Heterogeneous Researchers in a Two-Sector Representative Consumer Economy

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Abstract

Research activities have uncertain outcomes. The question asked in this paper is whether or not this uncertainty can be a central piece on the explanation of long run consumption growth paths. More specifically, we inquire how the existence of different research projects, with different degrees of uncertainty, contribute to unpredictable consumption growth paths. The proposed scenario is a two-sector representative consumer model with researchers that invest in different innovation projects. There is heterogeneity in terms of risk associated to research programs (researchers invest in projects with the same expected outcome but different volatility). This difference in volatility, combined with an adaptive learning – bounded rationality rule, implies an aggregate index of technology and a consumption growth rate that do not present a predictable pattern over time.

Keywords: Heterogeneous agents, Bounded rationality, Optimal control, Research activities, Volatility and chaos.

JEL classification: C61, O32, O33

1. Introduction

Belief and behavior heterogeneity of economic agents is an important field in today's economic research. The most influential work at this level respects to the explanation of asset prices fluctuations. Beginning with the work of Brock and Hommes (1998), several authors have tried to explain how the co-existence of fundamentalist traders and technical analysts contributes to a random and hardly predictable time series for asset prices. Heterogeneity combined with an adaptive belief system allows to find, in this kind of asset pricing models, time paths for asset prices that are erratic, that is, where periods of low volatility and high volatility alternate, where volatility clustering is evidenced and where some important empirical features about financial markets can be mimetized. Some important work concerning asset pricing heterogeneous agents was developed in Brock et al. (2001), Hommes et al. (2002), Gaunersdorfer et al. (2003), Azariadis and Kaas (2002), Chiarella and He (2002), Kurz and Schneider (1996), Kurz (1997*a*, 1997*b*), Kurz and Beltratti (1997) and Kurz and Motolese (2001).

Heterogeneity and adaptive beliefs are also an influential line of thought of contemporary macroeconomics, mainly in what concerns expectations and learning mechanisms. The most important references at this level are, on one hand, the bounded rationality approach of Sargent (1993) and the discussion of learning mechanisms by Evans and Honkapohja (2001). Other authors, like Barucci (1999), Nourri and Venditti (2001), Tuinstra and Wagener (2003) and Negroni (2003) study stability conditions of macroeconomic models with heterogeneous agents.

Heterogeneity analysis is today extended to a large number of economic issues. Besides asset pricing and macroeconomic stability, different individual behavior or expectations serves as a means to explain exchange rate fluctuations [De Grauwe and

Grimaldi (2002)], economic growth [Maliar and Maliar (2001), Becker and Tsyganov (2002)] or monetary policy [Kurz et al. (2003)].

The model to develop in this paper combines, as the previous references, a mechanism of bounded rationality and learning with the notion of agent heterogeneity. This model is an endogenous growth two sector model where a representative consumer maximizes utility. The source of heterogeneity is in technology generation [as in Kurz et al. (2003)] and not in consumer preferences as it became usual in this kind of model [it is the case of Becker and Tsyganov (2002)] – the representative consumer structure continuous to hold. Under such a scenario we observe that different risk in R&D activities can explain long run consumption growth rates that are erratic and impossible to predict.

Research activities are risky by nature; nevertheless, some are riskier than others. Individuals or firms engaged in research activities choose their research projects between a set of possibilities with equal expected returns but different volatilities. Sometimes, the ones that bet in higher risk activities are the most successful ones; in other occasions, the ones that play safe attain the best result. This constant switching in terms of the best performance strategy is the key ingredient for the non predictable long run time paths to encounter. Because technology producers cannot change from one research activity to another instantly (we consider a learning bounded rationality mechanism) there will always be a certain number of agents choosing some research investment strategy; this share changes according to accumulated past results concerning the innovation activity, such that in certain periods of time it increases and in others it declines.

The fundamental result is that heterogeneity in one economic sector is a source of randomness and unpredictability for the whole economic system. The production of final goods may not be associated to unpredictable outcomes, at least not in the same extent as the generation of knowledge, but final goods time trajectories become erratic trajectories in the moment that we consider a technological level that is determined by the dynamics of a heterogeneous agents – bounded rationality research sector.

The remainder of the paper has the following contents. Section 2 characterizes the main features of the model. It is constructed a two-sector model, where the first sector generates a homogeneous final good that can be indistinctly consumed or used in subsequent periods as capital, and the second sector is an R&D sector. Section 3 assumes a steady state scenario with no volatility. In this case the properties of the model are the ones common to the Romer-Jones endogenous growth model. In section 4 the dynamic analysis of the model is pursued through a numerical example. We understand with this example that a same set of parameters and initial values imply time paths of the most important economic aggregates that change each time the example is run. Finally, section 5 makes a few final comments. Two appendixes are also included: appendix A concerns to the proof of the propositions presented in section 3, while appendix B is destined to the presentation of the most important time paths of the numerical example in section 4.

2. A Multiple Research Projects Two-Sector Model

We begin by assuming a discrete time infinite horizon utility maximization problem for a given representative consumer. In this problem, variable c_t denotes the level of real consumption in each time moment, $\rho>0$ is a constant discount factor and $U(c_t)$ will represent the utility function. The utility function respects the following assumptions, *i*) U is continuous, concave and smooth (infinitely many times continuously differentiable);

ii) $\theta>1$ is a concavity parameter of the utility function that obeys the condition $U' = c_t^{-\theta}$.

The optimal control problem consists on the maximization of the flow of utility functions in expression (1),

$$\sum_{t=0}^{\infty} U(c_t).\frac{1}{(1+\rho)^t} \tag{1}$$

The maximization problem is constrained by the economy's production possibilities. Following the endogenous growth literature [in particular, Romer (1986, 1990) and Jones (1995, 2003)], we consider a two-sector environment where two kinds of economic goods are generated: final goods, that can be either consumed or used as capital in the generation of new goods, and technology. Variable k_t will define real per capita capital, which depreciates at a rate $\delta>0$, and A_t will represent the technological level of the economy. The capital accumulation constraint is the following,

$$\Delta k_t = A_t.f(k_t) - c_t - \delta.k_t \text{ , } \Delta k_t = k_{t+1} - k_t \text{ , } k_0 \text{ given.} \tag{2}$$

In (2), the production function is assumed to exhibit constant marginal returns. Thus, this function may be interpreted as an endogenous growth production function, similar to the one in Rebelo (1992). We impose the following condition to the function, $f' = f(k_t)/k_t = \zeta > 0$ constant (marginal and average returns are identical and constant in time).

The second sector generates technology, a non rival good that can be simultaneously used in the production of physical goods and in the generation of additional technology. We consider decreasing but positive marginal returns in the accumulation of technological knowledge [as in Jones (2003) we may interpret this statement as translating the existence of positive intertemporal technology spillovers]. Furthermore, technology generation depends solely on the previously accumulated knowledge. Therefore, given a parameter $\phi \in (0,1)$, the technology production function is a function $f^A(A_t)$, which obeys the condition $f^{A\,\prime} = \phi.[f^A(A_t)/A_t]$. For purposes of our posterior analysis we consider the following functional form: $f^A(A_t) = A_t^{\phi}$

In a homogeneous scenario regarding technological investment opportunities, the following dynamic rule reflects the accumulation of technological knowledge,

$$\Delta A_t = g.f^A(A_t) - \omega.A_t\ ,\ \Delta A_t = A_{t+1} - A_t\ ,\ A_0 \text{ given.} \tag{3}$$

In (3), parameter g is a positive productivity parameter and ω is an obsolescence rate for technology.

