Re-formulation of the Solow economic growth model whit the Richards population growth law.

Elvio Accinelli[*] and Juan Gabriel Brida[†]

Abstract

In standard economic growth theory it is usually assumed that labor force follows exponential growth. That is not a realistic assumption. In this paper we introduce a generalized logistic equation (Richards law) that describes more accurately population growth. Then we analyze the neoclassical Solow model with growth of population following the Richards law, and compares it with the classical model with exponential growth. We show that with the Richards law, the intrinsic rate of population growth plays no role in determining long run equilibrium per worker level of capital. We also present the closed-form solution of the model when the production function is Cobb-Douglas and we analyze the stability of the model, contrasting its long run equilibrium with the steady state of the traditional model.

Keywords: Population growth; Solow's growth model; Richards equation.

1 Introduction

The main purpose of this short paper is to improve the classical Solow model of economic growth by modifying its population growth law. One of the usual characteristics of any standard economic growth model is the assumption that labour force L grows at a constant rate $n > 0$. In continuous time it is natural to define this growth rate as:

$$r = \frac{\dot{L}}{L} = \frac{\frac{\partial L}{\partial t}}{L} \tag{1}$$

which implies that the labour force growths exponentially and for any initial level L_0, at time t the level of the labour force is

$$L(t) = L_0 e^{rt} \tag{2}$$

---

[*] Universidad Autonoma Metropolitana, Posgrado en Ciencias Económicas; Mexico E-mail adress: elvio@correo.xoc.uam.mx ; Tel.: (5255) 55947392, Fax: (5255) 55947392

[†] School of Economics and Management - Free University of Bolzano, Italy. E-mail adress: JuanGabriel.Brida@unibz.it Tel.: +39 0471 013492, Fax: +39 0471 013 009

The simple exponential growth model can provide an adequate approximation to such growth *only* for the initial period because, growing exponentially, labour force approaches infinity when t goes to infinity, which is clearly unrealistic. The exponential model does not accommodate growth reductions due to competition for environmental resources such as food and habitat. Verhulst [1] considered that a stable population would have a saturation level characteristic; this limit for the population size is usually called the *carrying capacity* of the environment (in this paper denoted by L_∞) and forms a numerical upper bound on the growth size. With the world population now at 6 billion, and increasing by about 100 million per year, we have clearly entered a zone where we can see, and may well encounter, limits on the human carrying capacity of the Earth (see [2], [3] and [4] for a description and methods of estimating human carrying capacity). It is a very well known stylized fact that since the 1950s, population growth rate is decreasing and it is projected to decrease to 0 during the next six decades. This decrease is particularly relevant in the group of developed countries but is also observable on a global scale. The decrease in the rate of growth is predominantly due to the aging of the population and, consequently, a dramatic increase in the number of deaths. From 2030 to 2050, the world population would grow more slowly than ever before in its history(see [5]).

Then, as described by Maynard Smith [6], a more realistic law of growth of the labour force $L(t)$ must verify the following properties:

1. when population is small enough in proportion to environmental carrying capacity L_∞, then L grows at a constant rate $r > 0$.
2. when population is large enough in proportion to environmental carrying capacity L_∞, the economic resources become more scarce and this affect negatively growth of the population.
3. population growth rate is decreasing to 0.

In this paper we assume that labour force $L(t)$ verify all these properties. In particular, in section 2 we introduce the Richards law that is a very general equation verifying the previous conditions frequently used to describe and analyze population processes. Under this population growth law in section 3 we obtain a generalization of the Solow model. In section 4 we find the exact solution and analyze the stability of the model. Finally, in section 5 we present some concluding remarks.

