Genericity analysis of split bifurcations*

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Abstract

This paper analyzes the genericity of bifurcations of one-parameter families of smooth (C^1) vector fields that are embedded in an underlying multidimensional parameter space. Bifurcations with crossing equilibrium loci are called "split bifurcations." They include, for example, the pitchfork bifurcation and the transcritical bifurcation in one-dimensional systems. In a regular parameter space where the system's Jacobian matrix with respect to endogenous variables and parameters has full rank at every equilibrium for all parameter values, there is a generic (open and dense) set of one-parameter C^1 families of vector fields without split bifurcations. It is not difficult to obtain a regular parameter space when there are enough parameters. A regional migration model (à la Fujita, Krugman and Venables 1999) featuring the pitchfork bifurcation is presented as an example.

Keywords and Phrases: Bifurcation; Genericity analysis; Regular parameterization; Migration dynamics

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1 Introduction

The study of bifurcations[1] sheds light on the complex behavior of nonlinear dynamical systems. The main economic applications are in macroeconomic dynamics (see, for example, Rosser 1991, Barnett and He 1999, and Antinolfi, Keister and Shell 2001) and regional migration (see, for example, Krugman 1991, Fujita and Mori 1997, and Fujita, Krugman and Venables 1999). Bifurcation systems are usually studied in parameter spaces of low dimensions, especially in one-dimensional parameter spaces. For example, many one-dimensional systems with one-dimensional parameter spaces are characterized to have the pitchfork bifurcation or the transcritical bifurcation. A common feature of these bifurcations is that they have crossing equilibrium loci. We call this type "split bifurcations." At a split bifurcation, an equilibrium may branch off into one among many possible loci and economic theory yields no prediction of which route it will take.

The bifurcation diagram of a system, however, changes, sometimes structurally, as we change the parameter space in which it resides. This is addressed quite early in the literature that split bifurcations are not stable. For example, it is well-known that "all bifurcations of one-parameter families at an equilibrium with a zero eigenvalue[2] can be perturbed into saddle-node bifurcations" (Guckenheimer and Holmes 1997, p. 149). Yet, a genericity analysis of bifurcations is still lacking. Although split bifurcations illustrate interesting dynamic patterns, still the following issues are interesting. First, there are countless parameters in a real world system, a one-parameter system is actually a reduction taken from an underlying multi-dimensional parameter space. The choice of parameters apparently affects the dynamic behavior of a system. What choices of parameters exhibits the most general patterns of a system? Second, it is interesting to study the effects of one parameter at a time, especially if the parameter is a dominant force changing the economy. Yet, parameters do not change one at a time in the real world. A more plausible case is that many parameters change simultaneously resulting in a path in the parameter space. Third, different types of equilibrium diagrams, bifurcation or not, can be generated depending on the choice of parameter paths. Given enough parameters, can we distinguish the generic case from those that occasionally or almost never occur? This paper answers the above

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[1] Following Perko (2001), a dynamical system has a bifurcation if its qualitative behavior changes. This means it is structurally unstable.

[2] A split bifurcation occurs at an equilibrium with at least one zero eigenvalue since its Jacobian with respect to endogenous variables does not have full rank.

questions by characterizing the generic pattern of parameterized families of vector fields along general paths of parameter change in a large enough parameter space. The differentiable approach in general equilibrium theory is instrumental here (Debreu 1970, 1976, Dierker 1974 and Mas-Colell 1985). It deals with topological properties of equilibria defined by smooth equations in "regular parameter spaces," where a system's Jacobian matrix with respect to endogenous variables and parameters has full rank at every equilibrium for all parameter values (see Mas-Colell 1985, 5.8.12). A regular parameter space has good topological properties so that a generic set from a class of systems can be characterized.

To capture the reduction from many parameters to a one-parameter system, we define a "one-dimensional parameterization" of a multi-parameter dynamical system to be a smooth (C^1) map from $[0,1]$ to the parameter space. It is a smooth multi-dimensional path (curve) in the original parameter space which takes a "slice" of the original system. It generates a one-parameter family of vector fields parameterized in $[0,1]$. More precisely, let $p : [0,1] \to \Theta \subset \Re^m$ be a C^1 path in $\Re^m$. Then p takes a slice from the following system

$$\dot{x} = f\left(x, \theta\right)$$

where $f : X \times \Theta \to \Re^n$, $x \in X \subset \Re^n$, and generates a one-parameter system $\dot{x} = f\left(x, p\left(t\right)\right)$ in $[0,1]$.

Therefore, generic properties of one-parameter systems in an underlying parameter space are those common to a generic set of paths. We show that, in a regular parameter space, (i) there is a generic (open and dense in the C^1 norm) set of paths without split bifurcations,[3] (ii) for one-dimensional systems (the variable is in $\Re$), the set of paths without bifurcations is open. The reason to focus on C^1 maps is that any continuous map can be approximated arbitrarily close by a C^1 map (see for example, Theorem 2.6 in Hirsch 1976, ch.2). It is not difficult to obtain a regular parameter space when there are enough parameters. In standard general equilibrium models, endowment and preferences serve respectively as regular parameter spaces. For models whose parameter spaces are not regular or difficult to check, we demonstrate a technique to augment the parameter space into a regular one. It is practically applied to a regional migration model (as in Fujita, Krugman and Venables 1999, ch.

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[3] We did not find this result in standard references on nonlinear dynamical systems such as Peixoto (1973), Wiggins (1988, 1990), Guckenheimer and Holmes (1983), and Perko (2001). One possible reason is that a regular parameter space is not used outside the differentiable approach in general equilibrium.

5) which features a pitchfork bifurcation.

The genericity of split bifurcations affects our perception of the real world. For example, suppose a pitchfork bifurcation is the genuine representation of a real world system. Having the parameter crossing the critical value may have an indeterminate result (when the two outer spikes are stable). The equilibrium will move along one of a few possible paths. Since economic theory gives no prediction here, it is left to chance which route it will take. However, when the parameter space is regular, the generic case is the saddle-node bifurcation or no bifurcation (see Section 2). Thus, there is no room for chance; the equilibrium will move along a determined path depending on the value of another parameter that is not specified in the one-parameter model (see Section 2 and Figures 2, 3 for details). Consequently, omitting parameters is one of the causes of the occurrence of split bifurcations.

Section 2 characterizes the generic set of one-parameter systems. Section 3 presents an example of a regular parameter space for a regional migration model. Section 4 concludes.