Our attention will focus on a setup with research heterogeneity. This means the existence of various investment alternatives regarding technology production. We assume that the economy is populated by a large number of researchers and that there are alternative research activities h=1, …, H. The distinction between research activities in our framework will be made considering different degrees of risk involved in the innovation process, i.e., all activities share the same expected outcome but diverse levels of volatility characterize the various possible outcomes. The heterogeneity will be translated through parameter g; we assume that different research projects imply distinct values for this technological productivity component. In this way, equation (3) splits in H equations, each one representing the time evolution of the accumulation of technological knowledge regarding each specific innovation process,

$$\Delta A_{h\,t} = g_{ht}.f^A(A_t) - \omega.A_{ht}\ ,\ \Delta A_{ht} = A_{ht+1} - A_{ht}\ ,\ A_{h0} \text{ given.} \tag{4}$$

Note that in (4) the accumulation of knowledge through a project of type h corresponds to the productivity of all the already existent knowledge when applied to type h

innovative activity; the obsolescence of this kind of technology contributes negatively to its accumulation.

For the productivity value g_{ht} we now assume that $\{g_{ht}, t=1, 2, \ldots\}$ is a Markov process. This Markov process is similar to the one in Kurz et al. (2003), i.e., the following dynamic rule is considered,

$$\ln(g_{ht+1}) = \lambda.\ln(g_{ht}) + \varepsilon_{ht+1}\ ,\ \varepsilon_{ht} \sim N(0,\sigma_h^2)\ iid \tag{5}$$

with λ a positive parameter. Note that the only source of heterogeneity is the standard deviation of the normal distribution. Two possibilities regarding technological research will represent two different knowledge accumulation rates because the volatility associated with each project is not equal. Given different time paths for g_{ht}, we guarantee that the accumulation of A_h through (4) differs among investment in technology decisions; furthermore, given that such accumulation process is dependent on a Markov process we will have stochastic time paths characterizing technology values and technology growth rates.

The index of technology available to the production of physical goods is an aggregate value, which may be thought as a weighted average of the technological level that results from each one of the H research activities. Let n_{ht} represent the share of researchers that at any time moment choose to follow the research strategy h; then we define

$$A_t = \sum_{h=1}^{H} n_{ht}.A_{ht} \tag{6}$$

The fractions n_{ht} are updated in time according to a bounded rationality rule or a learning mechanism. Researchers compare their results with the results of alternative research strategies and change to the best strategy, but this does not happen instantly or permanently. The adaptive learning rule that is here adopted follows the asset pricing literature that have introduced the concept of 'rational routes to randomness', namely Brock and Hommes (1997, 1998). This learning rule is based on discrete choice models, in the line of Mansky and McFadden (1981) and Anderson et al. (1993), which implies the following value for the assumed share,

$$n_{ht} = \frac{e^{\beta . a_{ht}}}{\sum_{i=1}^{H} e^{\beta . a_{it}}} \ , \quad \beta \geq 0, \quad h = 1,...,H \tag{7}$$

In expression (7), β is an intensity of choice parameter. It represents the degree of rationality with which researchers choose to change the reallocation of their effort to another research project. If $\beta \to \infty$ the degree of rationality is maximum, that is, individuals change strategies immediately in the presence of better results than the ones obtained with the chosen strategy. For β=0, researchers will never change strategy independently of the obtained results. We assume that β is a positive finite value, representing a bounded rationality behavior for researchers.

Variables a_{ht} are performance measures or fitness functions that translate the past performance of the chosen research strategy. These functions have as a central property the fact that older observations are less relevant than recent observations [this follows a same kind of rule adopted in Barucci (1999) for the study of expectational stability in macroeconomic models with heterogeneous beliefs]. So, we consider a factor τ>0 that discounts to the present past technological outcomes. Each a_{ht} function is then the sum of all the past technology values until some present moment T, according to (8).

$$a_{ht} = \sum_{t=0}^{T} A_{ht} \cdot \frac{1}{(1+\tau)^{T-t}} \ , \ h = 1,...,H \tag{8}$$

We are now in conditions to define formally our model.

Definition 1. Heterogeneous researchers two-sector model. *The representative consumer of the economy controls the time path of consumption in order to maximize the sequence of utility values in (1). The maximization problem is subject to a capital accumulation constraint, (2), and to a series of H technology generation rules, (4). Technological results vary according to a Markov process affecting technology productivity, (5), being innovation risk the source of heterogeneity. The number of researchers choosing an innovation strategy is determined by a bounded rationality rule, (7), where past results constitute the criteria underlying such choices, as indicated by (8). The level of technology that determines goods production is an average of the several technological achievements, as in (6).*

3. Dynamics and Steady-State Properties

The analytical treatment of the optimal control problem in definition 1 does not allow to obtain completely unequivocal results. This is because the different volatility assumption implies that the same parameter values may give place to different time trajectories for the main variables of the model. Because all research projects have the same expected outcome, projects with high and low risk alternate as the ones that produce more technological knowledge in a totally random way, and so we will not have any capacity to predict future results. In this section we study the dynamics of the model in the vicinity of the expected steady state. In the following section, we allow for volatility in research projects and making use of a numerical example we will display and discuss the unpredictability of technology and consumption long run growth rates.

Definition 2. Expected Steady State. *Defining $E(g_{ht})$ as the expected value of the stochastic variable g_{ht}, so that $E(g_{ht+1})= E(g_{ht})^{\lambda}$ and $E(g_{1t})= E(g_{2t})= \ldots =E(g_{Ht})$ for $g_{10}= g_{20}= \ldots = g_{H0}$, the expected steady state will be a long run locus in which the technology level is a constant value and the consumption-capital ratio is also constant.*

Having in mind definition 2, we can prove several propositions. To do this, we first encounter the optimality necessary conditions of the problem in definition 1. Consider a Hamiltonian function, a shadow-price for capital, p_{kt}, and a set of co-state variables for each one of the technology variables, p_{Aht}, h=1, …, H. The current-value Hamiltonian function is

$$\aleph(k_t, A_{ht}, c_t) = U(c_t) + p_{kt}.\left[A_t.f(k_t) - c_t - \delta.k_t\right] + \sum_{h=1}^{H} p_{Aht}.\left[g_{ht}.f^A(A_t) - \omega.A_{ht}\right] \tag{9}$$

Optimality necessary conditions are:

$$\frac{\partial \aleph}{\partial c_t} = 0 \Rightarrow c_t^{-\theta} = p_{kt} \tag{10}$$

$$\Delta p_{kt} = \rho.p_{kt} - \frac{\partial \aleph}{\partial k_t} \Rightarrow \Delta p_{kt} = \left(\rho + \delta - \zeta.A_t\right).p_{kt} \tag{11}$$

$$\begin{aligned} &\Delta p_{Aht} = \rho.p_{Aht} - \frac{\partial \aleph}{\partial A_{ht}} \Rightarrow \\ &\Delta p_{Aht} = \left(\rho + \omega - \phi.n_{ht}.A_t^{-(1-\phi)}.\sum_{i=1}^{H} g_{it}.p_{Ait}\right).p_{Aht} - p_{kt}.n_{ht}.f(k_t) \ , h = 1,\ldots,H \end{aligned} \tag{12}$$

$$\lim_{t\to+\infty} p_{kt}.\frac{1}{(1+\rho)^t}.k_t = 0; \ \lim_{t\to+\infty} p_{Aht}.\frac{1}{(1+\rho)^t}.A_{ht} = 0 \tag{13}$$

[transversality conditions]

Relation (10) can be used to change (11) to a dynamic equation relating to the growth path of consumption. We find a result that is common in endogenous growth literature,

$$\frac{\Delta c_t}{c_t} = \frac{1}{\theta}.\left[\zeta.A_t - (\rho + \delta)\right] \tag{14}$$

The consumption growth rate in (14) would be a constant value if the technological level were constant. Since every research project is subject to decreasing marginal returns, the expected value of A_t tends effectively to a long run steady state constant value, but the Markov process associated with the productivity of technological projects implies a consumption growth rate that would be around a constant value but that does not stabilize in such value. Periods of high and low volatility will alternate as the technological projects with high a low risk perform better, according to the learning process given by the bounded rationality mechanism. This fact will be highlighted in next section's example.