2 The Richards population growth law

The logistic equation is one of the simplest realistic model of population dynamics verifying all properties introduced in the previous section. To incorporate the carrying capacity L_∞ on the growth size, Verhulst (1838)[1] introduced the

---

[1] The logistic equation was first discovered and proposed to model population growth by Verhulst. (See Schtickzelle [7])

logistic growth equation as an extension of the exponential model augmented by a multiplicative factor, $1 - \dfrac{L}{L_\infty}$, which represents the fractional deficiency of the current size from the saturation level L_∞:

$$\dot{L} = rL\left(1 - \frac{L}{L_\infty}\right) \tag{3}$$

The Verhulst logistic equation is also referred to in the literature as the Verhulst-Pearl equation after Verhulst, who first introduced the curve, and Pearl [8], who used the logistic equation to describe population growth in the United States from 1790 to 1920. The equation depends on two parameters: the upper asymptote L_∞ and the rate parameter r. The rate parameter determines the rate at which growth initially accelerates and the upper asymptote determines the long run population level. The curve is sigmoidal with inflection point at half the value of the upper asymptote, $\frac{L_\infty}{2}$. This places an undesirable restriction on the shape of the curve and clearly limits the generality of the model. Notwithstanding this limitation, the logistic growth equation continue to be widely and frequently used to describe population processes.[2] To avoid the constraint of the logistic equation of having a very particular inflection point, a restriction that is really inconvenient when we want to fit the equation to observations, many have chosen to use models with additional parameters that generalize the logistic equation.[3] The Richards equation continues to be the most popular of these more flexible growth equations generalizing the logistic law since it was first proposed by Richards [22] to extend the Verhulst logistic growth equation to fit empirical data, but motivated by theoretical arguments. Nelder [23] seems to be the first in developing methods for fitting Richards model to data. The Richards model, valued for its accuracy, has been employed more than any other theoretical models for population modelling [24].

Richards law is the solution of the initial value problem:

$$\left\{\begin{array}{c} \dot{L} = rL\left(1 - \left(\dfrac{L}{L_\infty}\right)^{\delta}\right) \\ L(0) = L_0 \end{array}\right. \tag{4}$$

where $r > 0$ is the intrinsic growth rate per capita, L_∞ is the carrying capacity, and δ is a positive real number. Thus, the growth $\dot{L}(t)$ at time t, is assumed to be proportional to the size $L(t)$ at time t multiplied by a saturating function.

---

[2] See [10] for a review on logistic models and [9] for a review on the use of the logistic curve to fit human populations. In [11] and [12] the classical Solow model was extended by replacing the exponential population growth by the logistic model and in this paper we introduce a further generalization of these exercises.

[3] According to Renshaw [13], most theoretical models of population are extensions of the logistic equation. See, for example, [14], [15], [16], [17] and [20]. In two recent survey papers Buis [18] and Tsoularis and Wallace [19] revisited the previous works on generalizations of the logistic growth function and outlined some of their respective properties and restrictions. See [21] for a brief historical review of applications in economics of the logistic curve and its generalizations to model population growth.

Note that for $\delta = 1$, Richards equation reduces to the Verhulst logistic equation[4]. The Richards equation has been popular for several reasons. It has an additional parameter δ, which is a shape parameter that can make the equation equivalent to the logistic and to other well known models like Grompertz and von Bertalanfy equations [19]. Varying the shape parameter allows the point of inflection of the curve to be at any value between 0 and L_∞ and this make it more flexible to be fitted to data.[5] An additional reason in its use instead of other curves may be its availability as one of the standard equations offered for curve-fitting by statistical software packages. Often it is reported as providing good fits to observations. The Richards equation always gave closer fits and more accurate estimates of the characteristics of other original sigmoid curves, it is flexible in describing various asymmetrical sigmoid patterns and its parameters are numerically stable in statistical estimation (see [27] and [28]).

Richards equation is a separable differential equation and has solution

$$L(t) = \frac{L_\infty}{\left[1 + e^{d-\delta rt}\right]^{\frac{1}{\delta}}} \tag{5}$$

where $d = \ln\left[\left(\frac{L_\infty}{L_0}\right)^\delta - 1\right]$ is a parameter that indirectly defines the value of t at which $L = \frac{L_\infty}{2}$.[6]

The three main properties of the Richards logistic growth are:

1. $\lim\limits_{t\to+\infty} L(t) = L_\infty$, the population will ultimately reach its carrying capacity.
2. The relative growth rate is:

$$n(t) = \frac{\dot{L}(t)}{L(t)} = rL(t)\left(1 - \left(\frac{L(t)}{L_\infty}\right)^\delta\right) = r\frac{L_0 L_\infty \left(L_\infty^\delta - L_0^\delta\right) e^{\delta rt}}{\left[L_0^\delta e^{\delta rt} + \left(L_\infty^\delta - L_0^\delta\right)\right]^{\frac{2}{\delta}}} \tag{6}$$

   When t is small, $n(t)$ is close to r and decreases monotonically to 0 as t tends to infinity. The maximum relative growth rate is given by

$$n(\hat{t}) = \frac{rL_\infty\delta}{(1+\delta)^{1+\frac{1}{\delta}}} \tag{7}$$

   and is attached at the inflection point $\hat{t} = L_\infty\left(\frac{1}{1+\delta}\right)^{\frac{1}{\delta}}$.
3. The growth curve is sigmoidal and the inflection point is at the proportion $(1+\delta)^{-\frac{1}{\delta}}$ of the final size

A detailed description of how well the Richards equation describes a real population can be found in [23] and [25]. For generalizations of the Richards equation the reader can consult [29], [27] and [28].

---

[4] For this reason, some authors adopt the term *generalized logistic equation* to describe the Richards equation. See, for example [23] and [26].

[5] See reference [26]

[6] See reference [35]

3 The modified Solow model

There are two key elements to the model:

- The production function, i.e. how the inputs of capital K and labour L are transformed into outputs.
- How the labour and capital change over time.

As usual, we shall assume that:

1. the production function $F(K, L)$ satisfy the following conditions:
    (a) $F(\lambda K, \lambda L) = \lambda F(K, L)$, $\forall \lambda, K, L \in R^+$ (constant return to scale)
    (b) $F(K, 0) = F(0, L) = 0$, $\forall K, L \in R^+$
    (c) $\frac{\partial F}{\partial K} > 0$, $\frac{\partial F}{\partial L} > 0$, $\frac{\partial^2 F}{\partial K^2} < 0$, $\frac{\partial^2 F}{\partial L^2} < 0$
    (d) $\lim\limits_{K \to 0} \frac{\partial F}{\partial K} = \lim\limits_{L \to 0} \frac{\partial F}{\partial L} = +\infty$; $\lim\limits_{K \to +\infty} \frac{\partial F}{\partial K} = \lim\limits_{L \to +\infty} \frac{\partial F}{\partial L} = 0$ (Inada conditions)
2. the capital stock changes equal the gross investment $I = sF(K, L)$ minus the capital depreciation δK :
$$\dot{K} = sF(K, L) - \delta K \tag{8}$$
3. the labour force $L(t)$ verify the following properties:
    (a) $L(0) = L_0 > 0, \dot{L}(t) < 0, \forall t \geq 0$ and $\lim\limits_{t \to +\infty} L(t) = L_\infty$ (population is strictly increasing and bounded)
    (b) If $n(t) = \frac{\dot{L}(t)}{L(t)}$ then $\dot{n}(t) < 0, \forall t \geq 0$ and $\lim\limits_{t \to +\infty} n(t) = 0$ (the rate of growth of population is strictly decreasing to zero)

The last assumption is the unique difference with the original Solow model as presented in [34].

If $k = \frac{K}{L}$ is the capital per worker then $f(k) = F\left(\frac{K}{L}, 1\right) = F(k, 1)$ is the production function in intensive form. We have that:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \tag{9}$$

and then

$$\frac{\dot{k}}{k} = \frac{sF(K, L) - \delta K}{K} - n(t) = s\frac{F\left(\frac{K}{L}, 1\right)}{\frac{K}{L}} - \delta - n(t). \tag{10}$$

From this, we obtain the equation of motion for the modified Solow model which describes how capital per worker varies over time:

$$\dot{k} = sf(k) - (\delta + n(t))\, k \tag{11}$$

Note that in the original Solow model where labour force grows exponentially it is $n(t) = n$ (constant) and the equation of the motion is $\dot{k} = sf(k) - (\delta + n)\, k$. In this case there is a non zero steady state $\hat{k}_1$ that is globally asymptotically stable. The value $\hat{k}_1$ is the unique positive solution of equation

$$sf(k) = (\delta + n)\, k. \tag{12}$$

See, for example, reference [33]. In contrast, in the modified model the equation is not autonomous and there is only one steady state at zero.