2 The Genericity Analysis

To illustrate the role of an underlying parameter space, let's consider the following one-dimensional system with two parameters $(a, b) \in \Re^2$.

$$\dot{x} = a + bx - x^3$$

where $x \in \Re$. This system exhibits the standard form of the pitchfork bifurcation when $a = 0$ (see Figure 1; the solid and dashed lines indicate stable and unstable equilibria respectively). To show that this picture is not robust, we perturb a to 0.005 and obtain Figure 2 instead. The general contour is still the same and the stable and unstable regions change slightly, but the equilibrium loci do not cross each other. This is a saddle-node bifurcation. The same happens when we perturb a to -0.005 in Figure 3. The three-dimensional equilibrium diagram with two-dimensional parameter space (a, b) is plotted in Figure 4. Consider any path in the (a, b) space and the equilibrium diagram generated by taking a slice of the three-dimensional picture along the path. We can see that there is a pitchfork bifurcation only in some paths passing $(0, 0)$.

[Figure 1 Here]

[Figure 2 Here]
[Figure 3 Here]
[Figure 4 Here]

The transcritical bifurcation is not robust either. Consider the following system with $(a,b) \in \Re^2$.

$$\dot{x} = a + bx - x^2$$

where $x \in \Re$. The one-parameter family of vector fields with $a = 0$ contains a transcritical bifurcation at $b = 0$ (Figure 5). However, if we perturb a to 0.005, the resulting picture has no bifurcation (Figure 6). And when a is perturbed to -0.005, it has two saddle-node bifurcations (Figure 7). The three-dimensional equilibrium diagram is presented in Figure 8. The transcritical bifurcation only appears in some paths passing $(0,0)$.

[Figure 5 Here]
[Figure 6 Here]
[Figure 7 Here]
[Figure 8 Here]

These examples illustrate that split bifurcations, those with crossing equilibrium loci, are not robust in a parameter space of higher dimensions. This raises the following question: "what kind of dynamic behavior is typical given enough parameters?" To make more rigorous arguments, consider a general C^1 dynamical system.[4]

$$\dot{x} = f(x, \theta)$$

where $f : X \times \Theta \to \Re^n$, $x \in X \subset \Re^n$, X is an open subset of $\Re^n$, and $\Theta \subset \Re^m$ is an m-dimensional parameter space with elements θ. We examine one-parameter families of vector fields generated by one-dimensional parameterizations (1-parameterizations) in Θ.

Definition 1. A *1-parameterization* in Θ is a C^1 map $p : [0,1] \to \Theta$.

The space of 1-parameterizations is endowed with the C^1 norm. A 1-parameterization p defines a one-parameter family of vector fields $f(x, p(t))$ where $t \in [0,1]$. To describe bifurcations with crossing equilibrium loci, we need the following notions:

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[4] C^1 is required for the existence and uniqueness of a solution (Perko 2001, 2.2).

$E(p) = \{(x,t) \in X \times [0,1] \,|\, f(x, p(t)) = 0\}$ is the *equilibrium diagram* of f under 1-parameterization p. Each element $(x,t) \in E(p)$ is an equilibrium point of f for the parameter value $p(t)$.

An *equilibrium locus* from an equilibrium point $(x,t) \in E(p)$ is the image of a continuous map $e : [0,1] \to X \times [0,1]$ such that $e(0) = (x,t)$ and $e(z) \in E(p)$ for $z \in [0,1]$.

Definition 2. 1-parameterization p has a *split bifurcation* at $\left(\hat{x}, \hat{t}\right) \in E(p)$ (or at parameter value $\hat{t}$) if for any neighborhood around $\left(\hat{x}, \hat{t}\right)$ there are more than two distinct equilibrium loci from $\left(\hat{x}, \hat{t}\right)$.

It is necessarily a bifurcation since the number of equilibria changes in the neighborhood of $\hat{t}$. Split bifurcations include, for example, the pitchfork bifurcation and the transcritical bifurcation in one-dimensional systems. Next lemma presents a necessary condition for split bifurcations.

Lemma 1. *If p has a split bifurcation at $\left(\hat{x}, \hat{t}\right)$, then $D_{(x,t)} f\left(\hat{x}, p\left(\hat{t}\right)\right)$ has rank at most $n-1$.*

Proof. If $D_{(x,t)} f\left(x, p\left(\hat{t}\right)\right)$, an $n \times (n+1)$ matrix, has full rank then it has n linearly independent columns. Without loss of generality, suppose $D_{(x_{-n},t)} f\left(\hat{x}, p\left(\hat{t}\right)\right)$ has full rank (where $x_{-n} = (x_1, ..., x_{n-1})$). By the implicit function theorem, $f(x, p(t)) = 0$ can be locally solved by a C^1 function of x_n. This means $E(p)$ is a C^1 curve in a neighborhood of $\left(\hat{x}, \hat{t}\right)$. Therefore, in a small neighborhood, there can be only two distinct equilibrium loci from $\left(\hat{x}, \hat{t}\right)$. ▮

In the following, we show that, when the underlying parameter space is regular, there is an open and dense set of 1-parameterizations without split bifurcations.

Definition 3. The parameter space Θ is *regular* for f if $D_{(x,\theta)} f(x,\theta)$ has full rank whenever $f(x,\theta) = 0$ for all $(x,\theta) \in X \times \Theta$.

We say that 1-parameterization p is *regular* if $D_{(x,t)} f(x, p(t))$ has full rank (which is n) whenever $f(x, p(t)) = 0$ for $t \in [0,1]$. Therefore, regular 1-parameterizations do not have split bifurcations. We show, moreover, that they are generic in a regular parameter space.

Proposition 1. *In a regular parameter space Θ, the set of regular 1-parameterizations is open and dense.*

Proof. Since a perturbation (in the C^1 norm) yields small changes in $f(x, p(t))$ and $D_{(x,t)} f(x, p(t))$, openness is straightforward by their continuity. To show density, we need the following theorem (see Guillemin and Pollack 1974, p. 68, and Mas-Colell 1985, p. 320): For a C^r map $f : M \rightarrow N$ between manifolds, $y \in N$ is a *regular value* if $Df(x)$ has full rank whenever $f(x) = y$.

