

Chaotic expansion of powers and martingale representation

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Abstract: This paper extends a recent martingale representation result of [N-S] for a Lévy process to filtrations generated by a rather large class of semimartingales. As in [N-S], we assume the underlying processes have moments of all orders, but here we allow angle brackets to be stochastic. Following their approach, including a chaotic expansion, and incorporating an idea of strong orthogonalization from [D], we show that the stable subspace generated by Teugels martingales is dense in the space of square-integrable martingales, yielding the representation. While discontinuities are of primary interest here, the special case of a (possibly infinite-dimensional) Brownian filtration is an easy consequence.

1. INTRODUCTION

Recently, [N-S] established a martingale representation property for the filtration generated by a Lévy process $X = (X_t)$ having an exponentially decaying law. They showed that every square-martingale $M \in \mathcal{H}^2$ has a representation as an infinite sum of the form $M = \sum_{n=1}^{\infty} \int H_n dN_n$ for certain pairwise strongly orthogonal martingales N_n .² The series convergence takes place in \mathcal{H}^2 . The base martingales N_n are intrinsically associated to X , and, in their case, on a choice of an orthogonal polynomial. The result is an interesting contrast to the standard theory for filtrations generated by a finite-dimensional Brownian motion or Poisson process, where martingale representation takes the form of a finite sum. It is tantamount to the decomposition $\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i)$ of the Hilbert space \mathcal{H}^2 into the infinite orthogonal direct sum of the stable subspaces $\mathcal{S}(N_i)$ generated by N_i (consisting of all stochastic integrals $\int H dN_i \in \mathcal{H}^2$), rather than a finite direct sum as customary.

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²In this paper we use integer powers X^n frequently. To avoid confusion, we use subscripts to denote sequences of processes, such as H_n . If necessary, the time t -value is then denoted $H_n(t)$. (In the univariate case, we use the usual notation X_t .)

Lévy processes are very interesting, but the concepts and techniques introduced in [N-S] appear of wider applicability. Chief among them are their notion of *Teugels Martingales* $X^{(n)}$, whose strong orthogonalization gives the base martingales N_n , a chaotic representation of n -th power X^n in terms of $X^{(i)}$, and the idea that polynomials in X_{t_j} are dense in the space of square integrable random variables, given a suitable growth condition on X .

In a recent expository article, [D] discusses several approaches and results on martingale representation, including [N-S], those based on the Jacod-Yor Theorem, and an earlier general result in [D2] (and other cited references) for the filtration generated by a finite activity process. It appears that the [N-S] result is the first of its kind (Azéma martingales excepted) for an infinite activity process, let alone a discontinuous process of infinite variation, which Lévy processes often are. In connection with [N-S], [D] highlights a general strong orthogonalization procedure well-suited to our purposes as an alternative to the orthogonal polynomial method that [N-S] use to orthogonalize the Teugels martingales.

Our aim is to generalize the [N-S] results in two directions. First, we extend to processes X quite a bit more general than Lévy processes. These processes and their (generalized) Lévy measures $\nu = \nu(\omega, dt, dx)$ have moments of all order. Aside from stringent growth conditions, the main assumption is that $x^n * \nu$ be continuous and adapted to a Brownian filtration for all integers $n \geq 2$. In the Lévy case, $x^n * \nu$ is a constant times t . A more general example is a “Lévy processes with stochastic intensity λ_t ”, where ν takes the form $\nu = \lambda_t dt v(dx)$ for some ordinary Lévy measure v and a nonnegative Ito process (λ_t) . Here, simply, $x^n * \nu = a_n \int_0^\cdot \lambda_t dt$, where $a_n = \int x^n v(dx)$; so $x^n * \nu$ are stochastic but proportional.

Secondly, we extend the univariate treatment of [N-S] to the multivariate case, indeed to the case where the underlying filtration is generated by a countable number of independent processes X_n of the above general type. The [N-S] approach to representation as a convergent series in \mathcal{H}^2 is ideal for this purpose. Such an extension is already of interest when the processes X_n are independent Brownian motions, extending the standard finite-dimensional result to yield a unique representation for every martingale $M \in \mathcal{H}^2$ as $M = \sum_{n=1}^{\infty} \int H_n dX_n$ for some predictable processes H_n satisfying $\sum_{n=1}^{\infty} \mathbb{E} \int_0^T |H_n(t)|^2 dt < \infty$.

With regard to the standard finite-dimensional Brownian case, as derived in texts such as [E], [K-S], [Ø], and [P], [D] remarks that the approach of [Ø] appears the simplest. For the Brownian case, the Teugels martingales vanish, substantially simplifying the technique of [N-S]. In this case, the derivation in [N-S] becomes actually quite similar to that of [Ø]: both are based on denseness arguments, the former utilizing integer powers X^n and polynomials, the latter employing complex powers $e^{i\xi X}$ and the Fourier integral. It seems to us that, for the Brownian case, the technique of [N-S] is as simple, but more constructive.

We follow closely the approach and ideas of [N-S], aided also by an elaboration on strong orthogonalization in [D]. The more general development here calls for a somewhat different route at places, and furthering of some of the arguments and calculations in [N-S]. For instance, the chaotic expansions here involve multiple stochastic integrals with in general *stochastic integrands* - only in the (non-homogenous) Levy case (where ν is deterministic, i.e., a Poisson measure) are these integrands deterministic as in [N-S] and Wiener chaos.

The next section establishes notation, culminating in definitions of “power brackets” $[X]^{(n)}$ and $\langle X \rangle^{(n)}$, and the Teugels martingales $X^{(n)} := [X]^{(n)} - \langle X \rangle^{(n)} = x^n * (\mu - \nu)$.³ Section 3 sets forth the strategy, based on strong orthogonalization and a decomposition of \mathcal{H}^2 into an infinite orthogonal sum of stable subspaces, given a denseness hypothesis. Section 4 establishes some technical results based on the Burkholder-Davis-Gundy inequalities to ensure that various local martingales that later arise in the chaotic expansion as (iterated) stochastic integrals of Teugels martingales are in fact square-integrable martingales. Section 5 derives the needed L^2 denseness of polynomials for processes with an exponentially decreasing law. Section 6 presents an inductive chaotic expansion which basically shows (stopped) polynomials have representations as a sum of stochastic integrals of $X^{(i)}$ times functionals of the $\langle X \rangle^{(j)}$. These are put together in Section 7 to state and prove our main results. Section 8 is not needed for the main results, rather, by presenting an explicit chaotic expansion of powers X^n , it brings out the relevance of power brackets and provides motivation for the inductive definitions in Section 6. A final section concludes the paper.

2. NOTATION AND BASIC CONCEPTS

The notation below is for the most part standard, but we introduce some new ones too.

2.1. Stochastic basis. We fix throughout $0 < T \leq \infty$ and a complete right-continuous filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ such that $\mathcal{F} = \mathcal{F}_T$.

We denote by $\mathbb{F}(X_n)_{n=1}^k$ the completed filtration generated by a finite or infinite sequence $(X_n)_{n=1}^k$, $1 \leq k \leq \infty$, of measurable processes X_n .

Let L^0 denote the set of \mathcal{F} -measurable real-valued functions on Ω . For $p > 0$, we denote

$$L^p := L^p(\Omega, \mathcal{F}, \mathbb{P}) := \{\xi \in L^0 : \mathbb{E}|\xi|^p < \infty\}.$$

Of interest will be L^1 , L^2 , and random variables of finite moments, $L^* := \bigcap_{n=1}^{\infty} L^n$.

We denote by \mathcal{M} the set of uniformly integrable martingales $M = (M_t)_{t \in [0, T]}$ with $M_0 = 0$. Note, $M \in \mathcal{M}$ is closed by M_T . As is well-known, as $t \rightarrow T_-$, M_t converges to $\mathbb{E}(M_T | \bigvee_{0 \leq t < T} \mathcal{F}_t)$ a.s. and in L^1 . The localization of \mathcal{M} is denoted \mathcal{M}_{loc} .

2.2. Semimartingales. Let \mathcal{P} denote the set of predictable processes $H = (H_t)_{t \geq 0}$.⁴

If $X = (X_t)_{t \geq 0}$ is a semimartingale, its quadratic variation $[X, X]$ is abbreviated to $[X]$.⁵ And if H is a predictable X -integrable process, we denote the stochastic integral by

$$\int H dX := H \cdot X = \left(\int_0^t H_s dX_s \right)_{t \geq 0}.$$

³We use a different notation than [N-S]. Their equivalent of our $[X]^{(n)}$, $\langle X \rangle^{(n)}$, $X^{(n)}$ is $X^{(n)}$, $m_n t$, $Y^{(n)}$.

⁴By “ $t \geq 0$ ” we mean $t \in [0, T]$ if $T < \infty$ and $t \in [0, \infty)$ if $T = \infty$.

⁵One has, $[X] = [X^c] + \sum_{s \leq \cdot} (\Delta X)_s^2$, where X^c denotes the unique continuous local martingale such that $X_0^c = 0$ and $X - X^c$ is a purely discontinuous semimartingale. Also, $[X^c] = \langle X^c \rangle = [X]^c$. Here $[X]^c$, in contrast to X^c , stands for continuous finite-variation (not martingale) part of $[X]$. A similar notational ambiguity is to be tolerated for the symbols $X^{(n)}$ and $[X]^{(n)}$ introduced below.

Let \mathcal{A}^+ denote the set of adapted right-continuous increasing processes $A = (A_t)_{t \in [0, T]}$ such that $A_0 = 0$ and $A_T \in L^1$. Let $\mathcal{A} := \mathcal{A}^+ \ominus \mathcal{A}^+$ denote the set of adapted right-continuous processes of integrable variation. So, every $A \in \mathcal{A}$ has a unique decomposition $A = B - C$ for some $B, C \in \mathcal{A}^+$. Its total variation, denoted $\text{Var}(A)$, then equals $B + C$.

As is well-known, every $A \in \mathcal{A}$ has a unique Doob-Meyer decomposition $A = \widehat{A} + M$ with $\widehat{A} \in \mathcal{P} \cap \mathcal{A}$ and $M \in \mathcal{M}$. The compensator \widehat{A} is increasing if A is so.

2.3. Square-integrable martingales. As customary, we denote this Hilbert space by

$$\mathcal{H}^2 := \{M \in \mathcal{M} : M_T \in L^2\} = \{M \in \mathcal{M}_{\text{loc}} : [M]_T \in L^1\}.$$

Let $M, N \in \mathcal{H}^2$. The compensators of $[M]$ and M^2 coincide, and is denoted $\langle M \rangle$.⁶ We have $M^2 - [M]$, $[M] - \langle M \rangle \in \mathcal{M}$. One sets $\langle M, N \rangle := (\langle M + N \rangle - \langle M - N \rangle)/4$. (So, $\langle M, M \rangle = \langle M \rangle$.) The space \mathcal{H}^2 is endowed with the Hilbert norm⁷

$$\|M\|^2 := \mathbb{E} M_T^2 = \mathbb{E} [M]_T = \mathbb{E} \langle M \rangle_T. \quad (M \in \mathcal{H}^2)$$

Note, L^2 is isometric to $L^2(\Omega, \mathcal{F}_0, \mathbb{P}) \oplus \mathcal{H}^2$.

2.4. Infinite direct sum of strongly orthogonal stable subspaces. For $N \in \mathcal{H}^2$, set

$$L^2\langle N \rangle := \{H \in \mathcal{P} : \mathbb{E} \int_0^T H^2 d[N] < \infty\} = \{H \in \mathcal{P} : \mathbb{E} \int_0^T H^2 d\langle N \rangle < \infty\}.$$

Any $H \in L^2\langle N \rangle$ is N -integrable, $\int H dN \in \mathcal{H}^2$, and $\langle \int H dN \rangle = \int H^2 d\langle N \rangle$. Denote

$$\mathcal{S}(N) := \left\{ \int H dN : H \in L^2\langle N \rangle \right\} \subset \mathcal{H}^2. \quad (N \in \mathcal{H}^2)$$

As is well known, the subspace $\mathcal{S}(N)$ is a (closed) *stable subspace* of \mathcal{H}^2 .⁸ Given a sequence $(N_i)_{i=1}^\infty$ of pairwise *strongly orthogonal* martingales $N_i \in \mathcal{H}^2$, we denote the direct sum⁹

$$\begin{aligned} \bigoplus_{i=1}^\infty \mathcal{S}(N_i) &:= \left\{ \sum_{i=1}^\infty X_i : X_i \in \mathcal{S}(N_i) \text{ and } \sum_{i=1}^\infty \|X_i\|^2 < \infty \right\} \\ &= \left\{ \sum_{i=1}^\infty \int H_i dN_i : H_i \in \mathcal{P} \text{ and } \sum_{i=1}^\infty \mathbb{E} \int_0^T H_i^2 d\langle N_i \rangle < \infty \right\}. \end{aligned}$$

As $\bigoplus_{i=1}^\infty \mathcal{S}(N_i)$ is a (countable) direct sum of orthogonal closed subspaces, it is a closed subspace of \mathcal{H}^2 . (In fact, it is the stable subspace generated by $(N_i)_{i=1}^\infty$.¹⁰)

⁶The definition of $\langle M \rangle$ extends to $\mathcal{H}_{\text{loc}}^2$ by localization. Then, we get $\mathcal{H}^2 = \{M \in \mathcal{H}_{\text{loc}}^2 : \mathbb{E} \langle M \rangle_T < \infty\}$.