When the production function is of the Cobb-Douglas type:

$$F\,(K, L) = K^{\alpha} L^{1-\alpha}; 0 < \alpha < 1 \tag{13}$$

and the labour force grows follows the Richards law

$$\begin{cases} \dot{L} = rL\left(1 - \left(\dfrac{L}{L_{\infty}}\right)^{\delta}\right) \\ L(0) = L_0 \end{cases} \tag{14}$$

then the equation of motion for the modified Solow model, which describes how capital per worker varies over time is:

$$\dot{k} = sk^{\alpha} - (\delta + n(t))\, k \tag{15}$$

where $n\,(t)$ is given by (6).

4 Analysis of the model

Equation (15) is a Bernoulli type equation that can be transformed by the change of variables

$$u = k^{1-\alpha} \tag{16}$$

into the linear equation

$$\dot{u} = (1 - \alpha)\, [s - (\delta + n(t))\, u] \tag{17}$$

Recall that, given the continuous functions $a(t)$ and $b(t)$, the solutions of a linear differential equation:

$$\dot{x} = a(t)x + b(t) \tag{18}$$

are given by

$$x(t) = e^{A(t)}\left(x_0 + \int_0^t b(\tau) e^{-A(\tau)} d\tau\right) \tag{19}$$

where

$$A(t) = \int_0^t a(\tau) d\tau \tag{20}$$

and $x_0 = x(0)$ is the initial condition.[7] Note that the difference between two different solutions with initial conditions x_0 and $x_1$is given by

$$|x_0 - x_1|\, e^{A(t)} \tag{21}$$

Then, a solution of the linear equation 18 is stable if and only if the function $A(t)$ is bounded from above in $[0, +\infty)$. If, in addition, it is

$$\lim_{t\to+\infty} A(t) = -\infty \tag{22}$$

then the solutions are globally asymptotically stable. Note also that the solutions of (18) have an horizontal asymptote if there exists the limit:

$$\lim_{t\to+\infty} \frac{b(t)}{a(t)} = x_\infty \tag{23}$$

and, in such case, we have:

$$\lim_{t\to+\infty} x(t) = -x_\infty \tag{24}$$

In fact,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \frac{x_0 + \int_0^t b(\tau)e^{-A(\tau)}d\tau}{e^{-A(t)}} = \lim_{t\to+\infty} \frac{b(t)}{-a(t)} = -x_\infty \tag{25}$$

Now we can apply these remarks to equation (17). In this case we have:

$$\begin{array}{rcl} a(t) & = & -\left(1-\alpha\right)\left(\delta + n(t)\right) \\ b\left(t\right) & = & \left(1-\alpha\right)s \end{array} \tag{26}$$

and then it is

$$A(t) = -\left(1-\alpha\right)\left[\delta t - \int_0^t n\left(\tau\right)d\tau\right] \tag{27}$$

This function is bounded from above in $[0, +\infty)$ and tends to $-\infty$ as $t \to +\infty$, implying that their solutions are globally asymptotically stable.[8] Finally, we have that

$$\lim_{t\to+\infty} \frac{b(t)}{a(t)} = \lim_{t\to+\infty} \frac{\left(1-\alpha\right)s}{-\left(1-\alpha\right)\left(\delta - n\left(t\right)\right)} = -\frac{s}{\delta}$$

and then all the solutions have the horizontal asymptote $x = \frac{s}{\delta}$ as $t \to +\infty$. The change of variables $k = u^{\frac{1}{1-\alpha}}$ transforming solutions of equation (15) into solutions of (17) is continuous. Then the solutions of (15) are globally asymptotically stable and tend to the long run limit value $\left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$ as $t \to +\infty$. This

---

[7] Then, as recently pointed out by Barro and Sala-i-Martin [31] for the classical Solow model with Cobb-Douglas technology, also the modified model has a closed form solution. See also Irmen [32].