Transversality Theorem. *Suppose that $f : X \times S \rightarrow \Re^n$ is a C^r map where X, S are C^r boundariless manifolds with $r > \max\{0, \dim(X) - n\}$, let $f_s(x) = f(x, s)$, $f_s : X \rightarrow \Re^n$. If $c \in \Re^n$ is a regular value for f, then except for s in a set of measure zero in S, c is a regular value for f_s.*

For 1-parameterization p, construct the following map $\phi : E \times [0, 1] \times \Re^m \rightarrow \Re^n$,

$$\phi(x, t, a) = f(x, p(t) + a).$$

Apparently, we have $D_{(x,a)}\phi = D_{(x,\theta)} f(x, \theta)$ whenever $\theta = p(t) + a$. And $D_{(x,\theta)} f(x, \theta)$ has full rank whenever $f(x, \theta) = f(x, p(t) + a) = 0$ by that Θ is regular. This means $D_{(x,a)}\phi$, and hence $D_{(x,t,a)}\phi$, has full rank whenever $\phi(x, t, a) = 0$. By the Transversality Theorem, for all a except for a set of measure zero, $D_{(x,t)}\phi(x, t, a)$ has full rank whenever $\phi(x, t, a) = 0$. So, we can find $\bar{a}$ arbitrarily close to 0 such that $D_{(x,t)}\phi(x, t, \bar{a})$ has full rank whenever $\phi(x, t, \bar{a}) = 0$. Let $p'(t) = p(t) + \bar{a}$. Then p' is a regular 1-parameterization that is arbitrarily close to p. ∎

Thus, generic paths do not have split bifurcations.[5] What about the occurrence of bifurcations? Let θ define a dynamic economy with vector field $f(., \theta)$. An economy θ is *regular* if 0 is a regular value of $f(., \theta)$. In a regular parameter space, f has a bifurcation at θ if θ is not a regular economy (Rosser 1991, ch.2). It is easy to see this: When 0 is not a regular value of $f(., \theta)$, one of the eigenvalues of $D_x f(x, \theta)$ is 0. Since regular economies are of full Lebesgue measure in a regular parameter

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[5] The set of 1-parameters with split bifurcations is a nowhere dense set.

space (Debreu 1970, 1976), we can perturb θ into a regular one so that 0 is not an eigenvalue anymore. This results in a qualitative change in the dynamics.

The vector fields of regular economies, however, can be structurally stable or unstable. This is because we only have the full rank of the Jacobian matrix (with respect to endogenous variables), while structural stability is determined by finer properties such as its eigenvalues. Yet, for one-dimensional dynamical systems, we can show that 1-parameterizations with bifurcations are not generic: When $n = 1$, an equilibrium is hyperbolic[6] if and only if $D_x f$ has full rank. Moreover, a regular economy has a finite number of equilibria (Debreu 1970). This means θ is not a bifurcation value of f if and only if it is a regular economy (Perko 2001, 4.1, Theorem 3). We have the following results.

Lemma 2. *When $n = 1$, θ is not a bifurcation value if and only if it is a regular economy.*

Proposition 2. *When $n = 1$, the set of 1-parameterizations without bifurcations is open in Θ.*

Proof. 1-parameterization p has no bifurcations if and only if for all $t \in [0, 1]$, $\frac{\partial}{\partial x} f(x, p(t)) \neq 0$ whenever $f(x, p(t)) = 0$. A perturbation of p yields small changes in $f(x, p(t))$ and $\frac{\partial}{\partial x} f(x, p(t))$, so openness is straightforward by continuity. ▮

Therefore, the set of 1-parameterizations with bifurcations of a one-dimensional system is neither open nor dense since the complement set is open. In general (for all n), equilibrium diagrams of regular 1-parameterizations are generic. In the proof of Lemma 1, we show that for a regular 1-parameterization p, $E(p)$ is a C^1 curve in a neighborhood of any $(\hat{x}, \hat{t}) \in E(p)$. Thus, a generic equilibrium diagram contains in its interior open segments of nonintersecting smooth curves. This diagram is a one-dimensional differentiable manifold. When $n = 1$, this allows only the saddle-node bifurcation (Figures 2, 3 and 7) or no bifurcation (Figure 6).

Economic models often have regular parameter spaces. For example, endowment and preferences are commonly used as regular parameter spaces for exchange economies respectively, and fixed inputs in production are used for production economies.

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[6] An equilibrium point x of vector field $f(., \theta)$ is *hyperbolic* if none of the eigenvalues of $D_x f(x, \theta)$ has zero real part.

Unfortunately, sometimes it is difficult to check regularity in application, and sometimes a model may have a parameter space that is not regular. But in general it is not difficult to obtain a regular parameter space when there are enough parameters. We introduce a technique to augment a parameter space into a regular one. It is practically applied to a regional migration model (as in Fujita, Krugman and Venables 1999). Other examples can be found in Berliant and Zenou (2000) and Berliant and Kung (2004).

Suppose originally a model has a parameter space $\Theta_e \subset \Re^m$ with elements θ_e. The equilibrium of a model is defined by $f(x, \theta_e) = 0$ where $f : X \times \Theta_e \rightarrow \Re^n$ and $X \subset \Re^n$. We augment this economy with a parameter space Θ_r of at least n dimensions (with elements θ_r) such that the original system f is parameterized at a value $\bar{\theta}_r \in \Theta_r$ where $f(x, \theta_e) = \hat{f}\left(x, \theta_e, \bar{\theta}_r\right)$ for all $(x, \theta_e) \in X \times \Theta_e$. Let $\Theta = \Theta_e \times \Theta_r$ and define $\hat{f} : X \times \Theta \rightarrow \Re^n$. The desired Θ_r need to be regular for $\hat{f}(., \theta_e, .)$ for all $\theta_e \in \Theta_e$; that is, $D_{(x,\theta_r)}\hat{f}(x, \theta_e, \theta_r)$ has full rank whenever $\hat{f}(x, \theta_e, \theta_r) = 0$ for all $(x, \theta) \in X \times \Theta$. Consequently, Θ is regular since $D_{(x,\theta_e,\theta_r)}\hat{f}(x, \theta_e, \theta_r)$ has rank as high as the rank of $D_{(x,\theta_r)}\hat{f}(x, \theta_e, \theta_r)$ for any (x, θ). The task is reduced to finding a set of parameters of cardinality n no matter how many parameters there are originally.

The simplest example of regular parameters are "error terms". Consider the following dynamical system

$$\dot{x} = f(x, \theta) + \varepsilon$$

with errors $\varepsilon \in A$ where A is an open subset of $\Re^n$ containing the origin. Let $\hat{f}(x, \theta, a) = f(x, \theta) + a$, then $D_a\hat{f}(x, \theta, a)$ is the $n \times n$ identity matrix for all values of (x, θ, a), which means $D_{(x,a)}\hat{f}(x, \theta, a)$ always has rank n. So, $\Theta \times A$ is a regular parameter space.