⁷We also have $\|M\|^2 = \sup_{0 \leq t < T} \mathbb{E} M_t^2 \geq \frac{1}{4} \mathbb{E} M_*^2$, where $M_* := \sup_{0 \leq t \leq T} |M_t|$.

⁸Recall, a stable subspace \mathcal{K} is a *closed* subspace of \mathcal{H}^2 which is closed under stopping, or equivalently, closed under stochastic integration, i.e., $\mathcal{S}(M) \subset \mathcal{K}$ for every $M \in \mathcal{K}$.

⁹Recall, $M, N \in \mathcal{H}^2$ are strongly orthogonal if $\langle M, N \rangle = 0$. Then clearly, they are orthogonal in the Hilbert space sense, and every martingale in $\mathcal{S}(M)$ is strongly orthogonal to every martingale $\mathcal{S}(N)$.

¹⁰That is, $\bigoplus_{i=1}^\infty \mathcal{S}(N_i) =: \mathcal{K}$ is the smallest (the intersection of all) stable subspace(s) containing all N_i . Indeed, \mathcal{K} is stable, for if $N = \sum_i X_i \in \mathcal{K}$ with $X_i \in \mathcal{S}(N_i)$ and T is stopping time, then the stopped process $N^T = \sum_i X_i^T \in \mathcal{K}$ as each $X_i^T \in \mathcal{S}(N_i)$. Further, any stable subspace that contains all N_i also contains $\mathcal{S}(N_i)$, and so contains the closure of linear span of the $\mathcal{S}(N_i)$, which closure clearly equals \mathcal{K} .

2.5. Power brackets. For any semimartingale X , set $[X]^{(2)} := [X]$ and $[X]^{(n)} := \sum_{s \leq \cdot} (\Delta X)_s^n$ for $3 \leq n \in \mathbb{N}$. Note, $[X]^{(n+1)} = [X, [X]^{(n)}]$. Assume $\mathbb{E}[X]_T^{(2n)} < \infty$, i.e., $[X]^{(2n)} \in \mathcal{A}^+$, for all $n \in \mathbb{N}$. It is easy to see that $\text{Var}([X]^{(m)}) \leq [X]^{(m-1)} + [X]^{(m+1)}$ for any odd integer $m \geq 3$.¹¹ So, it follows $[X]^{(n)} \in \mathcal{A}$ for all $n \geq 2$. We denote the compensator of $[X]^{(n)}$ by $\langle X \rangle^{(n)}$. So, $\langle X \rangle^{(n)}$ is characterized as the unique predictable right-continuous finite variation process such that $[X]^{(n)} - \langle X \rangle^{(n)} \in \mathcal{M}$, and it is increasing when n is even.

2.6. Teugels martingales. Assume $\mathbb{E}[X]_T^{(2n)} < \infty$ for all $n \in \mathbb{N}$. Following [N-S], we define the *Teugels martingales* $X^{(n)}$ of order $n \geq 2$ by

$$X^{(n)} := [X]^{(n)} - \langle X \rangle^{(n)}, \quad n \geq 2.$$

As we saw, $X^{(n)} \in \mathcal{M}$, all n . (It is easy to see $X^{(n)} \in \mathcal{H}^2$ if all $\langle X \rangle^{(n)}$ are continuous.¹²)

We will not use random measures in the body of the paper. But, in order to elucidate these definitions as well as to relate to the Lévy measure notation adopted in [N-S], let $\mu = \mu(\omega, dt, dx)$ denote the integer-valued random measure associated to X and $\nu = \nu(\omega, dt, dx)$ be the compensator of μ .¹³ Since, $x^2 * \mu = \sum_{s \leq \cdot} (\Delta X)_s^2$, we have $[X] = [X^c] + x^2 * \mu$ and $\langle X \rangle = [X^c] + x^2 * \nu$. So, $X^{(2)} := [X] - \langle X \rangle = x^2 * (\mu - \nu)$. Let $n \geq 3$. Above, we saw $[X]^{(n)}$ is of integrable total variation and denoted is compensator $\langle X \rangle^{(n)}$. But $x^n * \mu = [X]^{(n)}$; so $x^n * \nu$ is also the compensator of $[X]^{(n)}$. Therefore, $x^n * \nu = \langle X \rangle^{(n)}$, and we have,

$$X^{(n)} = x^n * \mu - x^n * \nu = x^n * (\mu - \nu), \quad n \geq 2.$$

As for the definition of $X^{(1)}$, we denote the compensator of a special semimartingale X by $\langle X \rangle^{(1)}$ and set $X^{(1)} := X - \langle X \rangle^{(1)}$.¹⁴ (So, $X^{(1)} = X$ if and only if $X \in \mathcal{M}_{\text{loc}}$.)

¹¹Indeed, for odd $m \geq 3$, we have

$$\begin{aligned} \text{Var}([X]^{(m)}) &= \sum_{s \leq \cdot} |\Delta X|_s^m = \sum_{s \leq \cdot} 1_{|\Delta X|_s \leq 1} |\Delta X|_s^m + \sum_{s \leq \cdot} 1_{|\Delta X|_s > 1} |\Delta X|_s^m \\ &\leq \sum_{s \leq \cdot} 1_{|\Delta X|_s \leq 1} |\Delta X|_s^{m-1} + \sum_{s \leq \cdot} 1_{|\Delta X|_s > 1} |\Delta X|_s^{m+1} \leq [X]^{(m-1)} + [X]^{(m+1)}. \end{aligned}$$

¹²Indeed, the continuity of $\langle X \rangle^{(n)}$ implies $[X^{(n)}]_T = [[X]^{(n)}]_T = [X]_T^{(2n)} \in L^1$; hence $X^{(n)} \in \mathcal{H}^2$.

¹³Following the notation in Chapter II of [J-S], for a random measure $\nu(\omega, dt, dx)$ and an optional function $W = W(\omega, t, x)$, we set $(W * \nu)_t := \int_{[0, t] \times \{x \neq 0\}} W(s, x) \nu(ds, dx)$. For a Lévy process, $\nu = dt \nu_0(dx)$ is time and state-independent. More general examples are processes with ν of form $\lambda_t dt \nu_0(dx)$, for some, say, Itô process (λ_t) . These include Cox processes where $\nu_0(dx)$ is simply the Lévy measure of a Poisson process. As Cox processes are often thought of as “Poisson processes with stochastic intensity λ_t ”, the aforementioned more general examples may be thought of as “Lévy processes with stochastic intensity λ_t .”

¹⁴When the compensator $\langle X \rangle^{(1)}$ is continuous then the random measures μ and ν associated to X obviously coincide to those associated to $X^{(1)}$. In the case, the following three conditions are equivalent. (a) $X^{(1)} = x * (\mu - \nu)$, (b) $1_{|x| > 1} |x| * \nu < \infty$ a.s., (c) $X^{(1)} = N + M$ for some local martingale N with bounded jumps and some local martingale M of locally integrable variation. When (and only when) $X^{(1)} \in \mathcal{A}_{\text{loc}}$, i.e., $|x| * \nu < \infty$ a.s., we may also write $X^{(1)} = x * \mu - x * \nu$.

3. STRONG ORTHOGONALIZATION

Let $(M_i)_{i=1}^\infty$ be a sequence of martingales $M_i \in \mathcal{H}^2$. As in [D], we associate to it a sequence $(N_i)_{i=1}^\infty$ of pairwise strongly orthogonal martingales, which we call its *Strong Orthogonalization*. Set $N_1 := M_1$, and for $n \geq 2$ inductively define N_n as the orthogonal projection of M_n on the orthogonal complement of $\bigoplus_{i=1}^{n-1} \mathcal{S}(N_i)$. Note, this definition implies that N_i are pairwise strongly orthogonal and $\bigoplus_{i=1}^n \mathcal{S}(N_i)$ is a (closed) stable subspace.¹⁵ For example, if M_i are correlated Brownian motions, then N_i will be independent Brownian motions.

Remark. For almost all paths ω , $d\langle M_i, N_j \rangle(\omega)$ is a measure on $[0, T]$ which is absolutely continuous with respect to the measure $d\langle N_j, N_j \rangle(\omega)$ on $[0, T]$. So, the Radon-Nikodym derivative $\frac{d\langle M_i, N_j \rangle}{d\langle N_j, N_j \rangle}$ is well-defined, and one easily verifies that

$$M_i = N_i + \sum_{j=1}^{i-1} \int \frac{d\langle M_i, N_j \rangle}{d\langle N_j, N_j \rangle} dN_j.$$

This leads to an alternative definition of N_i : set $N_1 := M_1$, and having defined N_j inductively for $j < i$, use the above equation to define N_i . Note, $N_2 = M_2 - \int \frac{d\langle M_1, M_2 \rangle}{d\langle M_1 \rangle} dM_1$.

Remark. For $1 \leq k \leq \infty$, $\bigoplus_{i=1}^k \mathcal{S}(N_i)$ is not only the stable subspace generated by $(N_i)_{i=1}^k$, but also the stable subspace generated by $(M_i)_{i=1}^k$.

We denote the linear span of $\mathcal{S}(M_i)$, $i = 1, 2, \dots$, by¹⁶

$$\text{Span}(\mathcal{S}(M_i))_{i=1}^\infty := \bigcup_{n=1}^\infty \mathcal{S}(M_1) + \dots + \mathcal{S}(M_n).$$

The following is essentially a reformulation of the abstract martingale representation Theorem 3 of [D].¹⁷ Our strategy will be to apply it the Teugels martingales $X^{(i)}$ as the M_i .

Proposition 3.1. *Let $(M_i)_{i=1}^\infty$ be a sequence of martingales in \mathcal{H}^2 such that $\text{Span}(\mathcal{S}(M_i))_{i=1}^\infty$ is dense in \mathcal{H}^2 . Then, $\mathcal{H}^2 = \bigoplus_{i=1}^\infty \mathcal{S}(N_i)$, where $(N_i)_{i=1}^\infty$ is the strong orthogonalization of*

¹⁵These statements follow by a simple induction, using the fact if \mathcal{K} is a stable subspace then its orthogonal complement is (a stable subspace and is) strongly orthogonal to \mathcal{K} .

¹⁶In this paper, we denote the linear span of any subset \mathcal{K} of a vector space by $\text{Span}(\mathcal{K})$. So, $\text{Span}(\mathcal{K})$ is a the set of (finite) linear combinations of elements of \mathcal{K} , i.e., the smallest (intersection of all) linear subspace(s) containing \mathcal{K} . If \mathcal{K}_i , $i \in I$, is a family of linear subspaces of a vector space, we denote their linear span $\text{Span}(\mathcal{K}_i)_{i \in I}$. When the index set I is finite, we also denote $\text{Span}(\mathcal{K}_i)_{i=1}^n$ by $\mathcal{K}_1 + \dots + \mathcal{K}_n$. When \mathcal{K}_i are orthogonal subspaces of a Hilbert space, we emphasize the orthogonality by writing $\text{Span}(\mathcal{K}_i)_{i=1}^n$ as $\mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_n$ or $\bigoplus_{i=1}^n \mathcal{K}_i$. Note, when I is countable, and \mathcal{K}_i are closed, orthogonal subspaces, then their infinite direct sum $\bigoplus_{i=1}^\infty \mathcal{K}_i := \{\sum_{i=1}^\infty K_i : K_i \in \mathcal{K}_i; \sum_{i=1}^\infty \|K_i\|^2 < \infty\}$ equals the *closure* of $\text{Span}(\mathcal{K}_i)_{i=1}^\infty$.

¹⁷Prop 3.1 yields the same conclusion as Theorem 3 of [D] if \mathcal{H}^2 is separable. For, if $(M_i)_{i=1}^\infty$ is a dense sequence in \mathcal{H}^2 , then $\text{Span}(\mathcal{S}(M_i))_{i=1}^\infty$ is also dense in \mathcal{H}^2 , as it obviously contains all the M_i .

$(M_i)_{i=1}^\infty$. In other words, every martingale $M \in \mathcal{H}^2$ has a representation

$$M = \sum_{i=1}^{\infty} \int H_i dN_i$$

(as a convergent series in \mathcal{H}^2) for some predictable processes H_i satisfying

$$\sum_{i=1}^{\infty} \mathbb{E} \left(\int_0^T H_i^2 d\langle N_i \rangle \right) = \sum_{i=1}^{\infty} \mathbb{E} \left(\int_0^T H_i^2 d[N_i] \right) = \|M\|^2 < \infty.$$

Moreover, if $(H'_i)_{i=1}^\infty$ is another sequence with this property, then $\int |H'_i - H_i|^2 d\langle N_i \rangle = 0$ a.s., all i . In particular, the H_i are unique if $\langle N_i \rangle$ are strictly increasing.