Relatively to the expected steady state and the dynamics in the expected steady state vicinity, these are characterized by the following propositions (the correspondent proofs are presented in appendix A, in the end of the text).

Proposition 1. Expected Steady State Existence and Uniqueness. *The optimal control problem in definition 1 has a unique expected steady state as described in definition 2.*

The unique steady state mentioned in proposition 1 is (see the proof in appendix),

$$\begin{bmatrix} \overline{\psi} \\ \overline{A}_h \end{bmatrix} = \begin{bmatrix} (\zeta + 1/\theta).\overline{A} + \dfrac{\theta - 1}{\theta}.\delta + \dfrac{1}{\theta}.\rho \\ \left[\dfrac{E(g)}{\omega}\right]^{1/(1-\phi)} \end{bmatrix} \tag{15}$$

Expression (15) presents the steady state value of the consumption-capital ratio ($\psi_t = c_t/k_t$) and the expected steady state value of a h research project outcome. Note that every research project have a same expected outcome since the expected productivity of each project is the same [here it is defined by $E(g)$]. Note also that $\overline{A} = \overline{A}_h, \forall h = 1,..., H$, given the definition of A_t in (6).

Proposition 2. Stability properties. *The system relating variables ψ_t and A_{ht} exhibits saddle-path stability in the expected steady state vicinity, for any h=1,…,H.*

Proposition 3. Convergence properties. *The system relating variables ψ_t and A_{ht} presents, in the expected steady state vicinity and for any h=1,…,H, a saddle trajectory characterized by an increasing consumption-capital ratio in the presence of an increasing technological level.*

The three previous propositions are common to the class of two-sector models with capital constant returns and technology decreasing returns. The difference is that we have considered initially that researchers are distributed by different projects. The notion of expected steady state eliminates the importance of the existence of diverse innovation strategies, because those were distinguished only through different risk parameters. So, the research projects will all perform the same (we expect this) and consequently there are no incentives to change behavior, that is, the expected values of the shares n_{ht} will be constant values. The model was in this way reduced to a one-dimension technology model, in which $A_t=n_1.A_{1t}+ \ldots + n_H.A_{Ht}$.

4. GROWTH-PATHS: A NUMERICAL EXAMPLE

The expected steady state notion reduces the heterogeneous agents model to a homogeneous setup. In this section we return to the model with different risk in technology investments and look to a numerical example, through which we perceive that the same parameter values imply an infinite set of possibilities for the technology and consumption long run growth rates.

The numerical example takes the following set of parameter values: [$\phi\ \tau\ \omega\ \beta\ \sigma_1\ \sigma_2\ \zeta\ \lambda\ \rho\ \theta\ \delta$]=[0.25; 0.05; 0.06; 1; 0.1; 0.01; 1; 0.9; 0.04; 20; 0.01]. The initial values $g_{10}=g_{20}=1$ and $A_{10}=A_{20}=0.6$ are also considered. Note that in this example heterogeneity in research projects is limited to dimension 2: $H=2$. The only distinction between research projects is associated to the volatility parameter (ten times higher for research project 1 than for research project 2, meaning that the risk associated to activity 1 is considerably larger). To obtain reasonable results with this set of parameters (consumption growth rates around 3%) we calibrate the model by considering that the productivity parameter in equations (4) is equal to $g_{ht}/25$, with g_{ht} the stochastic variables defined through the dynamic rule in (5).

The previous parameter values and initial states allow to present the long run time trajectories for the several variables in the model. We focus the attention on four time paths:

a) the stochastic productivity variables, g_{1t} and g_{2t};

b) the aggregate technology growth rate;

c) the share of researchers engaged in scientific/technological activity 1, n_{1t};

d) the aggregate consumption growth rate.

The main feature of the results for the referred aggregates is that they change substantially each time the example is run. As stated in previous sections, the fact that any of two projects can perform better in each moment of time implies that it is not known in anticipation which is the project that will attract more researchers; since the rule to change research strategies is an adaptive rule, the time trajectories can follow substantially different paths for the same parameters and initial values of variables.

In appendix B we present several time paths, for the previously mentioned variables. The first set of figures (figures 1 to 3) is a set of three possible realizations of the productivity variables (g_{1t} and g_{2t}) trajectories over time. As we expected, the two series alternate over time as the best result regarding research productivity. The main regularity is the one imposed by the heterogeneity source: the first series present a well evident higher volatility. The series g_2 displays a lower research risk, but as assumed the two series present an equal expected outcome: $E(g_1)=E(g_2)=1$. The two time trajectories have differences for each one of the examples, given the stochastic component governing the Markov process. Nevertheless, there is a pattern: the higher volatility regarding the first research project, the same expected value, the reversion to the mean characteristic and the variability relating to the strategy that best performs are features present in any of the three first figures.

Figures 4 to 6 are the graphical representation of the growth rate of the technology variable. The technology variables, relating to each innovation project, evolve in time according to equation (4) and the aggregate technology variable is just an average of the technology results [remind (6)]. Thus, given the property of decreasing marginal returns, the long run value of this rate is, in the absence of random productivity, equal to zero. As displayed, the growth rate of A_t fluctuates around a constant value. The important evidence is that there is not an identifiable pattern of evolution in time for this variable. The bounded rationality setup contributes to periods of high and low volatility to coexist in a perfectly unpredictable way. The only element in common among the lines in figures 4 to 6 is the zero expected value.

We now turn to the graphical representation of the share of researchers affected to each of the two R&D projects. Figures 7 to 9 display the share of individuals working in knowledge creation that are associated with type 1 activities (symmetric lines would represent the share of individuals engaged in type 2 research activities). The adaptive learning process and the constant change in terms of the best performing strategy are the two key points explaining the absence of a pattern linking the three time paths in consideration. A same set of parameters gives place to a potentially infinite number of solutions for the time trajectory of n_t; furthermore, since there is not a productivity value that assumes itself as the best one for a long period of time, the variable under appreciation does not tend to stay near zero or near one for long periods of time, meaning this that one of the projects does not tend to concentrate all the researchers, and consequently researchers mobility is a frequent feature in our economic setup.

The growth rate of consumption is, in our model, the one in expression (14). We verify that this growth rate is a function of A_t and of a set of parameters. In this way, the behavior over time of the growth rate of c_t is qualitatively the same behavior of the technology aggregate variable. We have mentioned that A_t has an expected constant long run value and thus the expected long run value of the consumption growth rate is also constant. From (15) is true that $\overline{A} = \left[\frac{E(g)}{\omega}\right]^{1/(1-\phi)} = \left(\frac{1}{25\times 0.06}\right)^{1/(1-0.25)} = 0.582$ (note in this expression that the expected productivity value is divided by 25, according to the calibration aspect refered in the beginning of the section). The expected long run value of consumption growth is $\overline{c} = \frac{1}{20}.\left[1\times 0.582 - (0.04 - 0.01)\right] = 0.0266$. The consumption growth rate deviates from the average value in all the three presented figures [figures 10 to 12] but there is no regular pattern regarding the moments in which such deviations are more pronounced. The main feature is once more the absence of a predictable pattern. In this way, we have proposed an explanation to consumption growth unpredictability based on different degrees of uncertainty of the R&D activities.