3 Example: A Regional Migration Model

Economic activities are not distributed uniformly in space. Nonagricultural production often concentrates in a few regions, resulting in a core-periphery pattern. Studies explain questions such as why producers agglomerate and how one region come to dominate the others and become a manufacturing core. Krugman (1991), and Fujita, Krugman and Venables (1999) explain the emergence of the core-periphery pattern with the dynamics of a pitchfork bifurcation:[7] They consider an economy consisting

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[7] With boundary equilibria, it is also called the "tomahawk" bifurcation.

of two regions with equal resources. There are equal populations of immobile farmers in both regions who produce a homogeneous agricultural good, and also a population of mobile manufacturing workers that migrate between regions. Workers move to the region where they have a higher utility level. The transportation of manufactured goods across regions bears a cost while that of the agricultural good does not. Firms produce differentiated products with increasing returns to scale technologies and they compete monopolistically. There are two types of pecuniary externalities that generate the forces sustaining production agglomeration. They result in the positive feedback that comes from firms locating near each other. First, manufacturing production will concentrate where there is a large market with many workers consuming manufactured goods. Second, workers will move to where the production concentrates because the manufactured goods are cheaper there. This approach is known as the new economic geography.

With other parameters fixed, when the transportation cost is high, the symmetric equilibrium, where both regions have the same manufacturing populations, is the only equilibrium and it is stable. When the transportation cost is moderate, two other stable equilibria emerge; when this happens, one of the two regions attracts all the manufacturing production. When the transportation cost is low, the symmetric equilibrium becomes unstable and the only stable equilibria are the two core-periphery equilibria. We first introduce the static equilibrium where markets clear and agents do not move, then migration dynamics are considered to describe the dynamic equilibrium of the economy.

The original model is introduce in the following; more parameters are needed to provide a regular parameterization. To facilitate comparison, these parameters will be added later. There are two regions in the economy which are denoted by $i \in \{1, 2\}$. There are two types of commodities: a homogeneous agricultural good and differentiated manufactured goods. There is a continuum of manufactured goods of size $n \in \Re_+$, which is determined endogenously. Each manufactured good is denoted by $j \in [0, n]$. Let $p_i^A \in \Re_{++}$ denote the local price of the agricultural good, and let $p_i(j)$, where $p_i : [0, n] \to \Re_{++}$ is a measurable function, denote the local price of each manufactured good j in region i. There are two types of consumers: immobile farmers of population L_i^A in region $i \in \{1, 2\}$, and mobile workers of population L^M who migrate between regions. Each of them is endowed with one unit of labor.

Let $A \in \Re_+$ denote the quantities of the agricultural good, and let $m(j)$, where $m : [0, n] \to \Re_+$ is a measurable function, denote the quantity of manufactured good

j. All consumers have the same utility function

$$u(m, A) = M^{\mu} A^{1-\mu}$$

where $M = \left[\int_0^n m(j)^{\rho}\, dj\right]^{\frac{1}{\rho}}$ and $0 < \mu, \rho < 1$. A consumer in region i with income Y solves the following problem.

$$\begin{array}{c} \underset{A,\, m(j) \in \Re_{+}}{Max} \;\; u(m, A), \\ s.t.\; p_i^A A + \int_0^n p_i(j)\, m(j)\, dj = Y. \end{array} \tag{1}$$

The demand functions are

$$\begin{array}{c} \hat{A}_i(Y) = (1-\mu)\, Y / p_i^A, \\ \hat{m}_i(j, Y) = \mu Y G_i^{\frac{\rho}{1-\rho}} / p_i(j)^{\frac{1}{1-\rho}}, \end{array}$$

where $G_i = \left[\int_0^n p_i(j)^{\frac{\rho}{\rho-1}}\, dj\right]^{\frac{\rho-1}{\rho}}$ is the manufacturing price index.

The agricultural good is produced with labor by farmers with a one-to-one input-output ratio. The transportation of the agricultural good bears no cost. Thus, the equilibrium agricultural price is the same in both regions by no arbitrage; let $p_1^A = p_2^A = p^A$. Farmers retain all the revenue; they have income p^A.

Manufactured goods are produced by firms that employ mobile workers. Labor is the only input required. All firms have the same inverse production function

$$l = F + cq$$

where $F, c > 0$ are the fixed and the marginal input requirements. l units of labor are required for q units of output. The production technology exhibits increasing returns to scale due to the fixed costs. There is free entry into the market. Because of increasing returns to scale, each j-good is produced by and is the only product of an operating firm. Operating firms choose locations and engage in Chamberlinian monopolistic competition. Each firm chooses a location and charges a uniform free on board (f.o.b.) price for its product. Firms make decisions simultaneously. Let $w_i \in \Re_{++}$ denote the wage rate in region i. Suppose a firm locates in region i, charges price p, pays wage w_i, and sells output $q(p)$, where $q : \Re_{++} \to \Re$ is the demand of consumers. Its profit is

$$\pi_i(p) = pq(p) - w_i\left[F + cq(p)\right].$$

A firm in region i solves the following problem.

$$\underset{p \in \Re_{++}}{Max}\; \pi_i(p). \tag{2}$$

Because of the assumed constant elasticity utility function and the iceberg transportation cost (detailed later), the elasticity of demand facing a firm is independent of the locations of its consumers. (This is widely known; see Fujita et al. 1999.) A monopolistically competitive firm charges a price marked up from the marginal cost. The profit-maximizing price for a firm in region i is $p_i = cw_i/\rho$. Its maximized profit is

$$\pi_i = \frac{1-\rho}{\rho} cw_i \left[q - \frac{F}{(1-\rho)\, c} \right].$$

The transportation cost of manufacturing goods takes the Samuelson iceberg form. If one unit of good is shipped across regions, $1/T$ unit arrives. Since firms are identical and their behavior differs only in location, we label firms and their products with their locations. This simplifies the notation to $j \in \{1, 2\}$. Let p_i^j denote the price of region j products in region i, and $\hat{m}_i^j(Y)$ denote the demand for region j products of region i consumers (to replace $p_i(j)$ and $\hat{m}_i(j, Y)$). Let n_i denote the number of firms in region i. The total number of operating firms equals the total variety of products; $n_1 + n_2 = n$. Note that $G_i = \left[n_1 \left(p_i^1\right)^{\frac{\rho}{\rho-1}} + n_2 \left(p_i^2\right)^{\frac{\rho}{\rho-1}} \right]^{\frac{\rho-1}{\rho}}$.

A region i firm charges a free-on-board price $p_i = cw_i/\rho$. Thus, $p_i^i = p_i$ and $p_i^j = p_j T$ for $j \neq i$ by no arbitrage. Substituting Y with w_i, we have region i workers' indirect utility:

$$v_i = \mu^\mu (1-\mu)^{1-\mu} w_i G_i^{-\mu} \text{ for } i \in \{1, 2\}.$$

Workers are freely mobile. They choose a region that offers a higher utility level.