[8] Recall that the improper integral $\int_0^{+\infty} n\left(\tau\right)d\tau$ is finite.

limit value $\hat{k}_2 = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$ it is not a steady state, because it is not a solution of equation (15), but is the long run value of the per worker level of capital k. Note that this value do not depend on the intrinsic rate of population growth $n(t)$. Being the model asymptotically stable, small changes on the initial condition does not affect the long run economic performance. For any initial condition, capital per worker converges to the value $\hat{k}_2$.

If we contrast the long run levels of per worker level of capital $\hat{k}_1$ (exponential model) and $\hat{k}_2$ (logistic model),we can note that:

$$\hat{k}_2 = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}} > \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-\alpha}} = \hat{k}_1 \tag{28}$$

That is, if population growth decreases following the logistic law instead of growing exponentially, long run economic growth (represented by per capita levels of capital and output) is improved. This is also true for the long run levels of per worker level of output. Finally, let as remark that while in the classical Solow model the aggregate output $Y = F(K, L) = K^\alpha L^{1-\alpha} = Lk^\alpha$ goes unrealistically to infinity as $t \to +\infty$, in the model with logistic population growth this limit is finite, being labour force $L(t)$ convergent to L_∞ and per worker capital k convergent to $\hat{k}_2$.

5 Concluding remarks.

In growth theory it is usually assumed that population growth follows an exponential law. This is clearly unrealistic because, in particular, it implies that population goes to infinity when time goes to infinity. In this paper we have developed an improved version of the Solow growth model suggesting a more realistic approach by considering that population growth is strictly increasing and bounded and that its rate of growth is strictly decreasing to zero. In particular we use the Richards equation to model population growth and we integrate the equation of the motion in closed form. We conclude with the stability analysis of the model by finding a long run value (that is not a steady state) that is the limit of any solution when time tends to infinity. Then the process described by the model is asympotically stable: small changes on the initial condition of the growth process do not affect the long run performance. The paper shows that when population verifies the more realistic conditions, the intrinsic rate of population growth $n(t)$ plays no role in determining the long run equilibrium levels of per capita consumption, capital and output, while with exponential population growth an increase in the intrinsic rate of population growth leads to lower levels of these variables. It also shows that equilibrium per capita levels of consumption, capital and output are greater than those of the classical model. Thus, in the long run, economic growth is improved if labour force growth rate decreases. This is a motivation for policy makers to have an efficient population growth rate.

References

[1] Verhulst, P. F., Notice sur la loi que la population pursuit dans son accroissement. *Corresp. Math. Phys.* 10:113-121, (1838).

[2] Cohen, J.E., *How Many People Can the Earth Support?* Norton, New York, NY (1995)

[3] Cohen, J.E., Population Growth and Earth's Human Carrying Capacity, *Science* 269: 341-346, (1995).

[4] Arrow, K., B. Bolin, R. Costanza, P. Dasgupta, C. Folke, C. S. Holling, B.O. Jansson, S. Levin, K.G. MŠler, C. Perrings, D. Pimentel, Economic Growth, Carrying Capacity, and the Environment*, Science* 268: 520-521, (1995).

[5] Day, J. C., *Population Projections of the United States by Age, Sex, Race, and Hispanic Origin: 1995 to 2050*, U. S. Bureau of the Census, Current Population Reports, U.S. Government Printing Office, Washington D.C., 25, (1996).

[6] Maynard Smith, J., *Models in Ecology*, Cambridge University Press: Cambridge, (1974).

[7] Schtickzelle M., Pierre-François Verhulst (1804-1849). La première découverte de la function logistique*, Population*, Vol. 3, 541-556, (1981).

[8] Pearl, R. and L.J. Reed, On the rate of growth of the population of the United States since 1790 and its mathematical representation. *Proceedings of the National Academy of Sciences* 6: 275-288, (1920).

[9] Leach, D., Re-Evaluation of the Logistic Curve for Human Populations, *Journal of the Royal Statistical Society,* Series A (General), Vol. 144, No. 1, 94-103, (1981).

[10] Coke, K.L. and M. Witten, One-dimensional linear and logistic harvesting models, *Mathematical Modelling*, **7**, 301, (1986).