Two more items are subject to graphical representation in appendix B. These two items allow for a clearer picture about the unpredictability properties of our model. The first set of drawings [figures 13 to 15] relates to the graphical representation of A_2 (in the vertical axis) relatively to A_1 (in the horizontal axis). The second set [figures 16 to 18] is the set of stable trajectories between the consumption-capital ratio and each one of the two technology variables, according to the saddle-path expression derived in appendix A, (19).

Figures 13 to 15 represent the level of technology in research sector 2 for each level of technology in research sector 1. The two lines that cross the graphic correspond to the expected steady state values $\overline{A}_1 = \overline{A}_2 = 0.582$. The steady state point is the one in the intersection of the two lines. Volatility implies that there is not a unique equilibrium value but a large set of values that accumulate around the mentioned point. Confirming the information of the previous figures, larger deviations occur in the direction of higher technology values, and relatively high values for one technology variable tend to be accompanied by relatively high values of the other variable. The different shape of the line for each one of the three examples is evident in the figures.

Finally, figures 16 to 18 are saddle-path trajectories that obey equation (19) in appendix A. These trajectories indicate how the consumption-capital ratio, denoted by ψ_t, converges to the expected steady state point with the evolution of each one of the technology variables. The darker (and wider) lines are the ones relating to the pair of variables (ψ_t, A_{1t}) and the more compact (and clear) lines respect to the relation between (ψ_t, A_{2t}) - the difference in volatility between the two technology productivity values is the reason for the difference in shape between the two time trajectories. These trajectories have positive slopes [according to (19)] but they are not straight lines; they are collections of points that gravitate around the steady state but where it is identifiable a tendency for a positive relation between variables: relatively high values of the technology variables imply, generally speaking, a tendency for ψ_t higher values. Saddle-paths have different shapes but similar qualitative properties for simulations with a same set of parameters.

5. CONCLUSIONS

In an economy there are many types of R&D activities. Some have a certain degree of certainty relating expected outcomes; others involve a considerable degree of risk: results may be the expected ones, much better than expected or, in opposition, much worse. Having this observation in mind, we have a developed an optimal control problem for a representative consumer and a two-sector setup. The two economic sectors assumed were a final goods sector and a technological sector. The technological sector had the peculiarity of disaggregating research projects in a way that different uncertainty degrees in R&D projects were highlighted.

Combining the existence of distinct opportunities regarding innovation strategies with a rule of bounded rationality behavior for the agents engaged in the technology production process, we have attempted to put together an explanation for aggregate consumption growth paths volatility and unpredictability. It was shown that a same set of parameters and initial values of variables gives place to different consumption growth trajectories each time the example is concretized.

Considering equal expected values for research projects outcomes, we have also reduced the model to an expected two-sector endogenous growth model with technology homogeneity that is similar and that has the same steady state vicinity properties as the two-sector capital-technology model of the endogenous growth literature. The saddle-path stable trajectory that relates the joint convergence to the steady state of the consumption-capital ratio and of the technology variables can be displayed as a set of points that can be approximated by a positively sloped line but that does not evidence a regular and immutable pattern.

APPENDIX A – PROOF OF PROPOSITIONS

Proof of proposition 1. Consider the set of equations (4). Because there is decreasing marginal returns in the production of technology, the technology variables will tend to long run constant values, and thus $\Delta A_{ht}=0$ defines a steady state with a constant $\overline{A}$ value. Assuming the expected steady state scenario of definition 2, we consider a productivity value $E(g)$ that is constant and equal for all R&D projects. Furthermore, because $E(g_{t+1})= E(g_t)^{\lambda}$ given the considered Markov process, then $\overline{E}(g)=1$ defines a steady state point. in consequence, to determine the long run expected value of A_t we substitute in (4) g_{ht} by 1. Therefore, $\Delta A_{ht}=0 \Rightarrow \overline{A}_h = \omega^{-1/(1-\phi)}$, $\forall h$, and $\overline{A} = \overline{A}_h$.

As regarded, there is a unique steady state level for technology. In what concerns consumption and capital variables, expression (14) indicates that consumption grows at a constant rate in the steady state, given the constant value of $\overline{A}$. Relatively to capital, equation (2) implies that consumption and capital must grow at a same steady state rate, and thus a variable $\psi_t=c_t/k_t$ will grow at a constant long run growth rate. The equation that reflects the time evolution of this variable is

$$\Delta\psi_t = \left[\psi_t - \left(\zeta + \frac{1}{\theta}\right).A_t - \frac{\theta-1}{\theta}.\delta - \frac{1}{\theta}.\rho\right].\psi_t \qquad (16)$$

The unique steady state value of the consumption-capital ratio is $\Delta\psi_t = 0 \Rightarrow \overline{\psi} = \frac{\zeta + 1/\theta}{\omega^{1/(1-\phi)}} + \frac{\theta-1}{\theta}.\delta + \frac{1}{\theta}.\rho$.

The pair $(\overline{A}, \overline{\psi})$ is the unique steady state point of the system. □

Proof of proposition 2. Saddle-path stability implies a Jacobian matrix with eigenvalues that do not have all the same sign [more rigorously, the existence of a stable arm implies that some eigenvalues (at least one) exist in the interval (-2,0)]. Considering the several A_{ht} variables, the following system is a linearized version of the system composed by equations (4) and (16), in the vicinity of the steady state. Note that the matrix in the system is the Jacobian matrix.

$$\begin{bmatrix} \Delta\psi_t \\ \Delta A_{1t} \\ \Delta A_{2t} \\ \vdots \\ \Delta A_{Ht} \end{bmatrix} = \left[\begin{array}{c|cccc} \overline{\psi} & -(\zeta+1/\theta).n_1.\overline{\psi} & -(\zeta+1/\theta).n_2.\overline{\psi} & \cdots & -(\zeta+1/\theta).n_H.\overline{\psi} \\ \hline 0 & -(1-\phi.n_1).\omega & \phi.n_2.\omega & \cdots & \phi.n_H.\omega \\ 0 & \phi.n_1.\omega & -(1-\phi.n_2).\omega & \cdots & \phi.n_H.\omega \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \phi.n_1.\omega & \phi.n_2.\omega & \cdots & -(1-\phi.n_H).\omega \end{array}\right] . \begin{bmatrix} \psi_t - \overline{\psi} \\ A_{1t} - \overline{A} \\ A_{2t} - \overline{A} \\ \vdots \\ A_{Ht} - \overline{A} \end{bmatrix} \qquad (17)$$

The Jacobian matrix in (17) is a $(H+1)\times(H+1)$ square matrix with trace=$\overline{\psi} - \omega.(H-\phi)$ and determinant=$\overline{\psi}.(-\omega)^{-1}.(1-\phi)$. Computing eigenvalues, these are $\eta_1 = \overline{\psi} > 0$, $\eta_2 = -\omega.(1-\phi) < 0$ and $\eta_3 = ... = \eta_{H+1} = -\omega < 0$. As a result, there is one positive eigenvalue corresponding to the one-dimensional unstable arm of the system and H negative eigenvalues (that are smaller than 1 in absolute value), and thus the stable trajectory has dimension *H*. Because there are simultaneously stable and unstable trajectories, the equilibrium is defined by saddle-path stability. □