The above is the standard model of the new economic geography. However, it is difficult to check whether the parameters introduced so far constitute a regular parameter space. We augment the system with more parameters. Let Θ be an open subset of $\Re^3_{++}$, its elements are denoted by $\theta = (v, \gamma)$, where $v \in \Re^2_{++}$ and $\gamma \in \Re_{++}$. These parameters enter the model in the following way. (i) v parameterizes "regional fixed input": The fixed labor input of firms in region i is $F + v_i$. Note that although firms' profit function is changed to

$$\pi_i(p) = pq(p) - w_i \left[F + v_i + cq(p) \right],$$

their chosen prices (cw_i/ρ) are not affected. (ii) γ parameterizes "regional amenity": Workers have preferences over regions in the following way. If a worker lives in region 2, her utility function is unchanged. If she lives in region 1, her utility is factored up by $1/\gamma$. The new utility function of region 1 workers is

$$\frac{1}{\gamma} u\left(m_1^1, m_1^2, A_1\right).$$

This captures regional differences such as weather and the environment.

An *economy* is an 9-tuple $\left(L_1^A, L_2^A, L^M, \mu, \rho, c, F, T, \theta\right)$. Let L_i^M denote the worker population in region i, and A_i, m_i^j denote their consumptions of agricultural and manufactured goods respectively. Let A_{Ai}, m_{Ai}^j denote the consumption of farmers in regional i. Let q^j denote the output level of region j firms. An *allocation* in the economy is an 18-tuple $\left\{L_i^M, A_i, A_{Ai}, \left\{m_i^j, m_{Ai}^j\right\}_{j=1}^2, n_i, q^i\right\}_{i=1}^2$ (equal treatment of consumers and firms at the same location is implied; this is harmless since they will behave the same in equilibrium). A *feasible* allocation satisfies the following constraints:

$$L_1^M + L_2^M = L^M. \tag{3}$$

$$L_1^M m_1^1 + L_1^A m_{A1}^1 + L_2^M m_2^1 T + L_2^A m_{A2}^1 T - q^1 = 0. \tag{4}$$

$$L_1^M m_1^2 T + L_1^A m_{A1}^2 T + L_2^M m_2^2 + L_2^A m_{A2}^2 - q^2 = 0. \tag{5}$$

$$L_1^M A_1 + L_1^A A_{A1} + L_2^M A_2 + L_2^A A_{A2} = L^A. \tag{6}$$

Equation (3) balances the total worker population and the total demand for workers. Equations (4) and (5) balance the consumption of manufactured goods and their production. Equation (6) balances agricultural consumption and production.

Facing prices p^A, p_1, p_2, w_1, and w_2, the following conditions are satisfied in equilibrium. (Note that we have already imposed no-arbitrage on the transportation of goods.) The free entry of new firms drives the profit of operating firms down to zero.

$$\pi_1 = \pi_2 = 0. \tag{7}$$

Workers are freely mobile and identical, so their utility levels are equal if there are workers living in both regions.

$$v_1 = v_2, \text{ if } L_1^M, L_2^M > 0. \tag{8}$$

Note that workers' utility v_i is not defined if there are no workers in region i. To handle the boundary equilibria, we may define the potential wage as the limit of the equilibrium wage when worker population goes to zero. Then, the potential utility is derived accordingly. Having all workers in one region constitutes an equilibrium if none of them wants to move out; that is, the potential utility in the other region is not higher. Since the pitchfork bifurcation is interior, we focus on interior equilibria only: $L_1^M, L_2^M > 0$.

An equilibrium is a list of prices and a feasible allocation such that conditions (1), (2), (7), and (8) are satisfied. We simplify the system in the following. First, by (1), the demand of workers for the agricultural good and manufactured goods are $A_i = (1-\mu)\, w_i / p^A$ and $m_i^j = \mu w_i G_i^{\frac{\rho}{1-\rho}} \left(p_i^j\right)^{\frac{-1}{1-\rho}}$ respectively, and the demand of farmers for the two types of goods are $A_{Ai} = (1-\mu)$ and $m_{Ai}^j = \mu p^A G_i^{\frac{\rho}{1-\rho}} / \left(p_i^j\right)^{\frac{1}{1-\rho}}$. By (2), $p_i = c w_i / \rho$. Then by (7),

$$\begin{aligned} q^1 &= q^2 = \frac{\rho\,(F+v_i)}{c\,(1-\rho)}, \\ n_i &= \frac{L_i^M}{(F+v_i) + c\frac{\rho(F+v_i)}{c(1-\rho)}} = \frac{L_i^M\,(1-\rho)}{F+v_i}. \end{aligned}$$

Let $\bar{q} = \frac{\rho(F+v_i)}{c(1-\rho)}$, $\lambda = L_1^M / L^M$. $\lambda \in (0,1)$ by (3) and the fact that we study interior equilibria only. Equations (4), (5) and (6) are linearly dependent because of Walras' law; we take the last one as redundant. Plug the above results into (4) and (5), and we have

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda)\, L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho} T\right)^{\frac{1}{1-\rho}}} - \bar{q} = 0, \quad (9)$$

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda)\, L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_2}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_2}{\rho}\right)^{\frac{1}{1-\rho}}} - \bar{q} = 0. \quad (10)$$

Equation (8) can be replaced with

$$\frac{w_1}{\gamma} - w_2 G_2^{-\mu} G_1^{\mu} = 0. \quad (11)$$

Finally, normalizing prices to $w_1 + w_2 + p^A = 1$ and letting $p^A = 1 - w_1 - w_2$, the remaining is a system of three variables and three equations. Let $w = (w_1, w_2)$. Note that $w \in S = \left\{ w \in (0,1)^2 \,|\, w_1 + w_2 < 1 \right\}$.

Definition 4. A *static equilibrium* is a list $(w, \lambda) \in S \times (0,1)$ that satisfies (9), (10), and (11). (Note that $\lambda = L_1^M / L^M$, $p^A = 1 - w_1 - w_2$, and $\bar{q} = \frac{\rho(F+v_i)}{c(1-\rho)}$.)

It is common practice in the literature to use a two-stage equilibrium: At the first stage, mobile agents choose a region to reside; at the second stage, commodity markets reach an equilibrium given the current population distribution. Let f_1, f_2, g

denote the left-hand side functions of (9), (10), and (11), respectively. The material balance conditions $f_i(w, \lambda) = 0$, $i \in \{1, 2\}$, obtain at the second stage, and the migration balance condition $g(w(\lambda), \lambda) = 0$ obtains at the first, where $w(\lambda)$ is the commodity equilibrium for population λ. Given a population distribution, the adjustment to equilibrium price $w(\lambda)$ is assumed to take no time. Knowing the utility levels determined by the price equilibrium, workers migrate to regions that offer higher utility levels. This two-stage approach is only valid if there is a unique equilibrium at the second stage for any population distribution. A necessary condition for $f_i = 0$ to have a unique solution is that the index $|-D_w f(w, \lambda)| > 0$ at the equilibrium (Kehoe 1985, 1998).[8] We will use this condition later.

