The Optimal Control of Technology Choices

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Abstract

We may distinguish between two concepts of technology: a theoretical level of technology (that is, a technology possibilities frontier) and a level of technology in practice (that is, ready to use in production technology). Having these two concepts in mind, the paper develops an intertemporal optimization model in which we may control the theoretical knowledge frontier. If one wants to expand this frontier an obstacle arises: the resources devoted to create knowledge are diverted from the implementation of technology to productive uses. There is a trade-off between the two technology variables and we explore such a conflict under an optimal control framework. The paper also develops an application of this framework. An economic growth problem is built by putting together the previously presented setup and a capital accumulation constraint. The result is an endogenous growth model where long run growth depends on the technology choices made by a social planner.

Keywords: Optimal Control, Technology, Economic Growth.

JEL classification: O33, O41

I. Introduction

Optimal control models are applied to a wide variety of economic subjects. A representative agent is considered and her economic goal is to maximize an objective function or felicity function that is subject to one or more resource constraints. Objective functions may be profit functions for firms, policy functions for governments or utility functions for consumers. In every case, the felicity function is maximized today regarding all the future moments for a given finite or infinite horizon and the resource constraint is a dynamic constraint that links the several periods of time.

Take for instance the Ramsey (1928)-Cass (1965)- Koopmans (1965) framework for the analysis of the intertemporal consumption-capital accumulation choices. This is one of the most widely used framework in the study of economic issues along the past few decades. Despite the criticism that the optimization of consumption utility setup can be confronted with [see e.g. Kirman (1992) and Krussell and Smith (1998)] the truth is that it is the setup that we encounter supporting an important majority of theoretical studies in economics.[1] Economic growth models in particular are developed almost without relevant exceptions following this consumption utility optimal control setup. Since Lucas (1988), Romer (1990*a*), Rebelo (1991) or Caballé and Santos (1993) to more recent approaches, as Jones (1995), Bond, Wang and Yip (1996), Evans, Honkapohja and Romer (1998) and Ladrón-de-Guevara, Ortigueira and Santos (1999), the referred framework appears systematically to explain economic evolution in time.

The optimal control of consumption utility, as the optimal control of firm profits or the optimal control of policy goals, is based on a simple idea: economics work through trade-offs. In the case of consumption utility the trade-off exists between producing consumption goods and producing capital goods, and the goal of the representative agent is to make now the best choice in producing one and the other type of good for every time moment from today to some point in the future. The ultimate objective of the representative agent is to maximize consumption utility but, given the intertemporal nature of the problem, she cannot fulfill such an objective just by increasing the production of consumption goods. It is necessary to equate how to produce capital goods in order to allow for the possibility of maintaining the production of goods to consumption on perpetual grounds.

In its simplest form, that is, under a one-sector competitive framework, the utility model presents a saddle-path stable equilibrium. This result follows precisely from the existence of a trade-off or opportunity cost. In every situation where we put together an optimal control model in which it exists a trade-off between the control variable (in the case, consumption) and the state variable (physical capital) we will end up with such kind of equilibrium that is precisely the one that allows for more suggestive economic interpretations.

In this paper we concentrate on another specific economic trade-off, the one between technology creation and technology adoption. This distinction is not always a clear one. We follow notions at this level by Nelson and Phelps (1966), Romer (1990*b*) and Barro (1990). We will assume a theoretical level of technology or a basic science variable that relates to the idea of a stock of knowledge and / or techniques that exists in the economy and that can be used to produce new prototype goods, new ways to produce and new forms of managing the business activity. The notion of theoretical

[1] Just recall that the most outstanding works in contemporary macroeconomics make intensive use of the Ramsey framework. We cite a few of these: Blanchard and Fischer (1989), Stokey and Lucas (1989), Barro and Sala-i-Martin (1995), Romer (1996), Turnovsky (1997, 2000) and Evans and Honkapohja (2001).

technology level may be understood as a technological frontier that under the present state of the economic possibilities can be attained. In practice, the level of technology that is embodied in the capital used in production is commonly relatively far from the technological possibilities frontier. There is always a gap between the best-practice level of technology and the level of technology in practice. This last one is our second concept of technology. The index of technology in practice is the one we may include in an aggregate production function in order to explain economic growth or other economic subjects.