Proof of proposition 3. Each one of the negative eigenvalues η_2 to η_{H+1} has an associated eigenvector. These *H* eigenvalues compose a matrix from which we can derive the slope of the stable trajectory. For $\eta_3 = \ldots = \eta_{H+1} = -\omega$ the eigenvectors are $P_i = \left[0 \;\vdots\; -\frac{n_H}{n_1} \quad -\frac{n_H}{n_2} \quad \cdots \quad 1 \right]'$, *i*=3, …, *H*+1; for η_2 we have the following eigenvector: $P_2 = \left[\frac{(\theta - 1).\overline{\psi}}{\theta.[\overline{\psi} + \omega.(1-\phi)]} \;\vdots\; 1 \quad 1 \quad \cdots \quad 1 \right]'$. The matrix $P = [P_2 \quad P_3 \quad \cdots \quad P_{H+1}]$ is a (*H*+1)×*H* matrix that can be divided in two; the slope of the stable trajectory is given by $-\pi.\Pi^{-1}$, where π is the first line of *P* and Π is the square matrix composed by the lines 2 to *H*+1 of *P*. Some computation leads to a vector $[n_1 \quad n_2 \quad \cdots \quad n_H]$ as the first line of Π^{-1}. This is the only line of the matrix that one needs to calculate the slope results, because only the first element of π is different from zero. We have then,

$$\text{Slope of the stable trajectory} = \frac{\theta - 1}{\theta} \cdot \frac{\overline{\psi}}{[\overline{\psi} + \omega.(1-\phi)]} . [n_1 \quad n_2 \quad \cdots \quad n_H] \tag{18}$$

The slope of the stable trajectory is a set of positive values, given the constraints over parameter values that were established. Therefore, the stable trajectory is defined as

$$\psi_t - \overline{\psi} = \frac{\theta - 1}{\theta} \cdot \frac{\overline{\psi}}{[\overline{\psi} + \omega.(1-\phi)]} . [n_1.(A_{1t} - \overline{A}) + n_2.(A_{2t} - \overline{A}) + \ldots + n_H.(A_{Ht} - \overline{A})] \tag{19}$$

From (19) it is understandable that a convergence to the steady state through increasing values of technology levels imply the ratio consumption-capital will also exhibit an increasing behavior. This result is common to this kind of models and it is obvious and intuitive: a higher technology level means that a final good can be produced with lower quantities of capital and thus a higher share of final goods produced can be directed to consumption. Note that (19) is equivalent to (20), given the definition of aggregate technology level,

$$\psi_t = \overline{\psi} - \frac{\theta - 1}{\theta} . \frac{\overline{\psi}}{\left[\overline{\psi} + \omega .(1-\phi)\right]} . \overline{A} + \frac{\theta - 1}{\theta} . \frac{\overline{\psi}}{\left[\overline{\psi} + \omega .(1-\phi)\right]} . A_t \qquad (20)$$

Result (20) would be the one obtained directly if we had constructed the linearized model for the system (ψ_t, A_t). □

APPENDIX B – NUMERICAL EXAMPLE TIME TRAJECTORIES

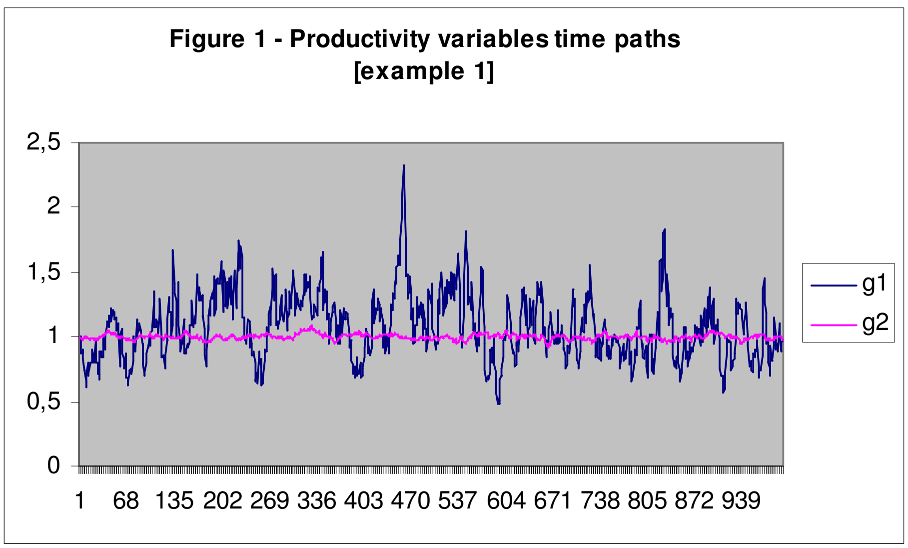


Figure 2 - Productivity variables time paths [example 2]

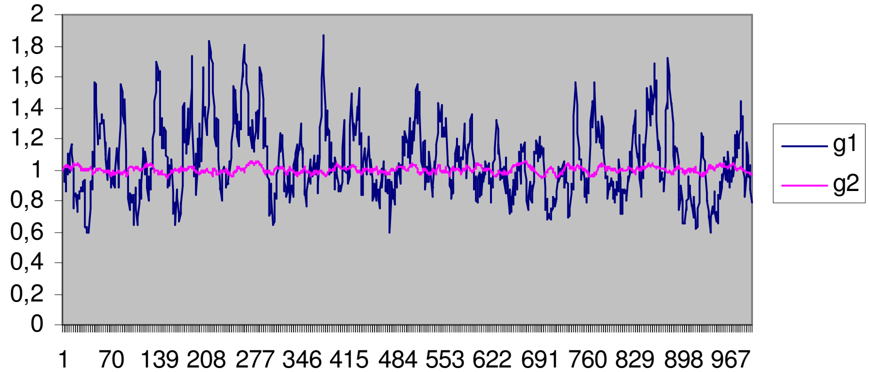


Figure 3 - Productivity variables time paths [example 3]

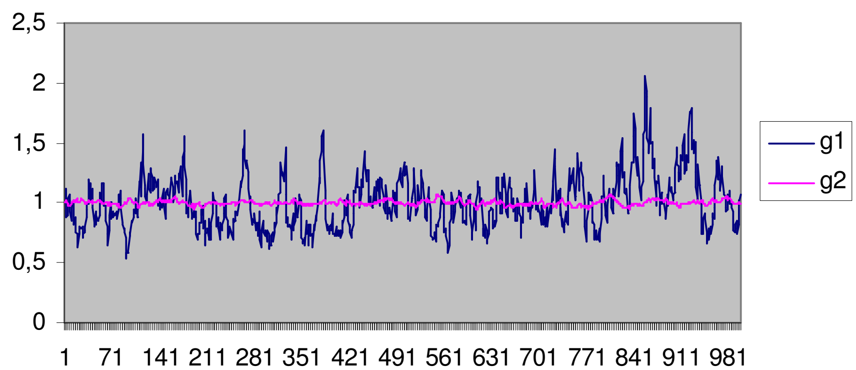
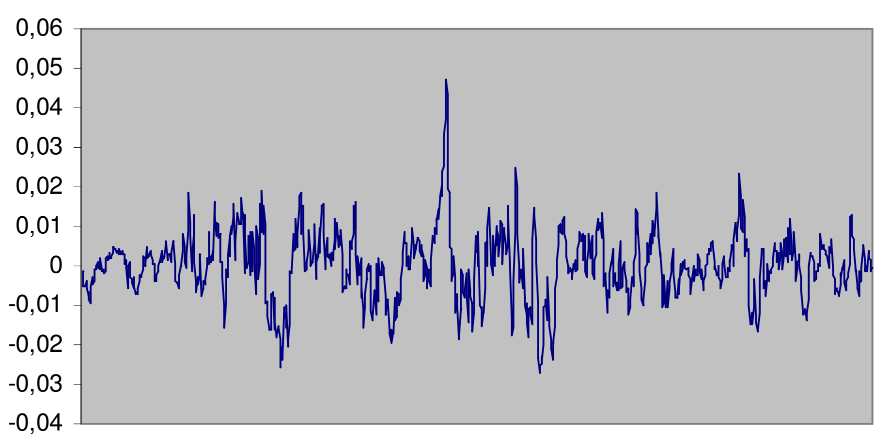