Adjustment dynamics are applied to describe the movement of mobile agents.

$$\dot{\lambda} = h(\lambda)$$

where $h(\lambda)$ is a vector field. A *dynamic equilibrium* of this economy is a profile of population ratio $\lambda \in (0, 1)$ such that $h(\lambda) = 0$. Desirable dynamics are C^1 and satisfy the following conditions:

(i) $h(\lambda) = 0$ if and only if $g(w(\lambda), \lambda) = 0$.
(ii) $D_\lambda h(\lambda) \lesseqqgtr 0$ if and only if $\frac{d}{d\lambda} g(w(\lambda), \lambda) \lesseqqgtr 0$.

We need h to be C^1 for the existence and uniqueness of a solution. Its equilibria should coincide with the static equilibria and its Jacobian matrix preserves the signs of the Jacobian of $g(w(\lambda), \lambda)$. These requirements rule out strange dynamics that alter the nature of the original static economy.

A common example is the replicator dynamics (Weibull 1995, Fujita et al. 1999, ch. 5): Let s denote the profile of population shares of m regions; $s \in (0, 1)^m$ and $\sum_{j=1}^{m} s_j = 1$.

$$\dot{s}_j = s_j \left[v_j(q, s) - \sum_{i=1}^{m} s_i v_i(q, s) \right] \text{ for } i = 1, \ldots, m$$

where $v_j(s)$ is the utility level in region j when the population distribution is s. One of the above equations is redundant since the population shares sum up to one. In

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[8] Note that since we deal with $w \in S$ instead of a simplex or a ball, our index is defined as $sign(|-D_w f|)$. The index theorem applies to this system (Mas-Colell 1985, 5.6.1).

our two region case (note that $\lambda = s_1$ and the equation for $\dot{s}_2$ is redundant), the replicator dynamics is

$$h\left(\lambda\right) = \lambda\left[v_1\left(\lambda\right) - \lambda v_1\left(\lambda\right) - \left(1-\lambda\right)v_2\left(\lambda\right)\right].$$

In the next Lemma, we show that Θ is regular. The augmented parameter space, with the new and original parameters combined, is also regular.

Lemma 3. *Θ is a regular parameter space for vector field $h\left(\lambda;\theta\right)$.*

Proof. The conditions imposed on h ensure that Θ is a regular parameter space for $h\left(\lambda,\theta\right)$ if and only if it is regular for $g\left(w\left(\lambda\right),\lambda,\theta\right)$. Define $F = \left(f_1, f_2, g\right)$, $F : \Re^3_{++} \to \Re^3$. We show that for all θ, for all $\left(w^*,\lambda\right)$ such that $F\left(w^*,\lambda\right) = 0$ (i.e., $w^* = w\left(\lambda\right)$), $\left|D_{(w,\lambda)}F\left(w^*,\lambda\right)\right| = 0$ if and only if $\frac{d}{d\lambda}g\left(w\left(\lambda\right),\lambda\right) = 0$.

Since $\left|D_w f\left(w\left(\lambda\right),\lambda\right)\right| \neq 0$, by the implicit function theorem,

$$D_\lambda w\left(\lambda\right) = -\left[D_w f\left(w\left(\lambda\right),\lambda\right)\right]^{-1} D_\lambda f\left(w\left(\lambda\right),\lambda\right).$$

And

$$\begin{aligned}\frac{d}{d\lambda}g\left(w\left(\lambda\right),\lambda\right) &= D_w g\left(w\left(\lambda\right),\lambda\right) D_\lambda w\left(\lambda\right) + D_\lambda g\left(w\left(\lambda\right),\lambda\right),\\ &= -D_w g\left(w\left(\lambda\right),\lambda\right)\left[D_w f\left(w\left(\lambda\right),\lambda\right)\right]^{-1} D_\lambda f\left(w\left(\lambda\right),\lambda\right) + D_\lambda g\left(w\left(\lambda\right),\lambda\right).\end{aligned}$$

Expand $\left|D_{(w,\lambda)}F\right|$ along its third row:

$$\begin{aligned}\left|D_{(w,\lambda)}F\right| &= \begin{vmatrix} D_w f & D_\lambda f \\ D_w g & D_\lambda g \end{vmatrix}\\ &= -D_\lambda g\left|D_w f\right| + \textstyle\sum_{k=1}^{2}\left(-1\right)^{k+1} D_{w_k} g\left|M_k\right|.\end{aligned}$$

where M_k is the matrix obtained from $D_w f$ by eliminating its kth column and adding $D_\lambda f$ as the last column.

Let x be the vector that solves

$$\left(D_w f\right)x = D_\lambda f.$$

Then $x = \left[D_w f\right]^{-1} D_\lambda f$, and by Cramer's rule, $x_k = \left|D_w f\right|^{-1}\left|N_k\right|$ where N_k is the matrix obtained from $D_w f$ by replacing its kth column with $D_\lambda f$. Notice that M_k and N_k differ only in the positions of their columns. N_k can be obtained from M_k

by switching the last column, $D_\lambda f$, with the preceding column $k-1$ times. Thus, $|N_k| = (-1)^{k-1}|M_k|$. So,

$$\begin{aligned} \left|D_{(w,\lambda)}F\right| &= -D_\lambda g\,|D_w f| + \textstyle\sum_{k=1}^{n}(-1)^{2k} D_{w_k} g\,|N_k| \\ &= -D_\lambda g\,|D_w f| + \textstyle\sum_{k=1}^{n} D_{w_k} g\,|D_w f|\,x_k \\ &= -D_\lambda g\,|D_w f| + |D_w f|\,D_w g\cdot x \\ &= -|D_w f|\left[D_\lambda g - D_w g\,[D_w f]^{-1} D_\lambda f\right] \\ &= -|D_w f|\,\frac{d}{d\lambda} g\left(w(\lambda),\lambda\right). \end{aligned}$$

Next, we show $\left|D_{(w,\lambda)}F\left(w^*,\lambda\right)\right| \neq 0$ for any θ.

$$D_\theta F\left(w,\lambda,\theta\right) = \begin{pmatrix} -\frac{\rho}{c(1-\rho)} & 0 & 0 \\ 0 & -\frac{\rho}{c(1-\rho)} & 0 \\ 0 & 0 & w_1 \end{pmatrix}.$$

So, $D_{(w,\lambda,\theta)}F = \left(D_{(w,\lambda)}F, D_\theta F\right)$ always has full rank at every equilibrium (w^*,λ) for all $\theta \in \Theta$ and Θ is regular for $g\left(w(\lambda),\lambda,\theta\right)$. ▮

The key to this construction is that these new parameters break the symmetric parameterization of the original model. Take the amenity factor γ for example. Suppose the model contains only two parameters T and γ (set $L_1^A = L_2^A$, $v = 0$). If $\gamma = 1$, which means workers do not particularly prefer one region to the other, then the equilibrium diagram along the transportation cost will have a pitchfork bifurcation. If we set $\gamma > 1$ or < 1, the resulting equilibrium diagram along T will look like Figures 2 or 3. Therefore, the pitchfork bifurcation is the result of a symmetric parameterization along transportation cost. It is not robust in a larger parameter space.