[11] Mingari Scarpello, G. and D. Ritelli, The Solow model improved through the logistic manpower growth law, *Annali Università di Ferrara -Sez VII -Sc. Mat.* 73, (2003).

[12] Brida, J.G., G. Mingari Scarpello and D. Ritelli, The Solow model with logistic manpower: a stability analysis, *SSRN Electronic Paper Collection WP* 785665 (http://ssrn.com/abstract=785665), (2005).

[13] Renshaw, E., *Modelling biological populations in space and time.* Cambridge University Press, (1991).

[14] Tabatabai,M., D. K. Williams and Z. Bursac, Hyperbolastic growth models: theory and application*, Theoretical Biology and Medical Modelling*, 2:14 (2005)

[15] Oliver, F. R., Another Generalisation of the Logistic Growth Function, *Econometrica*, Vol. 37, No. 1, 144-147, (1969).

[16] Chaddha, R. L. and S. S. Chitgopekar, A Generalization of the Logistic Curves and Long-Range Forecasts (1966- 1991) of Residence Telephones *The Bell Journal of Economics and Management Science*, Vol. 2, No. 2., 542-560, (1971).

[17] Turner, M., E. Bradley, K. Kirk, K. Pruitt, A theory of growth*, Mathematical Biosciences*, 29, 367-373, (1976).

[18] Buis, R., Sur l'interprétation de la loi logistique de croissance: Une relecture de la relation entre autocatalyse et croissance, *Acta Biotheoretica*, 45, 251-266, (1997).

[19] Tsoularis A and J. Wallace, Analysis of logistic growth models*, Mathematical Biosciences*, 179 (1): 21-55, (2002).

[20] Zwanzig , R., Generalized Verhulst Laws for Population Growth *, Proceedings of the National Academy of Sciences of the United States of America*, Vol. 70, No. 11., 3048-3051, (1973).

[21] Jarne, G., J. Sánchez-Chóliz and F. Fatás-Villafranca, "S-shaped" Economic Dynamics. The Logistic and Gompertz curves generalized, *The Electronic Journal of Evolutionary Modeling and Economic Dynamics*, 1048, http://beagle.u-bordeaux4.fr/jemed/1048/index.php, (2005).

[22] Richards, F. J., A flexible growth function for empirical use. *J. Exp. Botany* 10:290-300, (1959).

[23] Nelder, J. A., The fitting of a generalization of the logistic curve. *Biometrics* , 17: 89-110,(1961).

[24] Zeide B, Analysis of growth equations, *Forest Science*, 39:594-616, (1993).

[25] Turner, M., B. Blumenstein and J. Sebaugh, A generalization of the logistic law of growth*, Biometrics* 25: 577-580, (1969).

[26] Thomas, W. R., Pomerantz, M. J., and Gilpin, M. E., Chaos, asymmetric growth and group selection for dynamical stability. *Ecology* , 61:1312-1320, (1980).

[27] Birch, C.P.D., A New Generalized Logistic Sigmoid Growth Equation Compared with the Richards Growth Equation, *Annals of Botany,* 83: 713-723, (1999).

[28] Yin, X., J. Goudriaan, E. A. Lantinga, J. Vos, and H. J. Spiertz, A Flexible Sigmoid Function of Determinate Growth, *Annals of Botany*, 91: 361 -371, (2003).

[29] Blumberg, A. A., Logistic growth rate functions, *Journal of Theoretical Biology*, 21: 42-44, (1968).

[30] Donghan, C. An Improved Solow-Swan Model, *Chinese Quarterly Journal of Mathematics*, Vol.13, No.2, 72-78, (1998).

[31] R. J. Barro and X. Sala-i-Martin. *Economic Growth*, Cambridge University Press, (2004).

[32] Irmen, A., Malthus and Solow: a note on closed-form solutions, *Economic Bulletin*, **10**, 1 (2004).

[33] Simonovits, A., *Mathematical methods in dynamical economics*, MacMillan Press, (2000).

[34] Solow, R. M., A Contribution to the Theory of Economic Growth, *Quarterly Journal of Economics*, **70**, 65, (1956).

[35] Waltman, P.E., *A second course in elementary differential equations*, Dover Publications, INC., (2004).