Proof. Since $\bigoplus_{i=1}^n \mathcal{S}(N_i)$ is a stable subspace and contains M_n , we have $\mathcal{S}(M_n) \subset \bigoplus_{i=1}^n \mathcal{S}(N_i)$. Hence, $\text{Span}(\mathcal{S}(M_i))_{i=1}^\infty \subset \bigoplus_{i=1}^\infty \mathcal{S}(N_i)$. The first statement follows as the former is assumed dense and the latter is closed. The uniqueness statement follows because direct sum decomposition is unique; so, $\int H'_i dN_i = \int H_i dN_i$, implying $\int |H'_i - H_i|^2 d\langle N_i \rangle = 0$. \square

Remark. The H_i are unique on the support of the measure measure $d\langle N_i, N_i \rangle$ (as measure on $[0, T]$ for each ω .) There, H_i in fact equals the Radon-Nikodym derivative $\frac{d\langle M, N_i \rangle}{d\langle N_i \rangle}$.

Remark. When $d\langle N_i, N_i \rangle = \lambda_i dt$ for some positive predictable processes λ_i , we can normalize by replacing N_i with $\int \lambda_i^{-1/2} dN_i$. The new N_i still satisfy $\langle N_i, N_i \rangle_t = t$, so the condition on the H_i simplify to $\sum_{i=1}^\infty \mathbb{E} \left(\int_0^T H_i^2 dt \right) < \infty$, as in [N-S]. This is possible in the Lévy case, where the λ_i turn out to be positive constants. However, the condition does not hold in general (some N_i may even be zero); so, unlike [N-S], we do not normalize here.

Remark. It is easy show that the the strong orthogonalization of two sequences $(X_i)_{i=1}^\infty$ and $(M_i)_{i=1}^\infty$ of martingales in \mathcal{H}^2 coincide if $X_j := M_j + \sum_{i=1}^{j-1} \int H_{i,j} dM_j$ for some locally bounded predictable processes $H_{i,j}$.

4. MARTINGALES AND SEMIMARTINGALES OF FINITE MOMENTS

Here we define a set \mathcal{C}^* of semimartingales, to a subset of which our main results apply. Recall, $L^* := \bigcap_{n=1}^\infty L^n$. We begin with the definition of martingales of finite moments:

$$\mathcal{H}^* := \{M \in \mathcal{H}^2 : M_T \in L^*\} = \{M \in \mathcal{M}_{\text{loc}} : [M]_T \in L^*\}.$$

The equality is a direct consequence of the Burkholder-Davis-Gundy inequalities.¹⁸

¹⁸The Burkholder-Davis-Gundy inequalities, as stated on page 175, section IV.5 of [P], states that for any local martingale M , finite stopping time T , and $p \geq 1$, there are constants c and C such that

$$\mathbb{E} (M_T^*)^p \leq c \mathbb{E} [M_T]^{p/2} \leq C \mathbb{E} (M_T^*)^p.$$

In this paper, T is deterministic, but is allowed to equal infinity. On page 190, [P] also states the first inequality for $T = \infty$. Above, $M_T^* := \sup_{t \leq T} |M_t|$. However, by Doob's maximal inequality, we can replace M_T^* simply by $|M_T|$ (with a larger constant C).

Proposition 4.1. *Let $M, N \in \mathcal{H}^*$. Then $\int M_- dN \in \mathcal{H}^* \cap \mathcal{S}(N)$.*

Proof. Set $M_* = \sup_{t \in [0, T]} |M_t|$. By Schwartz inequality then Doob's maximal inequality,

$$\begin{aligned} \mathbb{E} \left[\int M_- dN \right]_T^n &= \mathbb{E} \left(\int_0^T M_-^2 d[N] \right)^n \leq \mathbb{E} (M_*^2 [N]_T)^n \\ &\leq (\mathbb{E} M_*^{4n})^{\frac{1}{2}} (\mathbb{E} [N]_T^{2n})^{\frac{1}{2}} \leq \left(\frac{4n}{4n-1} \right)^{2n} (\mathbb{E} M_T^{4n})^{\frac{1}{2}} (\mathbb{E} [N]_T^{2n})^{\frac{1}{2}} < \infty. \end{aligned}$$

Hence $[\int M_- dN]_T \in L^*$. Thus the local martingale $\int M_- dN$ is in fact in $\mathcal{H}^* \cap \mathcal{S}(N)$. \square

Clearly, $[X]^{(2n)} \leq [X]^n$ for any semimartingales X and $n \in \mathbb{N}$.¹⁹ So, if $M \in \mathcal{H}^*$, then $[M]_T^{(2n)} \in L^*$ for all $n \in \mathbb{N}$. Recall, the Teugels martingale is now defined as $M^{(n)} := [M]^{(n)} - \langle M \rangle^{(n)} \in \mathcal{M}$. Our approach relies on $\langle M \rangle^{(n)}$ being continuous. We define

$$\mathcal{H}^* := \{M \in \mathcal{H}^* : \langle M \rangle^{(n)} \text{ is continuous for all } n \geq 2\}.$$

For any $M \in \mathcal{H}^2$, we set

$$\mathcal{S}^*(M) := \mathcal{H}^* \cap \mathcal{S}(M).$$

Proposition 4.2. *Let $M, N \in \mathcal{H}^*$. Then $\int M_- dN \in \mathcal{S}^*(N)$.*

Proof. One readily shows by induction that $[\int M_- dN]^{(n)} = \int M_-^{2n} d[N]^{(n)}$. So $\langle \int M_- dN \rangle^{(n)} = \int M_-^{2n} d\langle N \rangle^{(n)}$, which is continuous. The desired result thus follows by Prop. 4.1. \square

The following consequence will be useful for multivariate representations.

Corollary 4.3. *Let $M', N' \in \mathcal{H}^2$. Let $M \in \mathcal{S}^*(M')$ and $N \in \mathcal{S}^*(N')$. Assume $[M', N'] = 0$. Then, $MN \in \mathcal{S}^*(M') \oplus \mathcal{S}^*(N')$.*

Proof. Clearly, $[M, N] = 0$. So, Prop 3.2 and integration by parts imply $MN \in \mathcal{S}^*(M) \oplus \mathcal{S}^*(N)$. But, $\mathcal{S}(M) \subset \mathcal{S}(M')$ and $\mathcal{S}(N) \subset \mathcal{S}(N')$ as $\mathcal{S}(M')$ and $\mathcal{S}(N')$ are stable subspaces. Hence, $MN \in \mathcal{S}^*(M') \oplus \mathcal{S}^*(N')$. \square

The following result will guarantee that the stochastic integrals of the Teugels martingales in the chaotic expansions below will actually be martingales belonging to \mathcal{H}^2 (even to \mathcal{H}^*).

Proposition 4.4. *Let $M \in \mathcal{H}^*$. Then $M^{(n)} \in \mathcal{H}^*$ and $\langle M \rangle_T^{(n)} \in L^*$ for all $n \in \mathbb{N}$, where $M^{(1)} := M$. Moreover, $[M^{(n)}] = [M]^{(2n)}$ and $\langle M^{(n)} \rangle = \langle M \rangle^{(2n)}$.*

Proof. Recall, $[X, A] = 0$ for all semimartingales X and continuous finite variation semimartingales A . As $\langle X \rangle^{(n)}$ is assumed continuous, this implies $[M^{(n)}] = [M]^{(2n)}$. But, $[M]^{(2n)} \leq [M]^n$. Therefore $[M^{(n)}]_T \in L^*$. Thus $M^{(n)} \in \mathcal{H}^*$. Hence $M_T^{(n)} \in L^*$, and $\langle M \rangle_T^{(n)} = [M^{(n)}]_T - M_T^{(n)} \in L^*$. Let $i \geq 2$. Clearly, $[M^{(n)}]^{(i)} = [M]^{(ni)}$. So $\langle M^{(n)} \rangle^{(i)} = \langle M \rangle^{(ni)}$ is continuous. Therefore $M^{(n)} \in \mathcal{H}^*$. \square

As $\langle M \rangle^{(n)}$ is continuous if $M \in \mathcal{H}^*$, for all semimartingales X , $[M^{(n)}, X] = [[M]^{(n)}, X]$.

Proposition 4.5. *Let $M, N \in \mathcal{H}^*$. If $[M, N] = 0$, then $[M^{(i)}, N^{(j)}] = 0$ for all $i, j \in \mathbb{N}$.*

¹⁹Indeed, we obviously have $\sum_{s \leq t} (\Delta X_s)^{2n} \leq (\sum_{s \leq t} (\Delta X_s)^2)^n$.

Proof. Note, for any two semimartingales X and Y , and $i + j \geq 3$, we have

$$\begin{aligned} [[X]^{(i)}, [Y]^{(j)}] &= \sum_{\cdot \leq s} (\Delta X_s)^i (\Delta Y_s)^j = \\ &= \sum_{\cdot \leq s} (\Delta X_s \Delta Y_s) (\Delta X_s)^{i-1} (\Delta Y_s)^{j-1} = [[X, Y], [[X]^{(i-1)}, [Y]^{(j-1)}]]. \end{aligned}$$

This implies $[[X]^{(i)}, [Y]^{(j)}] = 0$ if $[X, Y] = 0$. The result follows by applying to M and N and invoking the remark preceding the proposition on continuity of $\langle M \rangle^{(i)}$ and $\langle N \rangle^{(j)}$. \square

Let $\mathcal{A}^* \subset \mathcal{A}$ denote the set of *continuous* processes $A \in \mathcal{A}$ such that $\text{Var}(A)_T \in L^*$. As $A_* := \sup_{t \in [0, T]} |A_t| \leq \text{Var}(X)_T$, clearly then $A_t, A_* \in L^*$, all t .

Proposition 4.6. *Let $A, B \in \mathcal{A}^*$ and $M \in \mathcal{H}^*$. Then $AB \in \mathcal{A}^*$ and $\int AdM \in \mathcal{S}^*(M)$.*

Proof. Without loss of generality we may assume A and B are increasing. That $|A_T B_T| \in L^*$ then follows from Schwartz inequality. (Also $\int AdB \in \mathcal{A}^*$, as $|\int_0^T AdB| \leq |A_T B_T|$.) Similarly, $[\int_0^T AdM] \leq A_T^2 [M]_T$. So again by Schwartz inequality $\int AdM \in \mathcal{S}^*(M)$. \square

We now define $\mathcal{C}^* := \mathcal{A}^* + \mathcal{H}^*$. So, any semimartingale $X \in \mathcal{C}^*$ has a decomposition $X = A + M$, necessarily unique, with $A \in \mathcal{A}^*$ and $M \in \mathcal{H}^*$. Note, $X_T \in L^*$. We denote this compensator A by $\langle X \rangle^{(1)}$ and this martingale M by $X^{(1)}$. So,

$$X = \langle X \rangle^{(1)} + X^{(1)}, \quad X \in \mathcal{C}^*, \langle X \rangle^{(1)} \in \mathcal{A}^*, X^{(1)} \in \mathcal{H}^*;$$

As $\langle X \rangle^{(1)}$ is continuous, $[X, Y] = [X^{(1)}, Y]$ for any semimartingale Y . Hence $[X]^{(n)} = [X^{(1)}]^{(n)}$ for $n \geq 2$, implying $X^{(n)} = (X^{(1)})^{(n)}$.²⁰ Clearly, a process X belongs to \mathcal{C}^* if and only if it is a special semimartingale, its compensator belongs to \mathcal{A}^* , $X_0 = 0$, and $[X]_T \in L^*$. The above propositions and the preceding remarks clearly yield

Corollary 4.7. *Let $X, Y \in \mathcal{C}^*$. Then $XY, \int X_- dY \in \mathcal{C}^*$ and $\int X_- dM \in \mathcal{S}^*(M)$ for any $M \in \mathcal{H}^*$. Moreover, $X^{(n)} \in \mathcal{H}^*$ and $\langle X \rangle^{(n)} \in \mathcal{A}^*$ for all $n \in \mathbb{N}$. Furthermore, if $[X, Y] = 0$, then $[X]^{(i)}, [Y]^{(j)} = 0$ for all $i, j \in \mathbb{N}$.*

5. EXPONENTIALLY DECAYING LAWS AND L^2 -DENSENESS OF POLYNOMIALS

We first look at random variables, then processes. Define the subspace $L_* \subset L^*$ by

$$L_* := \{\xi \in L^0 : \mathbb{E} \exp(a|\xi|) < \infty \text{ for some } a > 0\}.$$

Using Schwartz inequality, one easily verifies that L_* is indeed a linear subspace.²¹

Given a finite or infinite sequence $(\xi_i)_{i=1}^k$, $k \leq \infty$ of random variables $\xi_i \in L^0$, we denote by $\mathcal{F}(\xi_i)_{i=1}^k$ the σ -algebra generated by the ξ_i . A *polynomial* in the ξ_i is a (finite) linear

²⁰Indeed $[X] = [X, X^{(1)}] = [X^{(1)}]$, and for $n \geq 3$, using induction,

$$[X]^{(n)} = [X, [X]^{(n-1)}] = [X, [X^{(1)}]^{(n-1)}] = [X^{(1)}, [X^{(1)}]^{(n-1)}] = [X^{(1)}]^{(n)}.$$

²¹Indeed, if $\xi = \xi_1 + \xi_2$ with $\mathbb{E} \exp(a_i |\xi_i|) < \infty$, then $\mathbb{E} \exp(a|\xi|) < \infty$, where $a = \frac{1}{2} \min(a_1, a_2)$.

combination of products $\xi_{i_1} \cdots \xi_{i_m}$, with $m \geq 0$ ranging over non-negative integers, $i_j \in \mathbb{N}$, and $i_j \leq k$ when $k < \infty$. (When $m = 0$, the product is empty, and by convention equals 1). As the indices i_j need not be distinct, this includes the monomials $\xi_{i_1}^{n_1} \cdots \xi_{i_m}^{n_m}$, $n_i \in \mathbb{N}$.