Figure 4 - A growth rate [example 1]



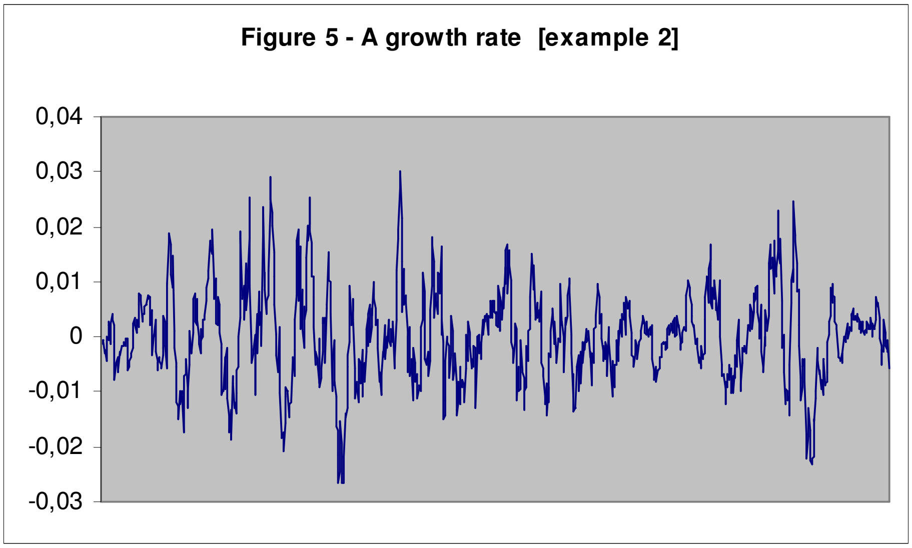


Figure 5 - A growth rate [example 2]
0,04
0,03
0,02
0,01
0
-0,01
-0,02
-0,03

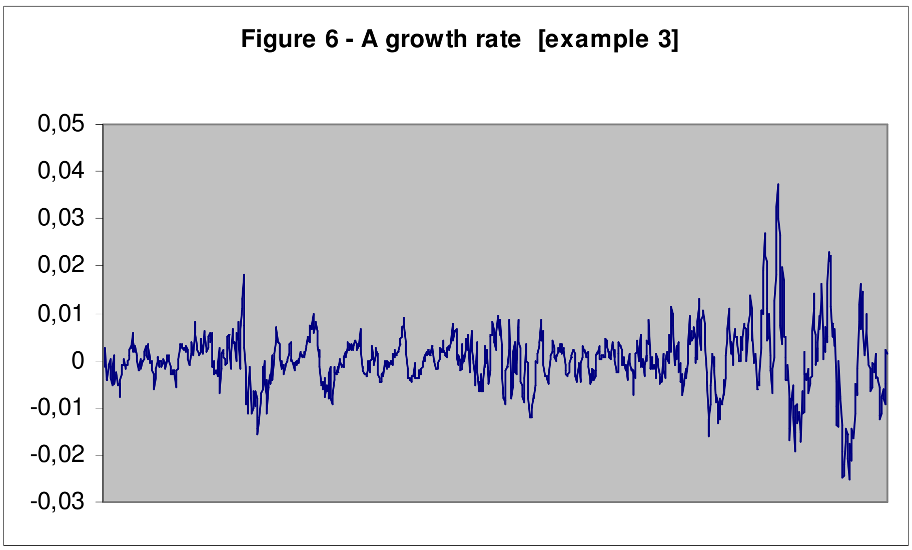


Figure 6 - A growth rate [example 3]
0,05
0,04
0,03
0,02
0,01
0
-0,01
-0,02
-0,03

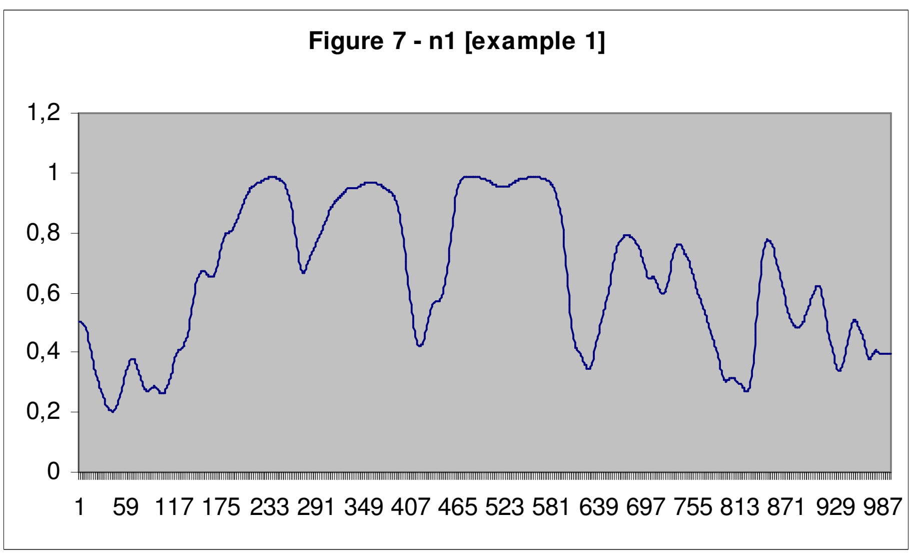


Figure 7 - n1 [example 1]
1,2
1
0,8
0,6
0,4
0,2
0
1 59 117 175 233 291 349 407 465 523 581 639 697 755 813 871 929 987

Figure 8 - n1 [example 2]

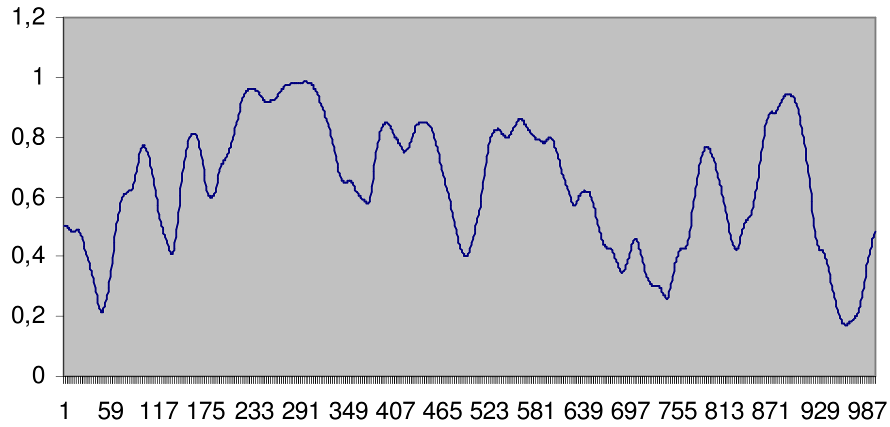


Figure 9 - n1 [example 3]