4 Conclusion

It is important to study an economic system in a parameter space of low dimension, especially when the chosen parameters are the main forces changing the economy. The study of bifurcations provides strong insight into the complex dynamical behavior of a system. However, the real world has many parameters that are omitted in a model, and the choice of parameters affects the equilibrium (bifurcation) diagram of a system. This raises the following question: what kind of dynamical behavior is typical given enough parameters? We characterize the generic pattern of one-parameter dynamical systems along general paths of parameter change in a large enough parameter space. The differentiable approach in general equilibrium theory is adopted and we show that, in a regular parameter space, there is a generic (open and dense) set of one-parameter C^1 families of vector fields without split bifurcations.

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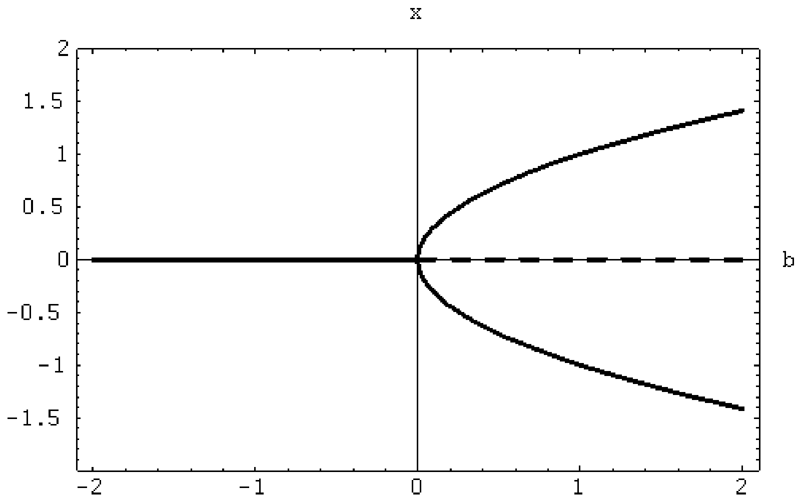

Figure 1: $a = 0$, $-2 < b < 2$.

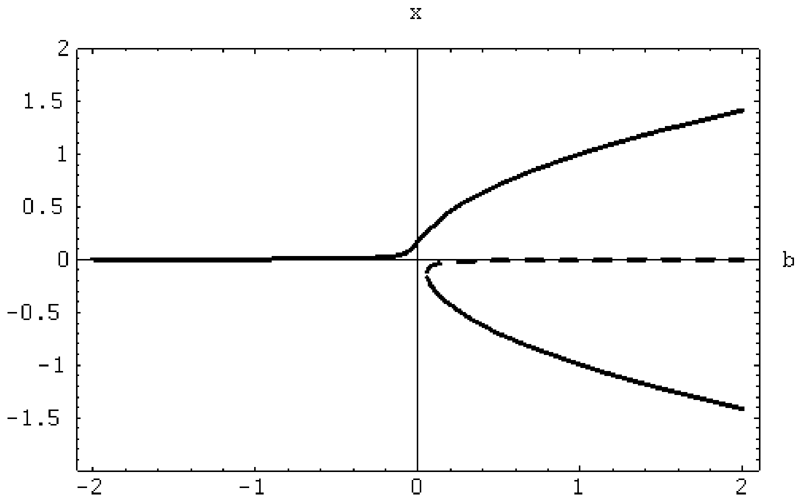

Figure 2: $a = 0.005$, $-2 < b < 2$.

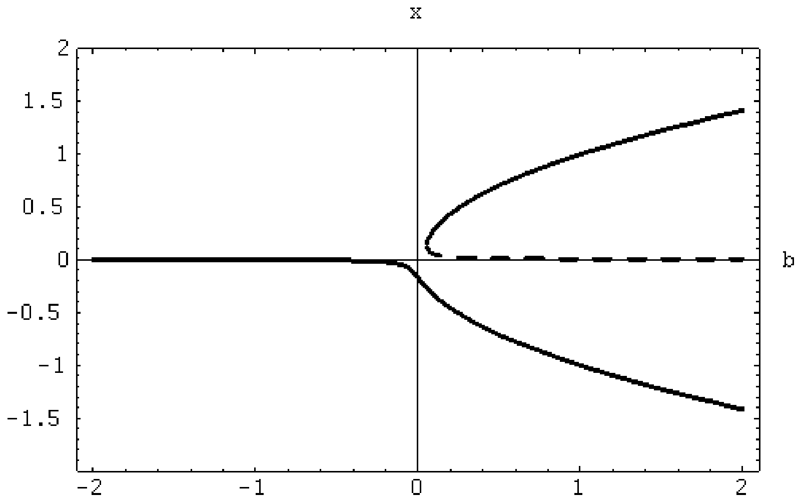

Figure 3: $a = -0.005$, $-2 < b < 2$.

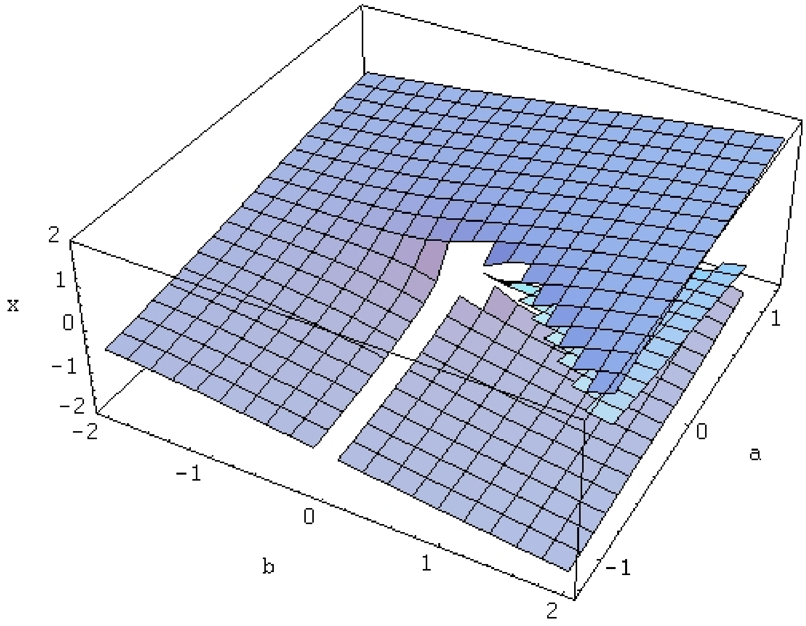

Figure 4: $-1.2 < a < 1.2$, $-2 < b < 2$.