Proposition 5.1. *Let $\xi_1, \dots, \xi_n \in L_*$. Assume $\mathcal{F} = \mathcal{F}(\xi_i)_{i=1}^n$. Then the set of polynomials in ξ_i , i.e., the linear space $\text{Span}\{\xi_{i_1} \cdots \xi_{i_m}\}_{1 \leq i_1, \dots, i_m \leq n, m \geq 0}$, is dense in L^2 .*

Proof. Let $\varphi \in L^2$ satisfy $\mathbb{E}(\varphi \xi_{i_1} \cdots \xi_{i_m}) = 0$ for all $m \geq 0$ and multi-indices $(i_1, \dots, i_m) \in \mathbb{N}^m$. (For $m = 0$ this means $\mathbb{E}\varphi = 0$.) It suffices to show $\varphi = 0$. Let $C_0^\infty(\mathbb{R}^n)$ denote the set of complex-valued smooth functions of compact support on \mathbb{R}^n . As is well known, the set $\{f(\xi) : f \in C_0^\infty(\mathbb{R}^n) \text{ is real valued}\}$ is dense in L^2 , where $\xi = (\xi_1 \cdots \xi_n)$.²² Therefore, it suffices to show $\mathbb{E}(\varphi f(\xi)) = 0$ for all $f \in C_0^\infty(\mathbb{R}^n)$. Define $u : C_0^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ by $u(f) := \mathbb{E}(\varphi f(\xi))$. Then, u is distribution, i.e., a continuous linear functional on $C_0^\infty(\mathbb{R}^n)$ under the latter's usual Frechet topology. We must show $u = 0$. For $x \in \mathbb{R}^n$, define $\hat{u}(x) = \mathbb{E}(\varphi \exp(-\sqrt{-1}x \cdot \xi))$. Then, \hat{u} is in $L_{\text{loc}}^1(\mathbb{R}^n)$, and considered as such as a distribution, it is the Fourier transform of u in the sense of distribution. Hence, it suffices to show $\hat{u} = 0$.

As $|\xi_i| \in L_*$, we have $|\xi| \leq |\xi_1| + \cdots + |\xi_n| \in L_*$. So, $\mathbb{E} \exp(a|\xi|) < \infty$ for some $a > 0$. Using Schwartz inequality yields $\mathbb{E}|\varphi \exp(-iz \cdot \xi)| < \infty$ for $z \in \mathbb{C}^n$ with $|\text{Im}(z)| < a/2$. This implies that the function $z \mapsto \mathbb{E}(\varphi \exp(-\sqrt{-1}z \cdot \xi))$ is holomorphic on $|\text{Im}(z)| < a/2$. It follows that \hat{u} , which is the restriction of this function to \mathbb{R}^n , is real analytic. But,

$$\frac{\partial^m \hat{u}}{\partial x_{i_1} \cdots \partial x_{i_m}}(0) = (-\sqrt{-1})^m \mathbb{E}(\varphi \xi_{i_1} \cdots \xi_{i_m}) = 0,$$

for all $m \geq 0$ by assumption. Since \hat{u} is analytic, it follows $\hat{u} = 0$, as desired. \square

The result extends to infinite sequences by L^2 -version of martingale convergence theorem:

Lemma 5.2. *Let $\xi_i \in L^2$, $i = 1, 2, \dots$. Assume $\mathcal{F} = \mathcal{F}(\xi_i)_{i=1}^\infty$. Then $\bigcup_{n=1}^\infty L^2(\Omega, \mathcal{F}(\xi_i)_{i=1}^n, \mathbb{P})$ is dense in L^2 .*

Proof. Set $\mathcal{F}_n := \mathcal{F}(\xi_i)_{i=1}^n$. Let $\theta \in L^2$. Set $\theta_n := \mathbb{E}[\theta | \mathcal{F}_n]$. By the martingale convergence theorem $\theta_n \rightarrow \theta$ a.s. and in L^1 . Moreover, since $\theta \in L^2$, the convergence also takes place in L^2 .²³ The desired result follows because θ_n belongs to $L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ by construction. \square

Proposition 5.3. *Let $\xi_i \in L_*$, $i = 1, 2, \dots$. Assume $\mathcal{F} = \mathcal{F}(\xi_i)_{i=1}^\infty$. Then the set of polynomials in ξ_i , i.e., the linear space $\text{Span}\{\xi_{i_1} \cdots \xi_{i_m}\}_{(i_1, \dots, i_m) \in \mathbb{N}^m, m \geq 0}$, is dense in L^2 .*

²²Indeed, L^p can be identified with $L^p(\mathbb{R}^n, \mathcal{B}, \mathbb{P} \circ \xi^{-1})$, where \mathcal{B} is the Borel σ -algebra on \mathbb{R}^n . Radon-integral theory then implies that compactly supported continuous functions of ξ are dense in L^p . But, such functions can be uniformly approximated by smooth functions of compact support, using convolution with a non-negative smooth function of small compact support and integral 1.

²³See, e.g., Theorem I.1.42 in [J-S]. The L^2 -convergence can be seen directly as follows. Note,

$$\mathbb{E}[\theta_n^2] = \mathbb{E}[(\mathbb{E}[\theta | \mathcal{F}_n])^2] \leq \mathbb{E}[\mathbb{E}[\theta^2 | \mathcal{F}_n]] = \mathbb{E}[\theta^2].$$

Hence, $\theta_n \in L^2(\Omega, \mathcal{F}(\xi_1, \dots, \xi_n), \mathbb{P})$. It remains to show $\mathbb{E}[(\theta_n - \theta)^2] \rightarrow 0$. Set $\varphi_n = (\theta_n - \theta)^2$. Then,

$$\mathbb{E}[\varphi_n] = \mathbb{E}[\theta^2] + \mathbb{E}[\theta_n^2] - 2\mathbb{E}[\theta_n]\mathbb{E}[\theta] \leq \mathbb{E}[\theta^2] + \mathbb{E}[\theta_n^2] + 2\sqrt{\mathbb{E}[\theta^2]}\sqrt{\mathbb{E}[\theta_n^2]} \leq 4\mathbb{E}[\theta^2].$$

Hence $\sup_n \mathbb{E}[\varphi_n] < \infty$. As $\varphi_n \rightarrow 0$ a.s. and $(\varphi_n)_{n=1}^\infty$ is a positive submartingale, it follows from the submartingale convergence theorem that $\mathbb{E}[\varphi_n] \rightarrow 0$, as desired.

Proof. By Prop. 5.1, polynomials in ξ_1, \dots, ξ_n are dense in $L^2(\Omega, \mathcal{F}(\xi_i)_{i=1}^n, \mathbb{P})$. Since the latter's topology coincides with its relative topology as a subset of L^2 , it follows that polynomials in ξ_1, ξ_2, \dots are dense in $\bigcup_{n=1}^{\infty} L^2(\Omega, \mathcal{F}(\xi_i)_{i=1}^n, \mathbb{P})$ in the L^2 topology. The desired result thus follows from the previous Lemma. \square

We now extend these results to continuous-time stochastic processes, first univariate. Set

$$\mathcal{C}_* := \{\text{left or right continuous processes } X = (X_t)_{t \in [0, T]} \text{ such that } X_t \in L_* \text{ for all } t\}.$$

Proposition 5.4. *Let $X \in \mathcal{C}_*$. Assume $\mathcal{F} = \mathcal{F}(X_t)_{t \in [0, T]}$. Then the linear space of random variables $\text{Span}\{X_{t_1} \cdots X_{t_n}\}_{(t_1, \dots, t_n) \in [0, T]^n, n \geq 0}$ is dense in L^2 .*

Proof. Let $(s_i)_{i=0}^{\infty}$ be a dense sequence in $[0, T]$, containing 0 and T . Set $\xi_i = X_{s_i}$. By right or left continuity of X , we have $\mathcal{F} = \mathcal{F}(\xi_i)_{i=1}^{\infty}$. Also, $\xi_i \in L_*$. The desired result therefore follows by Prop. 5.3. (More strongly, it follows that we may choose the t_i in $\{s_i\}_{i=0}^{\infty}$.) \square

Remark. By not requiring the t_i to be distinct, we are including products of powers $X_{t_j}^{i_j}$. Indeed, $\text{Span}\{X_{t_1} \cdots X_{t_n}\}_{(t_1, \dots, t_n) \in [0, T]^n, n \geq 0} = \text{Span}\{X_{t_1}^{i_1} \cdots X_{t_n}^{i_n}\}_{t_1 < \dots < t_n \in [0, T], i_1, \dots, i_n \in \mathbb{N}, n \geq 0}$. Clearly, these also equal $\text{Span}\{X_{t_1}^{i_1} (X_{t_2} - X_{t_1})^{i_2} \cdots (X_{t_n} - X_{t_{n-1}})^{i_n}\}_{t_1 < \dots < t_n \in [0, T], i_1, \dots, i_n \in \mathbb{N}, n \geq 0}$. The latter is the form stated and used in [N-S]. Here, we use the simpler first form.

Proposition 5.5. *Let $X_i \in \mathcal{C}_*$, $i = 1, 2, \dots$. Assume $\mathcal{F} = \mathcal{F}(X_i(t))_{t \in [0, T], i \in \mathbb{N}}$. Then the linear space $\text{Span}\{X_{i_1}(t_1) \cdots X_{i_n}(t_n)\}_{(t_1, \dots, t_n) \in [0, T]^n, (i_1, \dots, i_n) \in \mathbb{N}^n, n \geq 0}$ is dense in L^2 .*

Proof. Let $(s_i)_{i=0}^{\infty}$ be a dense sequence in $[0, T]$, containing 0 and T . Set $\xi_{ij} = X_j(s_i)$. By right or left continuity of X_i , we have $\mathcal{F} = \mathcal{F}(\xi_{ij})_{i,j=1}^{\infty}$. Also, $\xi_{ij} \in L_*$. Using any bijection of $\mathbb{N} \times \mathbb{N}$ onto \mathbb{N} , we may regard $(\xi_{ij})_{i,j=1}^{\infty}$ as one long sequence. The desired result therefore follows by Prop. 5.3. (More strongly, it follows that we may choose the t_i in $\{s_i\}_{i=0}^{\infty}$.) \square

Remark. The above specializes to a finite dimensional version by letting all except a finite number of X_i be zero.

Remark. Since we are not requiring i_1, \dots, i_n to be distinct, we are including products of the form $X_i(t_1) \cdots X_i(t_n)$ for each i as well as products of such expressions over different i .

Although L^2 is isometric to $L^2(\Omega, \mathcal{F}_0, \mathbb{P}) \oplus \mathcal{H}^2$, it is \mathcal{H}^2 that embodies the filtration structure, not L^2 . For our purposes it is more convenient to cast the last two propositions in terms of \mathcal{H}^2 . To this end, we utilize the following notation. For any $\xi \in L^1$, set

$$\overline{\mathbb{E}}(\xi | \mathbb{F}) := (\mathbb{E}(\xi | \mathcal{F}_t))_{t \in [0, T]} - \mathbb{E}(\xi | \mathcal{F}_0) \in \mathcal{M}.$$

Clearly, $\overline{\mathbb{E}}(\xi | \mathbb{F}) \in \mathcal{H}^*$ for $\xi \in L^*$. The previous two propositions respectively yield,

Corollary 5.6. *Let $X \in \mathcal{C}_*$. Assume $\mathbb{F} = \mathbb{F}(X)$. Then the linear subspace of martingales*

$$\text{Span}\{\overline{\mathbb{E}}(X_{t_1} \cdots X_{t_n} | \mathbb{F})\}_{(t_1, \dots, t_n) \in [0, T]^n, n \in \mathbb{N}}$$

is (contained in \mathcal{H}^ and) dense in \mathcal{H}^2 .*

Corollary 5.7. *Let $X_i \in \mathcal{C}_*$, $i = 1, 2, \dots$. Assume $\mathbb{F} = \mathbb{F}(X_i)_{i=1}^\infty$. Then the linear subspace*

$$\text{Span}\{\overline{\mathbb{E}}(X_{i_1}(t_1) \cdots X_{i_n}(t_n) | \mathbb{F})\}_{(t_1, \dots, t_n) \in [0, T]^n, (i_1, \dots, i_n) \in \mathbb{N}^n, n \in \mathbb{N}}$$

is (contained in \mathcal{H}^ and) dense in \mathcal{H}^2 .*

6. INDUCTIVE CHAOTIC EXPANSION OF STOPPED POLYNOMIALS

Throughout this section, let $X \in \mathcal{C}^*$. Let $\mathcal{A}_0^*(X)$ denote the set of simple functions, i.e., the linear span of (deterministic) processes of the form $1_{[0, t]}$, $0 < t \leq T$. For $n \geq 1$, set

$$\mathcal{A}_n^*\langle X \rangle := \{A \in \mathcal{A}^* : A \text{ is adapted to } \mathbb{F}(\langle X \rangle^{(i)})_{i=1}^n\}.$$

Note, if $\langle X \rangle^{(i)}$ are deterministic (as in the Lévy case) then any $A \in \mathcal{A}_n^*\langle X \rangle$ is deterministic.