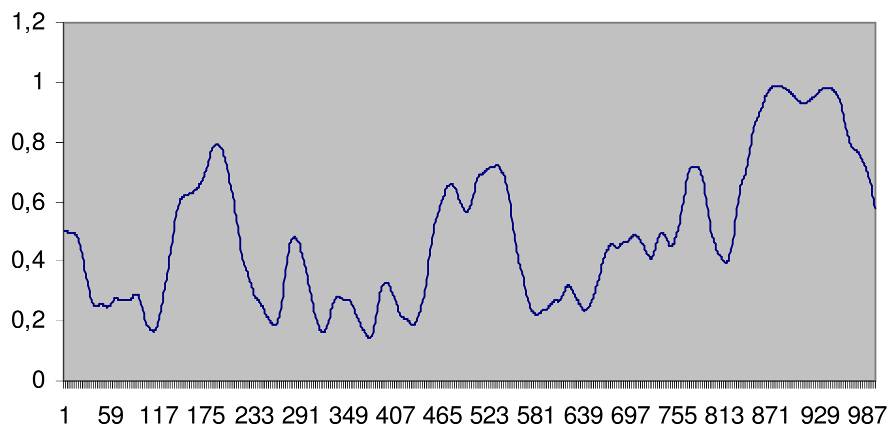
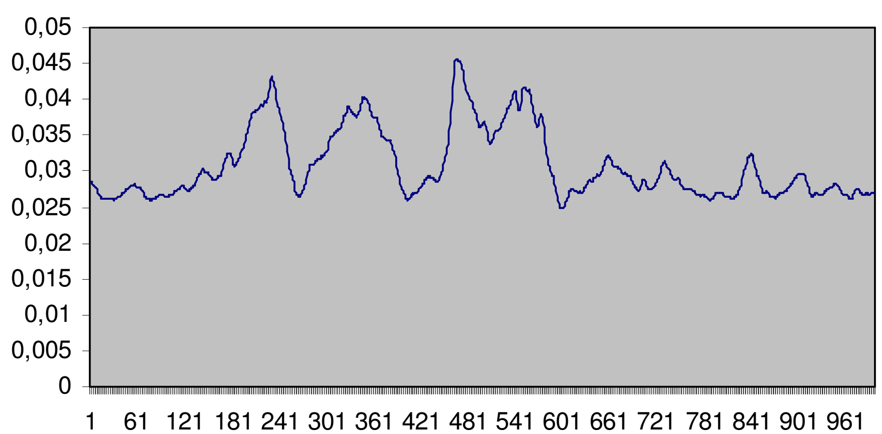


Figure 10 - c growth rate [example 1]



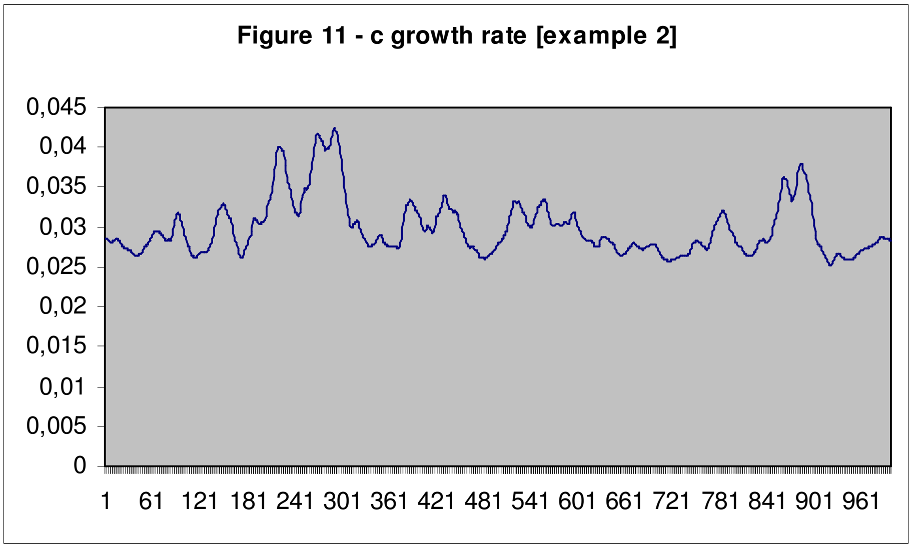


Figure 11 - c growth rate [example 2]
0,045
0,04
0,035
0,03
0,025
0,02
0,015
0,01
0,005
0
1 61 121 181 241 301 361 421 481 541 601 661 721 781 841 901 961

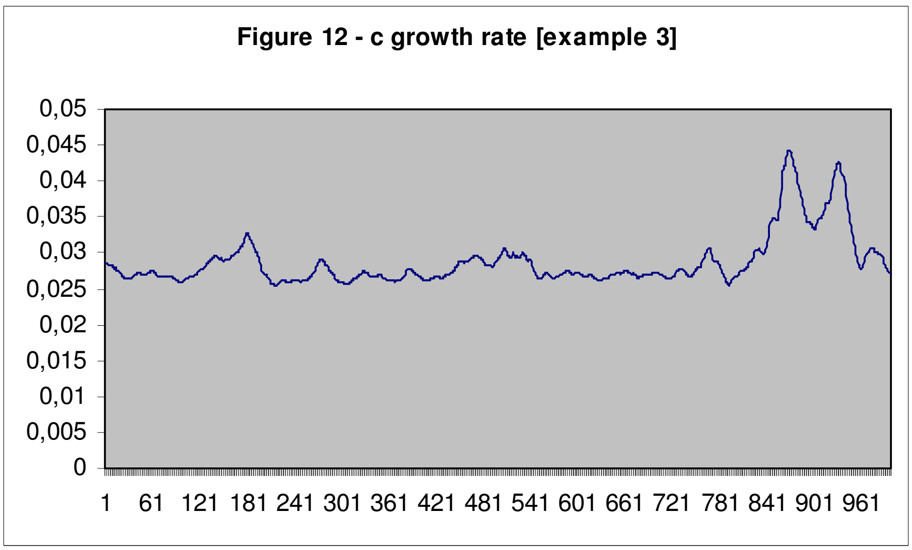


Figure 12 - c growth rate [example 3]
0,05
0,045
0,04
0,035
0,03
0,025
0,02
0,015
0,01
0,005
0
1 61 121 181 241 301 361 421 481 541 601 661 721 781 841 901 961