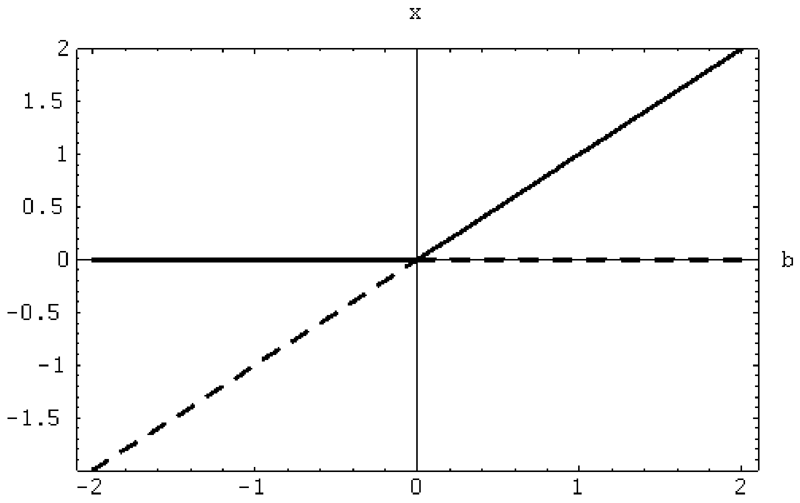

Figure 5: $a = 0,\ -2 < b < 2$.

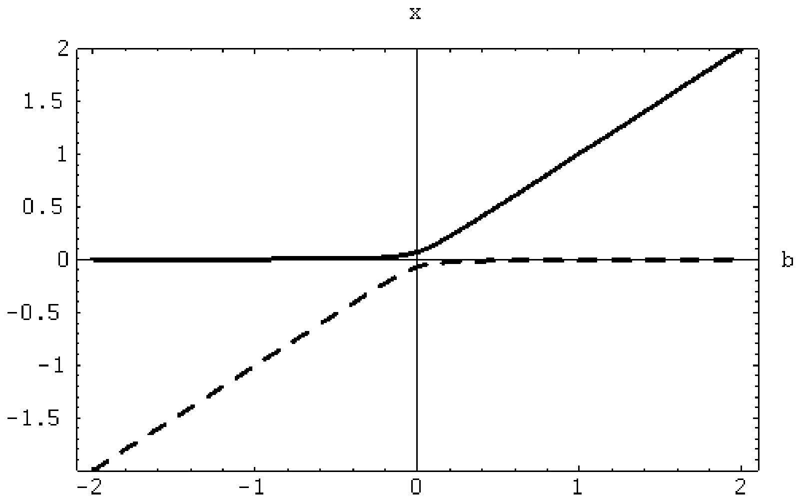

Figure 6: $a = 0.005,\ -2 < b < 2$.

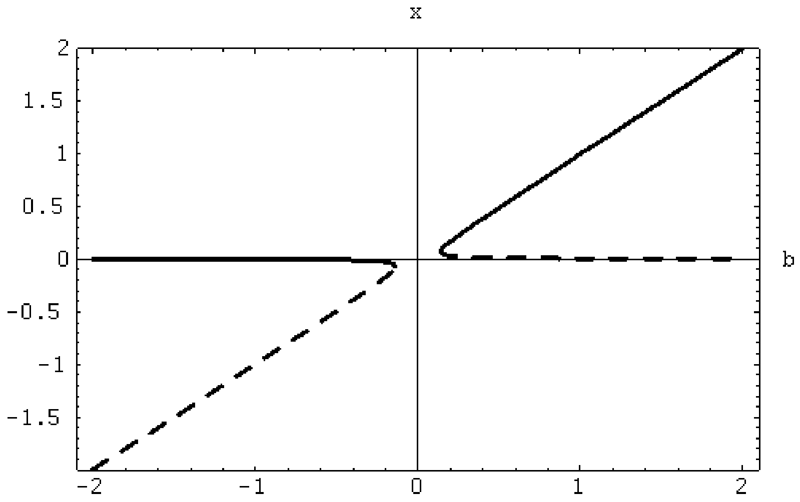

Figure 7: $a = -0.005$, $-2 < b < 2$.

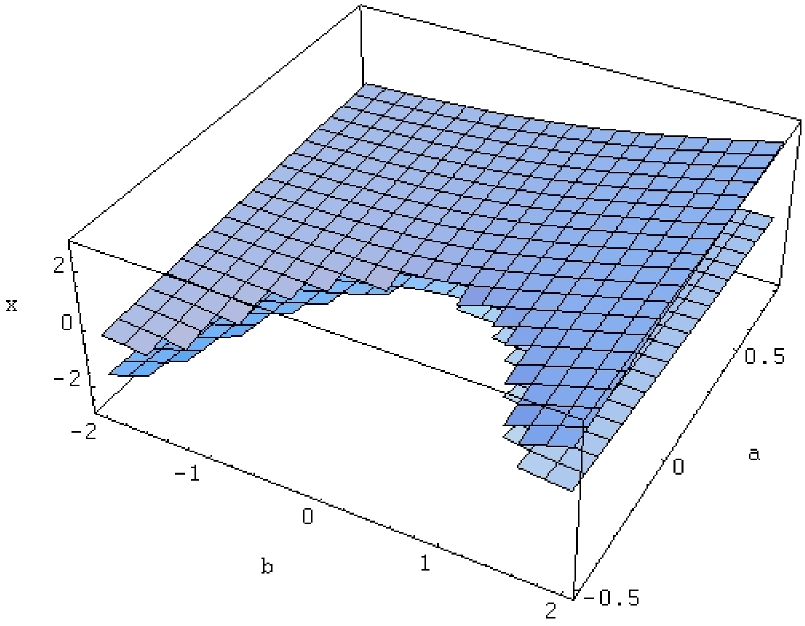

Figure 8: $-0.5 < a < 0.8$, $-2 < b < 2$.