We next define a sequence of linear subspaces $(\mathcal{C}_i^*(X))_{i=0}^\infty$ of \mathcal{C}^* and a sequence of linear subspaces of $(\mathcal{S}_i^*(X))_{i=1}^\infty$ of \mathcal{H}^* . We employ a joint inductive definition. Set $\mathcal{S}_0^*(X) := \mathbb{R}$,

$$\mathcal{S}_1^*(X) := \left\{ \int A dX^{(1)} : A \in \mathcal{A}_0^*(X) \right\};$$

$$\mathcal{C}_1^*(X) := \mathcal{A}_1^*\langle X \rangle + \{AM : A \in \mathcal{A}_0^*(X); M \in \mathcal{S}_1^*(X)\}.$$

Note, $X \in \mathcal{C}_1^*(X)$. For $n \geq 2$, we define inductively,

$$\mathcal{S}_n^*(X) = \text{Span}\left\{ \int Y_- dX^{(j)} : Y \in \mathcal{C}_i^*(X), i + j = n, 0 \leq i \leq n-1, 1 \leq j \leq n \right\};$$

$$\mathcal{C}_n^*(X) := \text{Span}\{AM : A \in \mathcal{A}_i^*\langle X \rangle; M \in \mathcal{S}_j^*(X), i + j = n, 0 \leq i, j \leq n\}.$$

For example, $X^{(9)} + \int \langle X \rangle^{(6)} d\langle X \rangle^{(3)} + \langle X \rangle^{(2)} \int (\int \langle X \rangle^{(1)} dX^{(2)}) dX^{(4)} \in \mathcal{C}_9^*(X)$.

Section 8 below presents an explicit (huge) decomposition $X^n = \sum_k A_k M_k \in \mathcal{C}_n^*(X)$. The A_k will be iterated (multiple) Stieltjes integrals of $\langle X \rangle^{(i)}$, and the M_k will be iterated stochastic integrals of products of such forms A against the Teugels martingales $X^{(j)}$. However, what is important for our main results is not the explicit form, but two key properties of \mathcal{C}_n^* : it is closed under multiplication and under stopping at deterministic times. (The latter is clear.) The following is a simple consequence of Section 4.

Proposition 6.1. *We have $\mathcal{S}_n^*(X) \subset \text{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$ and $\mathcal{C}_n^*(X) \subset \mathcal{C}^*$, all $n \in \mathbb{N}$.*

Proof. We use induction, case $n = 1$ being clear. Let $n \geq 2$, $M \in \mathcal{S}_n^* := \mathcal{S}_n^*(X)$, and $Y \in \mathcal{C}_n^* := \mathcal{C}_n^*(X)$. By linearity we may assume $M = \int Z_- dX^{(j)}$ for some $Z \in \mathcal{C}_i^*$, $i + j = n$, $i < n$, and $Y = AN$ for some $A \in \mathcal{A}_i^*$ and $N \in \mathcal{C}_j^*$, $i + j = n$. By induction, $Z \in \mathcal{C}^*$, and by Corollary 4.7, $X^{(j)} \in \mathcal{H}^*$. So by Corollary 4.7, $\int Y_- dX^{(j)} \in \mathcal{S}^*(X^{(j)})$. Therefore, $M \in \text{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$. If $j = n$ by what was just shown and otherwise by induction, we have $N \in \mathcal{C}^*$. So, $Y = AN \in \mathcal{C}^*$ by Corollary 4.7. \square

A principal and non-trivial property of $\mathcal{C}_n^*(X)$ is closedness under multiplication:

Proposition 6.2. *Let $Y \in \mathcal{C}_n^*(X)$, $Z \in \mathcal{C}_m^*(X)$, $n, m \geq 0$. Then $YZ, \int Y_- dZ \in \mathcal{C}_{m+n}^*(X)$.²⁴*

Proof. We use induction on $n+m$. The case $n+m=1$ is trivial. Assume $n+m \geq 2$. Note, if $A \in \mathcal{A}_i^*$ and $B \in \mathcal{A}_j^*$, then $AB \in \mathcal{A}_{i \vee j}^*$. This shows we may assume $Y \in \mathcal{S}_n^*$ and $Z \in \mathcal{S}_m^*$. By linearity we may further assume $Y = \int Y'_- dX^{(j)}$ for some $Y' \in \mathcal{C}_i^*$ with $i+j=n$, $i \geq 0$, $j \geq 1$, and $Z = \int Z'_- dX^{(l)}$ for some $Z' \in \mathcal{C}_k^*$ with $l+k=m$, $k \geq 0$, $l \geq 1$.

By induction we have $YZ' \in \mathcal{C}_{n+m-l}^*$. Therefore, $\int Y_- dZ$ is a sum of forms $\int AM_- dX^{(l)}$ for some $A \in \mathcal{A}_a^*$, $M \in \mathcal{S}_b^*$ with $a+b+l=n+m$, $a, b \geq 0$. Clearly, $AM \in \mathcal{C}_{n+m-l}^*$; so $\int AM_- dX^{(l)} \in \mathcal{S}_{n+m}^*$. It follows $\int Y_- dZ \in \mathcal{S}_{n+m}^*$. Next, we show $[Y, Z] \in \mathcal{C}_{m+n}^*$. We have,

$$\begin{aligned} [Y, Z] &= \int Y'_- Z'_- d[X^{(j)}, X^{(l)}] = \int Y'_- Z'_- d[X]^{(j+l)} \\ &= \int Y'_- Z'_- dX^{(j+l)} + \int Y'_- Z'_- d\langle X \rangle^{(j+l)} \\ &= \int Y'_- Z'_- dX^{(j+l)} + Y' Z' \langle X \rangle^{(j+l)} - \int \langle X \rangle^{(j+l)} d(Y' Z'), \end{aligned}$$

the last step by integration by parts and continuity of $\langle X \rangle^{(j+l)}$. By induction $Y' Z' \in \mathcal{C}_{n+m-j-l}^*$. Hence, the first term is in \mathcal{S}_{n+m}^* , the second term is in \mathcal{C}_{n+m}^* , and the third term is a sum of forms $\int \langle X \rangle^{(j+l)} d(AM)$ (or simpler forms $\int \langle X \rangle^{(j+l)} d(A) \in \mathcal{A}_{n+m}^*$) for some $A \in \mathcal{A}_c^*$ and $M \in \mathcal{S}_d^*$ with $c+d+j+l=n+m$, $c \geq 0$, $d \geq 1$. Set $B := \int \langle X \rangle^{(j+l)} dA$. Then $B \in \mathcal{A}_{j+l+c}^*$. Integrating by parts twice (bracket vanishing by continuities of $\langle X \rangle^{(j+l)}$, B)

$$\begin{aligned} \int \langle X \rangle^{(j+l)} d(AM) &= \int A \langle X \rangle^{(j+l)} dM + \int M_- \langle X \rangle^{(j+l)} dA \\ &= \int A \langle X \rangle^{(j+l)} dM + \int M_- dB = \int A \langle X \rangle^{(j+l)} dM + BM - \int BdM. \end{aligned}$$

All three terms are visibly in \mathcal{C}_{n+m}^* . Hence, $[Y, Z] \in \mathcal{C}_{n+m}^*$. We already showed $\int Y_- dZ$, and by symmetry $\int Z_- dY$, are in \mathcal{C}_{n+m}^* . Therefore, by Itô's product rule, so is YZ . \square

In particular, $X^n \in \mathcal{C}_1^*(X)$ as $X \in \mathcal{C}_n^*(X)$. If $Y \in \mathcal{C}_n^*$ and $s \in [0, T]$, then clearly the stopped process $Y_{\cdot \wedge s} := (Y_{t \wedge s})_{t \in [0, T]}$ is also in $\mathcal{C}_n^*(X)$. Therefore the product $X_{\cdot \wedge t_1} \cdots X_{\cdot \wedge t_n} \in \mathcal{C}_n^*(X)$.

We illustrate the significance of this for the case when $\langle X \rangle^{(n)}$ are deterministic here, and for the stochastic case in Section 7.2. We begin with the univariate case.

Corollary 6.3. *If $\langle X \rangle^{(i)}$ are deterministic for all $i \in \mathbb{N}$ then for all $(t_1, \dots, t_n) \in [0, T]^n$,*

$$\overline{\mathbb{E}}(X_{t_1} \cdots X_{t_n} \mid \mathbb{F}) \in \text{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n.$$

²⁴Moreover, by Itô's product rule we have, $[Y, Z] \in \mathcal{C}_{m+n}^*(X)$. The proof further shows, $YZ - [Y, Z], [Y, Z] - \langle Y, Z \rangle \in \text{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$, and $\langle Y, Z \rangle \in \mathcal{A}^*$.

Proof. Note, $X_{t_1} \cdots X_{t_n} = Y_T$, where $Y := X_{\cdot \wedge t_1} \cdots X_{\cdot \wedge t_n}$. So, it suffices to show $\overline{\mathbb{E}}(Y_T | \mathbb{F}) \in \text{Span}(\mathcal{S}^*(X^{(i)}))_{i=1}^n$. By the previous proposition, $Y \in \mathcal{C}_n^*(X)$ because each $X_{\cdot \wedge t_i} \in \mathcal{C}_1^*(X)$. So, by linearity, we may assume $Y = AM$ for some $A \in \mathcal{A}_i^*$ and $M \in \mathcal{S}_j^*$, $i + j = n$, $i, j \geq 0$. But, the assumption implies that A is deterministic. Therefore, $\overline{\mathbb{E}}(Y_T | \mathbb{F}) = A_T \overline{\mathbb{E}}(M_T | \mathbb{F}) = A_T M$. The desired result now follows from Prop. 6.1. \square

The multivariate case combines a similar argument with Cor. 4.3 and Prop. 4.5 as follows.

Lemma 6.4. *Let $X, Y \in \mathcal{C}^*$. Assume $[X, Y] = 0$ and $\langle X \rangle^{(n)}$ and $\langle Y \rangle^{(n)}$ are deterministic for all $n \in \mathbb{N}$. Then, for any $Z \in \mathcal{C}_n^*(X)$ and $W \in \mathcal{C}_m^*(Y)$, we have $[Z, W] = 0$ and*

$$\overline{\mathbb{E}}(Z_T W_T | \mathbb{F}) = \overline{\mathbb{E}}(Z_T | \mathbb{F}) \overline{\mathbb{E}}(W_T | \mathbb{F}) \in \text{Span}(\mathcal{S}^*(X^{(i)}), \mathcal{S}^*(Y^{(j)}))_{1 \leq i \leq n, 1 \leq j \leq m}.$$

Proof. By definition of \mathcal{C}_n^* and linearity, we may assume $Z = AM$ for some $A \in \mathcal{A}_k^*(X)$ and $M \in \mathcal{S}_l^*(X)$ such that $k + l = n$, $k, l \geq 0$, and similarly, $W = BN$ for some $B \in \mathcal{A}_a^*(Y)$ and $N \in \mathcal{S}_b^*(Y)$ such that $a + b = m$, $k, l \geq 0$. By Prop. 6.1, we have $M = \sum_{i=1}^l M_i$ and $N = \sum_{j=1}^b N_j$ for some $M_i \in \mathcal{S}^*(X^{(i)})$ and $N_j \in \mathcal{S}^*(Y^{(j)})$. And by Prop. 4.5 $[X^{(i)}, Y^{(j)}] = 0$. Applying Cor. 4.3, with $M' = X^{(i)}$ and $N' = Y^{(j)}$, we see that $M_i N_j \in \mathcal{S}^*(X^{(i)}) \oplus \mathcal{S}^*(Y^{(j)})$. We conclude $MN \in \mathcal{K} := (\text{Span}(\mathcal{S}^*(X^{(i)}), \mathcal{S}^*(Y^{(j)}))_{1 \leq i \leq n, 1 \leq j \leq m})$. As $MN \in \mathcal{M}$, we have $\overline{\mathbb{E}}(M_T N_T | \mathbb{F}) = MN$. (Both martingales have the same terminal value.) Now, the assumption implies A and B are deterministic. Hence,

$$\begin{aligned} \overline{\mathbb{E}}(Z_T W_T | \mathbb{F}) &= \overline{\mathbb{E}}(A_T M_T B_T N_T | \mathbb{F}) \\ &= A_T B_T \overline{\mathbb{E}}(M_T N_T | \mathbb{F}) = A_T B_T N M = A_T B_T \overline{\mathbb{E}}(M_T | \mathbb{F}) \overline{\mathbb{E}}(N_T | \mathbb{F}) \\ &= \overline{\mathbb{E}}(A_T M_T | \mathbb{F}) \overline{\mathbb{E}}(B_T N_T | \mathbb{F}) = \overline{\mathbb{E}}(Z_T | \mathbb{F}) \overline{\mathbb{E}}(W_T | \mathbb{F}). \end{aligned}$$