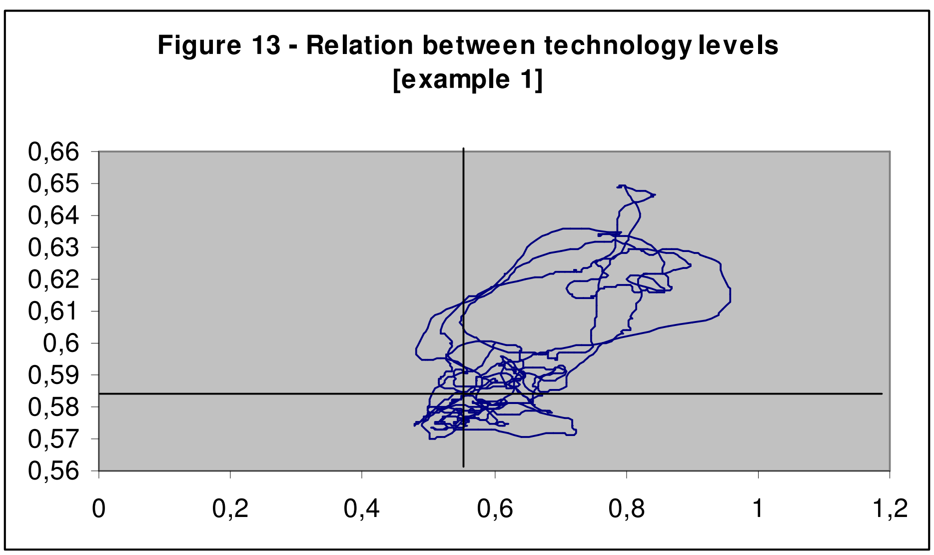


Figure 13 - Relation between technology levels [example 1]
0,66
0,65
0,64
0,63
0,62
0,61
0,6
0,59
0,58
0,57
0,56
0
0,2
0,4
0,6
0,8
1
1,2

Figure 14 - Relation between technology levels [example 2]

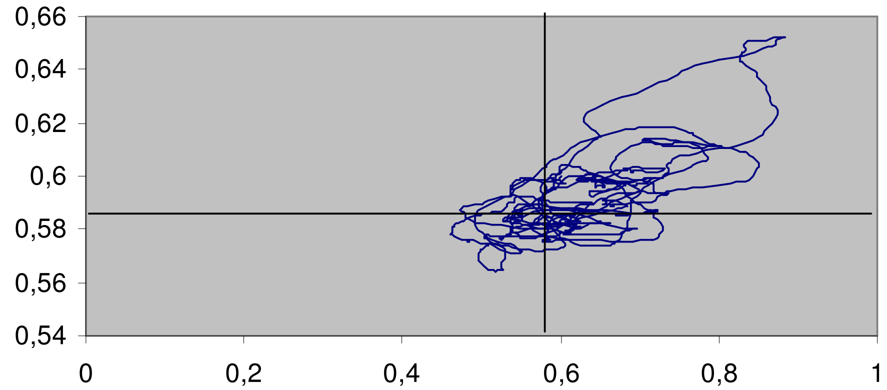


Figure 15 - Relation between technology levels [example 3]

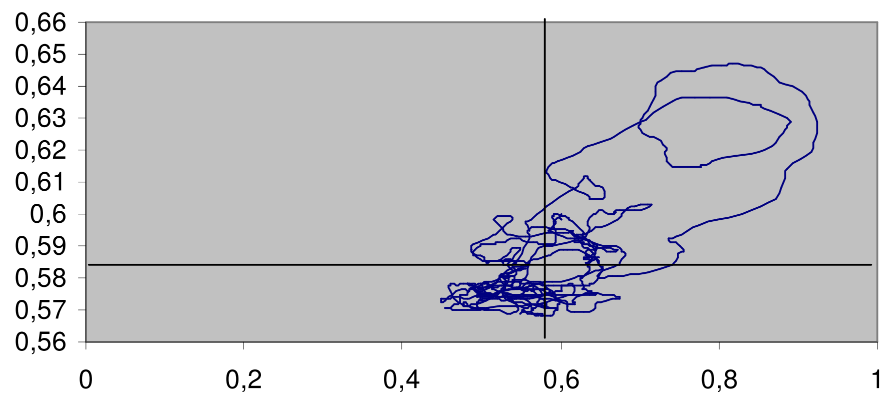


Figure 16 - Stable trajectories (consumption-capital ratio / A1, A2) [example 1]

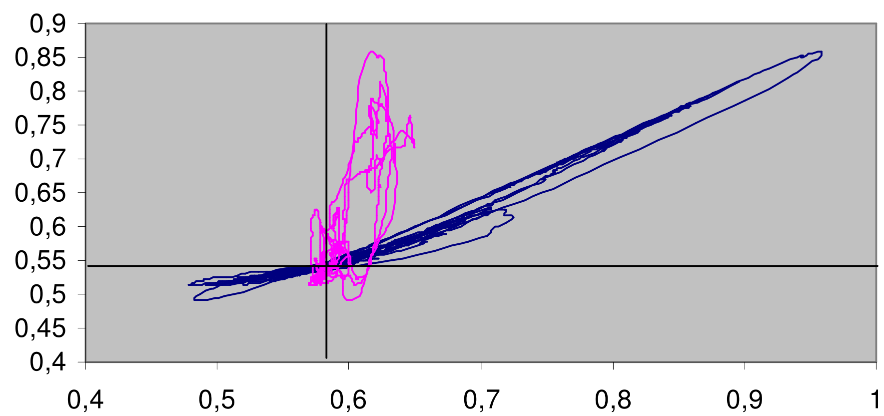


Figure 17 - Stable trajectories (consumption-capital ratio / A1, A2) [example 2]

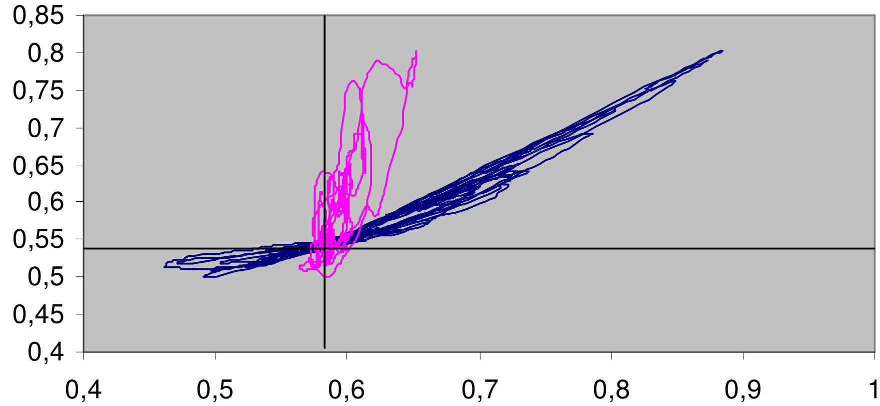
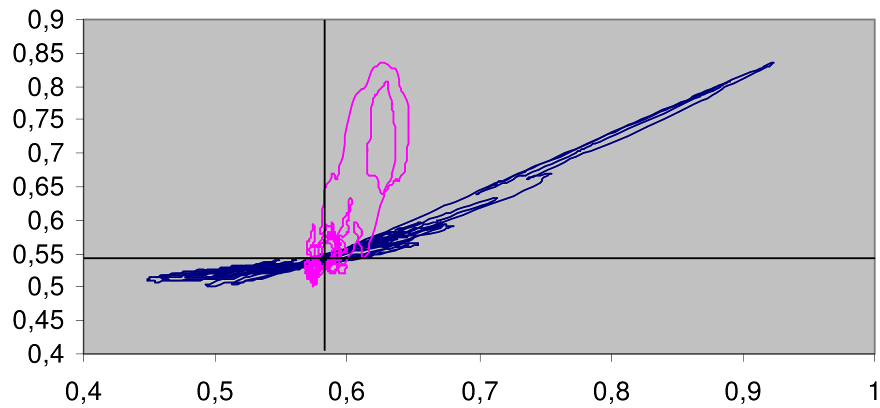


Figure 18 - Stable trajectories (consumption-capital ratio / A1, A2) [example 3]



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