only the first result since the other proofs are similar.

Corollary 19. *Suppose L is a lattice, T is a proper interval in $\Re$ and $f: L \to T$. Then f is sub$*$ (resp., strictly sub$*$) on L where $*$ is an associative, continuous and strictly increasing binary operation on T if and only if f is subadditivizable (resp., strictly subadditivizable) by some g on T such that $g(T) \subseteq \Re$ is closed under addition.*

Proof. By Theorem 16, $*$ is an associative, continuous and strictly increasing binary operation on T if and only if there is a continuous, strictly increasing g on T such that $g(T) \subseteq \Re$ is closed under addition and $t * \tau = g^{-1}(g(t) + g(\tau))$ for each $t, \tau \in T$. Also, f is sub$*$ (resp., strictly sub$*$) if and only if $g \circ f$ is subadditive (resp., strictly subadditive). ∎

Corollary 20. *Suppose L is a lattice, T is a proper compact interval in $\Re$ and $f: L \to T$. Then f is sub$*$ (resp., strictly sub$*$) on L where $*$ is a triangular, continuous binary operation on T if and only if f is subadditivizable (resp., strictly subadditivizable) by some g on T such that $g(T)$ is $[-\infty, 0]$ or $[0, \infty]$.*

Corollary 21. *Suppose L is a lattice, T is a proper interval in $\Re$ and $f: L \to T$. Then f is sub$*$ (resp., strictly sub$*$) on L where $*$ is a medial, commutative, idempotent, con-*

uous and strictly increasing binary operation on T if and only if f is subadditivizable (resp., strictly subadditivizable) by some real-valued g on T.

Differentiable Subadditivizable Functions. In this section we explore when a twice-differentiable function f on an interval L in $\Re^n$ is subadditivizable (resp., strictly subadditivizable). The next result is well known, e.g., [TV68], [To78].

Lemma 22. *A twice-differentiable real-valued function f on an open interval L in $\Re^n$ is subadditive if and only if the off-diagonal elements of its Hessian matrix $\nabla^2 f$ are nonpositive on L. If the off-diagonal elements of $\nabla^2 f$ are negative on L, then f is strictly subadditive.*

This result immediately implies the following known [TV68, Ve69] sufficient condition for a twice-differentiable function to be subadditivizable.

Lemma 23. *If L and T are respectively open intervals in $\Re^n$ and $\Re$, $f: L \to T$ is twice-differentiable, $g: T \to \Re$ is twice-differentiable and strictly increasing, and the off-diagonal elements of $(g' \circ f)\nabla^2 f + (g'' \circ f)\nabla f \nabla f^{\mathrm{T}}$ are nonpositive (resp., negative) on L, then f is subadditivizable (resp., strictly subadditivizable).*

This result has several applications that we now explore. A real-valued function f on a lattice L is called *k-subadditive* (resp., *strictly k-subadditive*) if f is subadditive (resp., strictly subadditive) when $k=0$ or if $\operatorname{sgn}(k)e^{kf}$ is subadditive (resp., strictly subadditive) when $k \neq 0$. Thus, for $k \neq 0$, f is a (strictly) k-subadditive function if and only if it is (strictly) subadditivizable by $g(t)=\operatorname{sgn}(k)e^{kt}$. By Corollary 21, a (strictly) k-subadditive function is (strictly) sub$*$ with $* = *_k$ being defined by

$$t *_k \tau = \ln\left(\frac{e^{kt}+e^{k\tau}}{2}\right)^{\frac{1}{k}}, \text{ for all } t, \tau \in \Re,$$

which is known as the *exponential mean of order k*. Since $\lim_{k\downarrow-\infty} t *_k \tau = t \wedge \tau$, $\lim_{k\to 0} t *_k \tau = \dfrac{t+\tau}{2}$, and $\lim_{k\uparrow\infty} t *_k \tau = t \vee \tau$, the k-subadditive functions range from the submeet ($k=-\infty$) to the subjoin ($k=\infty$) functions passing through the subadditive functions ($k=0$) as k increases from $-\infty$ to ∞.

The next result characterizes twice differentiable k-subadditive functions. Its proof is an immediate consequence of Lemmas 22 and 23 and the fact that a k-subadditive function is subadditivizable by $g(t)=\operatorname{sgn}(k)e^{kt}$.

Lemma 24. *If f is a twice-differentiable real-valued function on an open interval L in $\Re^n$ and k is a real number, then f is k-subadditive if and only if the off-diagonal elements of the matrix $K = \nabla^2 f + k\nabla f \nabla f^{\mathrm{T}}$ are nonpositive on L. If the off-diagonal elements of K are negative on L, then f is strictly k-subadditive.*

We give two immediate consequences of this Lemma. Let H denote the square matrix whose diagonal elements are zero and whose off-diagonal elements are one.

Corollary 25. *If f is a twice-differentiable real-valued function on an open interval L in $\Re^n$, $\ell \in \Re$, $m > 0$, and the off-diagonal elements of $\nabla^2 f - \ell H$ and $\nabla f \nabla f^{\mathrm{T}} + mH$ are nonpositive on L, then f is strictly subadditivizable.*

Proof. If $\ell < 0$, then f is strictly subadditive by Lemma 22. Suppose $\ell \geq 0$. Since $m > 0$, there is a $k > 0$ so that $\ell - km < 0$. Then the hypotheses assure that the off-diagonal elements of $K - (\ell - km)H$ are nonpositive on L, whence the off-diagonal elements of K are negative. Thus by Lemma 24, f is strictly k-subadditive and hence strictly subadditivizable. ∎

Corollary 26. *If f is a twice continuously differentiable real-valued function on an open interval containing a compact interval L in $\Re^n$ and if the off-diagonal elements of $\nabla f \nabla f^{\mathrm{T}}$ are negative on L, then f is strictly subadditivizable on L.*

Proof. By the compactness of L, the continuity of $\nabla^2 f$ and ∇f on L, and the negativity of the off-diagonal elements of $\nabla f \nabla f^{\mathrm{T}}$ on L, $\nabla^2 f$ is bounded below and the off-diagonal elements of $\nabla f \nabla f^{\mathrm{T}}$ are bounded away from zero. Thus for large enough $k > 0$, the off-diagonal elements of K are negative on L. Consequently, by Lemma 24, f is strictly k-subadditive and thus strictly subadditivizable on L. ∎

The following example applies these two Corollaries and shows that a strictly superadditive function may be strictly subadditivizable.

Example. Let $D = [1, 10] \times [\frac{1}{2}, 1]$. Let $f(s_1, s_2) = -\frac{s_1}{s_2}$ on D. Then the mixed partial derivative of f is $s_2^{-2} > 0$ on D, so f is strictly superadditive. However, by Corollary 26, f is strictly subadditivizable. If we extend the domain of f to $D' = [1, \infty) \times [\frac{1}{2}, 1]$, the same conclusion follows by Corollary 25.

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