Since as we showed above $MN \in \mathcal{K}$, and A_T, B_T are deterministic, $A_T B_T N M \in \mathcal{K}$. \square

A straightforward generalization using induction gives

Lemma 6.5. *Let $Y_1, \dots, Y_m \in \mathcal{C}^*$. Assume $[Y_j, Y_k] = 0$ if $j \neq k$ and $\langle Y_j \rangle^{(k)}$ are deterministic all j, k . Let $Z_j \in \mathcal{C}_{n_j}^*(Y_k)$, $1 \leq j \leq m$. Then, $[Z_j, Z_k] = 0$ for $j \neq k$, and*

$$\overline{\mathbb{E}}(Z_1(T) \cdots Z_m(T) | \mathbb{F}) = \overline{\mathbb{E}}(Z_1(T) | \mathbb{F}) \cdots \overline{\mathbb{E}}(Z_m(T) | \mathbb{F}) \in \text{Span}(\mathcal{S}^*(Y_j^{(k_j)}))_{1 \leq j \leq m, 1 \leq k_j \leq n_j}.$$

Corollary 6.6. *Let $X_i \in \mathcal{C}^*$, $i \in \mathbb{N}$. Assume $[X_i, X_j] = 0$ if $i \neq j$ and $\langle X_i \rangle^{(j)}$ are deterministic all $i, j \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, $(t_1, \dots, t_n) \in [0, T]^n$, and $(i_1, \dots, i_n) \in \mathbb{N}^n$,*

$$\overline{\mathbb{E}}(X_{i_1}(t_1) \cdots X_{i_n}(t_n) | \mathbb{F}) \in \text{Span}(\mathcal{S}^*(X_i^{(j)}))_{i \in \mathbb{N}, 1 \leq j \leq n}.$$

Proof. Let m be the number of (distinct) elements in the set $\{i_1, \dots, i_n\}$. By a permutation if necessary, we may assume that $i_1 = \dots = i_{n_1}$, $i_{n_1+1} = \dots = i_{n_2}$, \dots , $i_{n_{m-1}} = \dots = i_{n_m} = i_n$, with $n_1 + \dots + n_m = n$, $n_j \geq 1$. (So non-distinct elements are put next to each other). Set $Y_j := X_{n_j}$, $j = 1, \dots, m$. Define the product $Z_j(t) := Y_j(t \wedge t_{n_j}) \cdots Y_j(t \wedge t_{n_{j+1}-1})$. By Prop 6.2, $Z_j \in \mathcal{C}_{n_j}^*(Y_k)$. Moreover, clearly, $Z_1(T) \cdots Z_m(T) = X_{i_1}(t_1) \cdots X_{i_n}(t_n)$. The desired result now follows directly from the previous lemma. \square

Remark. Cor. 6.6 generalizes Cor. 6.3: simply set $X_1 = X$ and $X_i = 0$ for $i \geq 2$.

7. SQUARE-INTEGRABLE MARTINGALE REPRESENTATION

We set $\mathcal{C} := \mathcal{C}^* \cap \mathcal{C}_*$. (Recall, \mathcal{C}^* is the set of semimartingales of finite moments with continuous angel brackets, and \mathcal{C}_* is the set of processes with exponentially decreasing law.)

7.1. Lévy and Infinite-dimensional Brownian filtrations. We begin with an extension of the [N-S] result to Lévy processes which may be non-stationary.

Theorem 7.1. *Let $X \in \mathcal{C}$ be such that $\langle X \rangle^{(i)}$ are deterministic for all $i \geq 1$. Let $(N_i)_{i=1}^\infty$ denote the strong orthogonalization of $(X^{(i)})_{i=1}^\infty$. Assume $\mathbb{F} = \mathbb{F}(X)$. Then*

$$\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i).$$

Proof. Let $(t_1, \dots, t_n) \in [0, T]^n$, $n \in \mathbb{N}$. By Cor. 6.3, $\overline{\mathbb{E}}(X_{t_1} \cdots X_{t_n} | \mathbb{F}) \in \text{Span}(\mathcal{S}(X^{(i)}))_{i=1}^n$. Hence by Cor. 5.6, $\text{Span}(\mathcal{S}(X^{(i)}))_{i=1}^n$ is dense in \mathcal{H}^2 . Prop. 3.1 now yields the result. \square

Remark: A curious consequence is that the continuous martingale part X^c is in $\bigoplus_{i=1}^\infty \mathcal{S}(N_i)$. It somehow indicates that the discontinuous martingale part can be recovered in the limit from stochastic integral of $X^{(n)}$, $n \geq 2$. This is readily seen when X is a linear combination of n independent Poisson processes. Then, in fact, $X^{(1)} \in \text{Span}\{X^{(2)}, \dots, X^{(n+1)}\}$.

For a Brownian motion or a Poisson process the result simplifies to $\mathcal{H}^2 = \mathcal{S}(X^{(1)})$.

We need the Brownian case in our main results. Let us define a *Brownian martingale* as a continuous martingale such that $\langle B \rangle$ is deterministic. The law of B is then Gaussian, implying $B \in \mathcal{C}$. Clearly $B^{(n)} = 0$ for $n \geq 2$, as B is continuous. When $T < \infty$, a Brownian motion is a Brownian-martingale. In general, if W is a Brownian motion, then $\int H dW$ is a Brownian martingale for any *deterministic* process $H \in L(W)$, i.e., with $\int_0^T H_t^2 dt < \infty$. Any Brownian martingale B with strictly increasing $\langle B \rangle$ is of this type.²⁵

By a *Poisson-martingale* we mean a martingale $P \in \mathcal{C}$ such that $\langle P \rangle$ is deterministic and $P^{(2)} = P$. Clearly then, $P^{(n)} = P$, $\langle P \rangle^{(n)} = \langle P \rangle$, and $[P]^{(n)} = [P]$ for all $n \geq 2$. A non-stationary compensated Poisson process P with intensity (λ_t) is a Poisson martingale if $\int_0^T \lambda_t dt < \infty$. Then, $\langle P \rangle = \int \lambda dt$. The stationary case of constant λ implies $T < \infty$.

As Brownian and Poisson martingales both satisfy $N_n = 0$ for $n \geq 2$, Theorem 7.1 yields

Corollary 7.2. *Let B be either a Brownian martingale or a Poisson martingale. Assume $\mathbb{F} = \mathbb{F}(B)$. Then $\mathcal{H}^2 = \mathcal{S}(B)$.*

Remark. Suppose B is a standard Brownian motion or a Poisson process of intensity 1. As $\langle B \rangle_t = t$, we have $B \in \mathcal{H}^2$ only if $T < \infty$; so the above does not directly apply when $T = \infty$. When $T = \infty$ and $\mathcal{F} = \bigvee_{t < \infty} \mathcal{F}_t$, a standard argument using the corollary for finite T yields for every martingale $M \in \mathcal{H}^2$ a unique representation $M = \int H dB$ for some

²⁵Indeed, then $B = \int H dW$, where $W := \int K dB$, $K := \sqrt{d\langle B \rangle/dt}$, and $H := 1/K$.

predictable process H satisfying $\mathbb{E} \int_0^\infty H_t^2 dt < \infty$.²⁶ When $T = \infty$, a similar observation applies to Theorem 7.1 in the stationary Lévy case, for then $\langle N_i \rangle$ equals a constant times t .

We now turn to multivariate Lévy filtrations. The argument is similar, but the statement utilizes a notation of iterated countable direct sums which we explain first. Suppose we have a doubly indexed family $(\mathcal{K}_{ij})_{i,j \in \mathbb{N}}$ of closed, pairwise orthogonal subspaces \mathcal{K}_{ij} of \mathcal{H}^2 . If we choose any bijection between $\mathbb{N} \times \mathbb{N}$ and \mathbb{N} , we can identify this family with a sequence of closed orthogonal subspaces, and then take their direct sum. Clearly, the resulting subspace is independent of the choice of bijection. We denote this direct sum by

$$\bigoplus_{i,j=1}^{\infty} \mathcal{K}_{ij} := \left\{ \sum_{i,j=1}^{\infty} N_{ij} : N_{ij} \in \mathcal{K}_{ij}; \sum_{i,j=1}^{\infty} \|N_{ij}\|^2 < \infty \right\} \subset \mathcal{H}^2.$$

The order of the summation is irrelevant: we can interchange sums and write $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} N_{ij} = \sum_{i,j=1}^{\infty} N_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} N_{ij}$. Each inner sum is in \mathcal{H}^2 . This, corresponds to writing,

$$\bigoplus_{j=1}^{\infty} \bigoplus_{i=1}^{\infty} \mathcal{K}_{ij} = \bigoplus_{i,j=1}^{\infty} \mathcal{K}_{ij} = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} \mathcal{K}_{ij}.$$

Theorem 7.3. *Let $X_i \in \mathcal{C}$, $i \in \mathbb{N}$ be such that $[X_i, X_j] = 0$ for $i \neq j$ and $\langle X_i \rangle^{(j)}$ are deterministic for all $i, j \in \mathbb{N}$. Assume $\mathbb{F} = \mathbb{F}(X_i)_{i=1}^{\infty}$. Then*

$$\mathcal{H}^2 = \bigoplus_{i,j=1}^{\infty} \mathcal{S}(N_{ij}),$$

where, for each i , the sequence $(N_{ij})_{j=1}^{\infty}$ is the strong orthogonalization of $(X_i^{(j)})_{j=1}^{\infty}$.

Proof. Let $(t_1, \dots, t_n) \in [0, T]^n$ and $(i_1, \dots, i_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$. By Corollary 6.6, we have $\overline{\mathbb{E}(X_{i_1}(t_1) \cdots X_{i_n}(t_n) | \mathbb{F})} \in \text{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^{\infty}$. Hence by Corollary 5.7, $\text{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^{\infty}$ is dense in \mathcal{H}^2 . The desired result now follows by Prop. 3.1, applied to the doubly indexed sequence of martingales $(X_i^{(j)})_{i,j=1}^{\infty}$. \square

As a consequence we obtain an infinite-dimensional extension of the standard finite-dimensional martingale representation theorems for Brownian motions and Poisson processes.

Corollary 7.4. *Let $(B_i)_{i=1}^{\infty}$ be sequence martingales such that $[B_i, B_j] = 0$ for $i \neq j$, and for each i , B_i is either a Brownian or a Poisson martingale. Assume $\mathbb{F} = \mathbb{F}(B_i)_{i=1}^{\infty}$. Then,*

$$\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i).$$

Moreover, if all B_i are Brownian martingales, then every martingale in \mathcal{H}^2 is continuous.

²⁶Indeed, for each $n \in \mathbb{N}$, the Corollary applied to the case $T = n$, yields a predictable process H_n defined on $[0, n]$ such that $M = \int H_n dB$ on $[0, n]$ and $\mathbb{E} \int_0^n H_n^2(t) dt = \mathbb{E}[M]_n$. Moreover, by uniqueness $H_n = H_m$ on $[0, n \wedge m]$. One now defines H to equal H_n on $[0, n]$. Then $M = \int H dB$ and $\mathbb{E} \int_0^\infty H_t^2 dt = \|M\|^2 < \infty$.

Proof. The first statement follows because $N_{ij} = 0$ for $j \geq 2$ and $N_{i1} = B_i$. As for the continuity statement, let $M \in \mathcal{H}^2$. Write $M = M^c + M^d$ for the continuous-discontinuous decomposition. If all B_i are continuous, then M^d is strongly orthogonal to all B_i , and hence also orthogonal to $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) = \mathcal{H}^2$, implying $M^d = 0$. \square

Remark. The above specializes to the standard finite-dimensional case by taking all but a finite number of B_i equal to zero.

Remark. The assumption $[B_i, B_j] = 0$, $i \neq j$ can be weakened to correlated Brownian motions (such as $[B_i, B_j] = \rho_{ij}t$). The conclusion is then expressed in terms of the orthogonalization of the B_i , which will be independent Brownian motions.

7.2. The main result. We now generalize the results of the previous section to stochastic $\langle X \rangle^{(n)}$, beginning with the univariate case.

Theorem 7.5. *Let $X \in \mathcal{C}$ and $(B_i)_{i=1}^{\infty}$ be a sequence of Brownian martingales such that $[B_i, B_j] = 0$ for $i \neq j$, $[X, B_i] = 0$ all i , and $\langle X \rangle^{(n)}$ is adapted to $\mathbb{F}(B_i)_{i=1}^{\infty}$ all n . Let $(N_i)_{i=1}^{\infty}$ denote the strong orthogonalization of $(X^{(i)})_{i=1}^{\infty}$. Assume $\mathbb{F} = \mathbb{F}(X, B_1, B_2, \dots)$. Then*

$$\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) \oplus \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j).$$

Proof. Note, $[X^{(j)}, B_i] = 0$ for all i, j , for $j = 1$ by assumption, and for $j \geq 2$ because $X^{(j)}$ is purely discontinuous and B_i is continuous. This implies $[B_i, N_j] = 0$, which in turn implies $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i)$ and $\bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$ are orthogonal subspaces. Therefore, $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) + \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$ is a closed subspace of \mathcal{H}^2 ; so it suffices to show it is dense.

Corollary 5.7 applied to the sequence (X, B_1, B_2, \dots) implies that the linear span of martingales of the form $\overline{\mathbb{E}}((X_{t_1} \cdots X_{t_n})(B_{i_1}(s_1) \cdots B_{i_m}(s_m)) | \mathbb{F})$ is dense in \mathcal{H}^2 , as the indices run over $(t_1, \dots, t_n) \in [0, T]^n$, $n \in \mathbb{N}$, and $(s_1, \dots, s_m) \in [0, T]^m$, $(i_1, \dots, i_m) \in \mathbb{N}^m$, $m \in \mathbb{N}$. As in Prop. 3.1, we have, $\text{Span}(\mathcal{S}(X^{(j)}))_{j=1}^n \subset \bigoplus_{j=1}^n \mathcal{S}(N_j) \subset \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$. Therefore it is sufficient to show that

$$\overline{\mathbb{E}}((X_{t_1} \cdots X_{t_n})(B_{i_1}(s_1) \cdots B_{i_m}(s_m)) | \mathbb{F}) \in \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) + \text{Span}(\mathcal{S}(X^{(j)}))_{j=1}^n.$$

Set $Y := X_{\wedge t_1} \cdots X_{\wedge t_n}$. Note, $X_{t_1} \cdots X_{t_n} = Y_T$. Set $\varphi := B_{i_1}(s_1) \cdots B_{i_m}(s_m)$. We must show $\overline{\mathbb{E}}(\varphi Y_T | \mathbb{F}) \in \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) + \text{Span}(\mathcal{S}(X^{(j)}))_{j=1}^n$. By Prop. 6.2, $Y \in \mathcal{C}_n^*(X)$. So, Y is a sum of terms of the form AM , where $A \in \mathcal{A}_l^*(X)$ and $M \in \mathcal{S}_k^*(X)$, $l + k = n$, $0 \leq l, k \leq n$. Note that φA_T is in L^* and is also $\mathcal{G} := \mathcal{F}(B_i)_{i=1}^n$ -measurable because both φ and A_T have these two properties. Therefore, it is sufficient to show that for all $M \in \mathcal{S}_k^*(X)$, $k \leq n$, and all \mathcal{G} -measurable $\xi \in L^*$, we have $\overline{\mathbb{E}}(\xi M_T | \mathbb{F}) \in \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i) + \text{Span}(\mathcal{S}(X^{(j)}))_{j=1}^n$.

Let $\mathbb{G} := \mathbb{F}(B_i)_{i=1}^{\infty}$. Set $N := \overline{\mathbb{E}}(\xi | \mathbb{G})$. Cor. 7.4, applied to the filtration \mathbb{G} implies N is continuous and $N = \sum_{i=1}^{\infty} \int H_i dB_i$ for some \mathbb{G} -predictable processes H_i satisfying

$\sum_{i=1}^{\infty} \mathbb{E} \int_0^T H_i(t)^2 d\langle B_i \rangle < \infty$. But, H_i are a-priori \mathbb{F} -predictable too. So, in fact, we have $N \in \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i)$. In particular, N is also an \mathbb{F} -martingale. Hence $N = \overline{\mathbb{E}}(\xi | \mathbb{F})$.

Since N is continuous and $X^{(n)}$ are purely discontinuous for $n \geq 2$, we have $[N, X^{(n)}] = 0$. This is also true for $n = 1$, as $[X, B_i] = 0$ by assumption. Hence, $[N, M] = 0$. Since $N, M \in \mathcal{H}^* \subset \mathcal{H}^2$, it follows $NM \in \mathcal{M}$. Hence $\overline{\mathbb{E}}(\xi M_T | \mathbb{F}) = NM$, as both sides are martingales with the same value at T (namely ξM_T).

Now, $NM = \int NdM + \int M_-dN$. By Prop. 4.2, $\int M_-dN \in \mathcal{S}(N)$. Since $\bigoplus_{i=1}^{\infty} \mathcal{S}(B_i)$ is a stable subspace and contains N , it contains $\mathcal{S}(N)$. Therefore, $\int M_-dN \in \bigoplus_{i=1}^{\infty} \mathcal{S}(B_i)$. So, it remains to show $\int NdM \in \text{Span}(\mathcal{S}(X^{(j)}))_{j=1}^n$. Since $M \in \mathcal{S}_k^*(X)$, it is a sum of terms of form $\int AdX^{(i)}$ with $A \in \mathcal{A}^*$ and $i \leq k \leq n$. But, by Cor. 4.7, $X^{(i)} \in \mathcal{H}^*$ and $AN \in \mathcal{C}^*$. Hence, by Cor. 4.7, $\int ANdX^{(i)} \in \mathcal{S}(X^{(i)}) \subset \text{Span}(\mathcal{S}(X^{(j)}))_{j=1}^n$, as desired. \square

Remark. Lévy case is special case: simply take $B_i = 0$ for all i . The Brownian case of Corollary 7.4 is also a special case: simply take $X = 0$.

Remark. Since B_i are continuous, the assumption $[X, B_i] = 0$ is equivalent to $[X^c, B_i] = 0$. It is easy to see that that this assumption can be weakened to the following: $X^c = M + N$ for some $M, N \in \mathcal{H}^2$ such that $[M, B_i] = 0$ for all i and N is adapted to $\mathbb{F}(B_i)_{i=1}^{\infty}$.

Remark. We assumed $X_0 = 0$ throughout. This assumption is relaxed simply by requiring $X - X_0 \in \mathcal{C}$ instead of $X \in \mathcal{C}$.

Open Question: Assume X^c is a Brownian motion, $\langle X \rangle^{(n)}$ are adapted to $\mathbb{F}(X^c)$, and $\mathbb{F} = \mathbb{F}(X)$. If $\langle X \rangle^{(n)}$ are deterministic, then, as previously remarked, $X^c \in \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$. The question is to what extent this holds in general. It holds in the simple case where $X - X^c$ is a linear combination of independent Cox processes. When it holds, the conclusion of above theorem sharpens to $\mathcal{H}^2 = \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$ from $\mathcal{H}^2 = \mathcal{S}(X^c) \oplus \bigoplus_{j=1}^{\infty} \mathcal{S}(N_j)$.

The above result extends to the multivariate case by arguments already visited. For completeness, we include the proof.

Theorem 7.6. *Let $X_i \in \mathcal{C}$, $i \in \mathbb{N}$. Let $(B_i)_{i=1}^{\infty}$ be a sequence of Brownian martingales. Assume $[X_i, X_j] = [B_i, B_j] = 0$ for $i \neq j$, and for all i, j , $[X_i, B_j] = 0$, and $\langle X_i \rangle^{(j)}$ are adapted to $\mathbb{F}(B_k)_{k=1}^{\infty}$. Assume further that $\mathbb{F} = \mathbb{F}(X_i, B_i)_{i=1}^{\infty}$. Then*

$$\mathcal{H}^2 = \bigoplus_{k=1}^{\infty} \mathcal{S}(B_k) \oplus \bigoplus_{i,j=1}^{\infty} \mathcal{S}(N_{ij}),$$

where, for each i , the sequence $(N_{ij})_{j=1}^{\infty}$ is the strong orthogonalization of $(X_i^{(j)})_{j=1}^{\infty}$.

Proof. As above, we have $[X_i^{(j)}, B_k] = 0$, all i, j, k , and by Prop 4.5, we also have $[X_i^{(j)}, X_k^{(l)}] = 0$, all i, j, k, l . Hence all B_k and N_{ij} are strongly orthogonal to each other. Therefore $\bigoplus_{k=1}^{\infty} \mathcal{S}(B_k) \oplus \bigoplus_{i,j=1}^{\infty} \mathcal{S}(N_{ij})$ is a closed subspace of \mathcal{H}^2 , and it suffices to show it is dense.

Corollary 5.7 applied to the sequence $(X_i, B_i)_{i=1}^\infty$ implies that the linear span of martingales of the form $\overline{\mathbb{E}}((X_{j_1}(t_1) \cdots X_{j_n}(t_n))(B_{i_1}(s_1) \cdots B_{i_m}(s_m)) | \mathbb{F})$ is dense in \mathcal{H}^2 , as the indices run over $(t_1, \dots, t_n) \in [0, T]^n$, $(j_1, \dots, j_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$, and $(s_1, \dots, s_m) \in [0, T]^m$, $(i_1, \dots, i_m) \in \mathbb{N}^m$, $m \in \mathbb{N}$. As in Prop. 3.1, we have, $\text{Span}(\mathcal{S}(X_i^{(j)}))_{j=1}^n \subset \bigoplus_{j=1}^n \mathcal{S}(N_{ij})$. Therefore it suffices to show that

$$\overline{\mathbb{E}}((X_{j_1}(t_1) \cdots X_{j_n}(t_1))(B_{i_1}(s_1) \cdots B_{i_m}(s_m)) | \mathbb{F}) \in \bigoplus_{k=1}^\infty \mathcal{S}(B_k) + \text{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n.$$

Set $Y_t := X_{j_1}(t \wedge t_1) \cdots X_{j_n}(t \wedge t_n)$, and $\varphi := B_{i_1}(s_1) \cdots B_{i_m}(s_m)$. As $X_{j_1}(t_1) \cdots X_{j_n}(t_1) = Y_T$, we must show $\overline{\mathbb{E}}(\varphi Y_T | \mathbb{F}) \in \bigoplus_{k=1}^\infty \mathcal{S}(B_k) + \text{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$. As in the proof of Cor. 6.6, we break j_1, \dots, j_n into distinct elements, which by a permutation we may assume are next to each other. As such, we can write $Y = Y_1 \cdots Y_l$, where each Y_i is of the form $X_{j_i}(t \wedge t_{k_1}) \cdots X_{j_i}(t \wedge t_{k_{j_i}})$. By Prop. 6.2, each $Y_i \in \mathcal{C}_{m_i}^*(X_i)$ for some $m_i \geq 1$ with $\sum_i m_i = n$. So, $Y_i \in \mathcal{C}_n^*(X_i)$. So, each Y_i is a sum of terms of the form $A_i M_i$, where $A_i \in \mathcal{A}_{l_i}^*(X_i)$ and $M_i \in \mathcal{S}_{k_i}^*(X_i)$, $l_i + k_i = n$, $0 \leq l_i, k_i \leq n$. Note that $\varphi A_1(T) \cdots A_l(T)$ is in L^* and is also $\mathcal{G} := \mathcal{F}(B_i)_{i=1}^n$ -measurable because φ and all $A_i(T)$ have these two properties. Therefore, it is sufficient to show that for all $M_i \in \mathcal{S}_{k_i}^*(X_i)$, $k_i \leq n$, $i \leq l$ ($l \leq n$) and all \mathcal{G} -measurable $\xi \in L^*$, we have $\overline{\mathbb{E}}(\xi M_1(T) \cdots M_l(T) | \mathbb{F}) \in \bigoplus_{i=1}^\infty \mathcal{S}(B_i) + \text{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$.

Let $\mathbb{G} := \mathbb{F}(B_i)_{i=1}^\infty$. Set $N := \overline{\mathbb{E}}(\xi | \mathbb{G})$. As in the proof of Theorem 7.5, it follows that N is continuous and is actually \mathbb{F} -martingale; so $N = \overline{\mathbb{E}}(\xi | \mathbb{F})$. As before, the continuity of N and the assumption imply that $[N, X_i^{(j)}] = 0$, all i, j . Hence, $[N, M_i] = 0$, all i . Moreover, as $[X_i^{(k)}, X_j^{(l)}] = 0$ by Prop 4.5 for $i \neq j$, we get $[M_i, M_j] = 0$ for $i \neq j$. As $N, M_i \in \mathcal{H}^*$, these imply that $M := M_1 \cdots M_l$ and NM are martingales. Hence, $\overline{\mathbb{E}}(\xi M_1(T) \cdots M_l(T) | \mathbb{F}) = NM$, as both sides are martingales with the same value at T .

Now, $NM = \int NdM + \int M_- dN$. By Prop. 4.2, $\int M_- dN \in \mathcal{S}(N)$. Since $\bigoplus_{i=1}^\infty \mathcal{S}(B_i)$ is a stable subspace and contains N , it contains $\mathcal{S}(N)$. Therefore, $\int M_- dN \in \bigoplus_{i=1}^\infty \mathcal{S}(B_i)$. So, it remains to show $\int NdM \in \text{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$. But, $\int NdM = \int NM_{2-} \cdots M_{l-} dM_1 + \cdots + \int NM_{1-} \cdots M_{l-1-} dM_l$. Since $M_i \in \mathcal{S}_{k_i}^*(X_i)$, it is a sum of terms of the form $\int A_i dX_i^{(j_i)}$ with $A_i \in \mathcal{A}^*$ and $j_i \leq k_i \leq n$. But, by Cor. 4.7, $X_i^{(j_i)} \in \mathcal{H}^*$ and also all the products $M_1 \cdots M_{l-1} A_l N$, \dots , $M_2 \cdots M_l A_1 N$ are in \mathcal{C}^* . Hence, by Cor. 4.7, $\int M_1 \cdots M_{l-1} A_l N dX_l^{(j_l)} \in \mathcal{S}(X_l^{(j_l)}) \subset \text{Span}(\mathcal{S}(X_l^{(j)}))_{j=1}^n$, \dots , $\int M_2 \cdots M_{l-1} A_l N dX_1^{(j_1)} \in \mathcal{S}(X_1^{(j_1)}) \subset \text{Span}(\mathcal{S}(X_1^{(j)}))_{j=1}^n$. Hence, $\int NdM \in \text{Span}(\mathcal{S}(X_i^{(j)}))_{i,j=1}^n$, as desired. \square

8. EXPLICIT CHAOTIC EXPANSION OF POWERS

The following binomial expansion shows the relationship between integer powers and the power brackets. We set $[X]^{(1)} := X$ for any semimartingale. (Recall, $[X]^{(2)} := [X]$.)

Proposition 8.1. *Let X be a semimartingale with $X_0 = 0$. Then, for all $n \in \mathbb{N}$ we have,*

$$(8.1) \quad X^n = \sum_{i=0}^{n-1} \binom{n}{i} \int X_-^i d[X]^{(n-i)}.$$

Proof. By Itô's formula, and binomial expansion of $X^n = (X_- + \Delta X)^n$, we have

$$\begin{aligned} X^n - n \int X_-^{n-1} dX - \frac{1}{2}n(n-1) \int X_-^{n-2} d[X]^c \\ = \sum_{s \leq \cdot} (X_s^n - X_{s-}^n - n\Delta X_s X_{s-}^{n-1}) = \sum_{s \leq \cdot} \sum_{i=0}^{n-2} \binom{n}{i} X_{s-}^i (\Delta X_s)^{n-i}. \end{aligned}$$

For $i \leq n-3$, $\sum_{s \leq \cdot} X_{s-}^i (\Delta X_s)^{n-i} = \int X_-^i d[X]^{(n-i)}$. For $i = n-2$, the term $\int X_-^{n-2} d[X]^c$ combines with the term $\sum_{s \leq \cdot} X_{s-}^{n-2} (\Delta X_s)^2$ to give $\int X_-^{n-2} d[X]$. The formula follows. \square

Note, the leading term (corresponding to $i = 0$) is $[X]^{(n)}$.

We can substitute the same formula for X_-^i on the right-hand-side of Eq. (8.1). Repeating this procedure clearly leads to iterated integrals. We adopt the following notation. If H is a locally bounded predictable process, and X and Y are semimartingales, we denote

$$\int^- H dX := \left(\int H dX \right)_-, \quad \int \int^- H dX dY := \int \left(\int^- H dX \right) dY.$$

Note, $\int X_- dY = \int \int^- dX dY$ if $X_0 = 0$. For semimartingales Y_1, \dots, Y_n define inductively

$$\int \int^- \dots \int^- H dY_1 \dots dY_{n-1} dY_n := \int \left(\int^- \dots \int^- H dY_1 \dots dY_{n-1} \right) dY_n.$$

We denote multi-indices by $I = (i_1, \dots, i_p) \in \mathbb{N}^p$, and for integers $1 \leq p \leq n$, we set

$$\mathbb{N}_n^p := \{I = (i_1, \dots, i_p) \in \mathbb{N}^p : i_1 + \dots + i_p = n\}, \quad p, n \in \mathbb{N}.$$

Proposition 8.2. *Let X be a semimartingale with $X_0 = 0$. Then, for all $n \in \mathbb{N}$ we have,*

$$X^n = \sum_{p=1}^n \sum_{I \in \mathbb{N}_n^p} \frac{n!}{i_1! \dots i_p!} \int \int^- \dots \int^- d[X]^{(i_1)} \dots d[X]^{(i_{p-1})} d[X]^{(i_p)}.$$

Proof. Cases $n = 1, 2$ are clear, as the formula reads $X = \int d[X]^{(1)}$ and $X^2 = \int d[X]^{(2)} + 2 \int [X]_-^{(1)} d[X]^{(1)}$. For $n \geq 3$, each summand in Eq. (8.1) involving X_-^i , $i \geq 2$, can be expanded by Eq. (8.1) itself. Substituting and regrouping yields,

$$X^n = [X]_n + \sum_{i=1}^{n-1} \binom{n}{i} \int [X]_i d[X]_{n-i} + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \binom{n}{i} \binom{i}{j} \int \int^- X_-^j d[X]_{i-j} d[X]_{n-i}.$$

If $n = 3$, we are done. For $n \geq 4$, substituting for X_-^j , $j \geq 2$ from (8.1) and regrouping,

$$X^n = [X]_n + \sum_{i=1}^{n-1} \binom{n}{i} \int [X]_i d[X]_{n-i} + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \binom{n}{i} \binom{i}{j} \int \int^- [X]_j d[X]_{i-j} d[X]_{n-i}$$

$$+ \sum_{i=3}^{n-1} \sum_{j=2}^{i-1} \sum_{k=1}^{j-1} \binom{n}{i} \binom{i}{j} \binom{j}{k} \int \int^- \int^- X_-^k d[X]_{j-k} d[X]_{i-j} d[X]_{n-i}.$$

If $n = 4$, we are done. For $n \geq 5$, we continue substituting from (8.1) in this way, and clearly this procedure terminates by the n -th step, yielding then the desired result. \square

Combing the two propositions, one finds a similar iterated integral formula for $[X^n, X^m]$.²⁷

Substituting $[X]^{(i)} = \langle X \rangle^{(i)} + X^{(i)}$ into the term $\int \int^- \dots \int^- d[X]^{(i_1)} \dots d[X]^{(i_{p-1})} d[X]^{(i_p)}$, we get sums of expressions of form $\int \int^- \dots \int^- dY^{(i_1)} \dots dY^{(i_{p-1})} dY^{(i_p)}$, where each $Y^{(i)}$ can be either $\langle X \rangle^{(i)}$ or $X^{(i)}$. If $Y^{(i_p)}$ is $X^{(i_p)}$, then the quantity belongs to $\mathcal{S}_p^*(X)$. Otherwise, if $q < p$ is the largest integer such that $Y^{(i_q)}$ is $X^{(i_q)}$, then we are dealing with an expression of the form $\int \dots \int M_- d\langle X \rangle^{(i_{q+1})} \dots d\langle X \rangle^{(i_p)}$, where M of the form $M = \int Y_- dX^{(i_q)} \in \mathcal{S}_q^*(X)$, with $Y \in \mathcal{C}_{q-1}^*(X)$. In the proof of Proposition 6.2, we integrated by parts such expressions and used induction to show it belongs to $\mathcal{C}_p^*(X)$. The next result reports the explicit outcome of such repeated integration by parts, under a slightly more general setting, which applies to the present case with the A_j standing for the various $\langle X \rangle^{(i_j)}$.

Proposition 8.3. *Let M, A_1, \dots, A_n be semimartingales. Assume that $M_0 = 0$ (or all $A_i(0) = 0$) and all A_i are continuous and of finite variation. (So, $[A_i, M] = 0$.) Then*

$$\int \dots \int M_- dA_1 \dots dA_n = \sum_{p=0}^n \sum_{0=i_0 < i_1 < \dots < i_p \leq n} (-1)^p \left(\int A_{i_0, i_1} \dots A_{i_{p-1}, i_p} dM \right) A_{i_p, n},$$

where $A_{i,j}$ for $0 \leq i < j \leq n$ is defined by $A_{i,i} = 1$, ($A_{i-1,i} = A_i$) and

$$A_{i,j} := \int \dots \int A_{i+1} dA_{i+2} \dots dA_j. \quad (0 \leq i < j \leq n)$$

Proof. (Outline.) For $n = 1$ the formula reads $\int M_- dA_1 = MA_1 - \int A_1 dM$, which follows by integration by parts. For $n = 2$, we substitute this expression in $\int \int M_- dA_1 dA_2$. The first term $\int MA_1 dA_2$ is integrated by parts to give $M \int A_1 dA_2 - \int \int A_1 dA_2 dM$. The second term $-\int \int A_1 dM dA_2$ is likewise integrated by parts. The result is

$$\int \int M_- dA_1 dA_2 = M \int A_1 dA_2 - \int \int A_1 dA_2 dM - \int A_1 dM A_2 + \int A_1 A_2 dM.$$

For $n \geq 3$, one proceeds in a similar manner using integration by parts and induction.²⁸ \square

²⁷Namely, using the two propositions and the easily verified fact that $[[X]^{(i)}, [X]^{(j)}] = [X]^{(i+j)}$, we get

$$[X^n, X^m] = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{p=1}^{i+j} \sum_{I \in \mathbb{N}_{i+j}^p} \binom{n}{i} \binom{m}{j} \frac{(i+j)!}{i_1! \dots i_p!} \int \int^- \dots \int^- d[X]^{(i_1)} \dots d[X]^{(i_{p-1})} d[X]^{(i_p)} d[X]^{(n+m-i-j)}.$$

²⁸We point out that the continuity and finite variation assumption on A_i can be relaxed to $[A_i, M] = 0$ at the expense of left limits in the expressions. We also note that this is really an ordinary calculus result.

Note, the term corresponding to $p = 0$ is $M \int \cdots \int A_1 dA_2 \cdots dA_n$, while that corresponding to $p = n$ is $(-1)^n \int A_1 \cdots A_n dM$. As an example, say $n = 12 + 1$, $p = 4$, and $(i_1, i_2, i_3, i_4) = (2, 6, 7, 10)$. Then the corresponding term is

$$\int \left(\int A_1 dA_2 \left(\int \int \int A_3 dA_4 dA_5 dA_6 \right) A_7 \int \int A_8 dA_9 dA_{10} \right) dM \left(\int \int A_{11} dA_{12} dA_{13} \right).$$

The explicit form of the $\mathbb{F}(\langle X \rangle^{(i)})_{i=1}^n$ -adapted processes $A \in \mathcal{A}_n^*(X)$ appearing in the chaotic expansion of $X^n \in \mathcal{C}_n^*(X)$ is now clear: such A are products of iterated integrals of $\langle X \rangle^{(i)}$.

9. CONCLUDING REMARKS

The martingale representation result of [D] for finite activity processes mentioned in the introduction is seemingly of a quite different form than that of [N-S] or those here. However, the two forms can be tentatively reconciled through the language of random measures. Recast in this terms, Theorem 9 of [D] basically states that in the finite activity case a martingale can be represented as $W * (\mu - \nu)$ for a suitable predictable function $W(\omega, t, x)$. The [N-S] series representation $\sum_{n=1}^{\infty} H_n dN_n$ can be heuristically brought to this same form, once we consider that the Teugels martingale are given by $x^i * (\mu - \nu)$ and their strong orthogonalization N_n are basically of the form $(\sum_{i=1}^n K_{ni} x^i) * (\mu - \nu)$ for some predictable (constant in the Lévy case) processes $K_{n,i}$. In a loose sense, this then gives a representation of the form $W * (\mu - \nu)$ with the predictable function W given by the formal power series $W = \sum_{i=1}^{\infty} L_i x^i$, where, formally, $L_i = \sum_{n=i}^{\infty} H_n K_{n,i}$.

In closing, we pose an open question. We assumed throughout that angle brackets are continuous. This is a natural assumption and often met in practice. It is essentially a quasi-left-continuity assumption requiring all jumps be unpredictable. However, it may still be interesting to investigate the relaxation of this requirement within the present setting.

REFERENCES

- [D] Davis, M.H.A.: Martingale Representation and All That. *Advances in Control, Communication Networks, and Transportation Systems: In Honor of Pravin Varaiya*, E.H. Abed (Ed.), Systems and Control: Foundations and Applications Series, Birkhauser, Boston, (2005).
- [D2] Davis, M.H.A.: The representation of martingales of jump processes. *SIAM J. Control & Optimization* 14, 623-638, (1976).
- [D3] Davis, M.H.A.: Martingales of Wiener and Poisson processes. *J. Lon. Math. Soc.* (2), 13, 336-338, (1976).
- [E] Elliott, R.J.: *Stochastic Calculus and Applications*. Springer, (1982).
- [J-S] Jacod, J. and Shiryaev, A.N.: *Limit Theorems for Stochastic Processes*. Springer Verlag, Berlin (1987).
- [K-S] Karatzas, I. and Shreve, E: *Brownian Motion and Stochastic Calculus*. Springer-Verlag, second edition (1991).
- [N-S] Nualart, D. and Schoutens, W.: Chaotic and predictable representations for Lévy processes. *Stochastic Processes and their Applications* 90, 109-122, (2000).
- [Ø] Øksendal, B: *Stochastic Differential Equations*. Springer-Verlag, 6th ed. (2004).
- [P] Protter, P.: *Stochastic Integration and Differential Equations*. Springer-Verlag, Berlin (1992).