The question of technology creation versus technology adoption is present in many theoretical studies as Reinganum (1989), Aghion and Tirole (1994) or Parente and Prescott (1994). The novelty in the present study is that we assume that we may control, although only partially, technology creation. Obviously, this has to be a constrained control problem, that is, one is able to produce more or less technology depending on the way we allocate the available resources. The proposed trade-off is the following: resources are allocated to the creation of technology or to the adoption of best-practice technology to productive uses. If the economy (some kind of representative agent) chooses to pursue a higher rate of technology growth, then the level of technology that can potentially be used in the production process will not be as higher; on the other hand, if one wants inventive activities to have a more effective use in production, it is necessary to choose a lower technology growth. As such, the translation of these ideas to an intertemporal maximization problem will mean, as in the consumption utility case, a model where saddle path stability prevails and thus some meaningful economic results will be accomplished.

The remainder of the paper is organized as follows. In section II we present the basic assumptions underlying the model, revealing as well the technology choices that the social planner faces. Section III makes a first approach to the dynamics of the model, through the characterization of long run results. Section IV is concerned with transitional dynamics. In section V, we complete the model introducing a physical capital accumulation constraint, with the goal of transforming the control problem into an economic growth problem. Finally, section VI concludes.

II. Assumptions about Technology Choices

We begin by assuming a social planner with two goals in mind. To promote basic science (to achieve the highest possible growth rate for the theoretical level of technology) and to apply basic science to productive uses. Let $T(t)$ be the level of theoretical technology and $A(t)$ the level of ready-to-use technology.[2]

Assumption 1. *The social planner intends to achieve the highest possible values for variables* $\tau(t) \equiv \dot{T}(t)/T(t) - a(.)$ *and* $\phi(t) \equiv A(t)/T(t)$. *The first is the (controllable part of the) rate of technological progress and the second a measure of the gap between applied and basic technology indexes.*

With respect to the rate of technological progress we consider that a part of it is controllable, that is, $\tau(t)$ emerges as a control variable of our problem, while the other variable, $a(.)>0$, is an exogenous variable representing all the factors that cannot be controlled in the way technological progress happens. This last variable is certainly a function of the economy's human capital level, physical capital level and other factors external to our analysis.

Assumption 1 tells us that the social planner has a choice to make. The following definition presents the space in which she makes her choices.

Definition 1. Technology choice set and technology choice points. *The technology choice set, V, represents the set of alternatives that the social planner has. Set V must obey to the following minimal requirements:*

i) $V \subseteq \Re^2_+$

ii) V is closed

iii) V is convex

iv) $V \neq \varnothing$

v) $\mathbf{0} \in V$

For a given moment in time, a technology choice point is any $\mathbf{v} = \{\phi(t), \tau(t)\} \in V$.

Technology choice points obey to conventional preference relation rules. This means that the choice between any two points $\mathbf{v}^1$, $\mathbf{v}^2 \in V$ obeys to completeness, transitivity, continuity, monotonicity and convexity axioms. While the first three have a strictly operational use, the last two give important guidance about how the social planner makes her choices. The monotonicity assumption implies that the planner

[2] We take both variables as time dependent. See Nelson and Phelps (1966) for further insights about the meaning of the two variables.

prefers more to less, both in terms of technology growth and in terms of the reduction of the technology gap. Convexity means that it is preferable to achieve a point where some technology growth and some basic / applied technology gap reduction are accomplished than a point where $\tau(t)$ is high but $\phi(t)$ is low or the opposite.

Definition 2. Objective function. *The previous remarks imply a felicity function characterizing the social planner choices. For two technology choice points* $\mathbf{v}^1, \mathbf{v}^2 \in V$, *if there is a preference relation* $\mathbf{v}^1 \succsim \mathbf{v}^2$, *meaning that* $\mathbf{v}^1$ *is "at least as good as"* $\mathbf{v}^2$ *then we define a real valued function* $\upsilon : V \longrightarrow \Re$ *that obeys* $\upsilon(\mathbf{v}^1) \geq \upsilon(\mathbf{v}^2)$ *for all* $\mathbf{v}^1 \succsim \mathbf{v}^2$. *Function* υ *is the social planner objective function.*

Assumption 2*. The objective function* $\upsilon(\mathbf{v})$ *in definition 2 is assumed to have the following properties:*

i) υ *is continuous, concave and smooth (infinitely many times continuously differentiable).*

ii) υ *is homogeneous of degree one.*

iii) The intertemporal elasticities of substitution are positive and constant: $\upsilon_\phi . \frac{\phi(t)}{\upsilon} = \theta > 0$ *and* $\upsilon_\tau . \frac{\tau(t)}{\upsilon} = \mu > 0$.

iv) There is decreasing marginal felicity: $\theta < 1$ *and* $\mu < 1$.

Later we will find it useful to adopt an explicit functional form for the social planner objective function. Function $\upsilon(\mathbf{v})$ in (1) obeys to the properties in assumption 2.

$$\upsilon(\mathbf{v}) = \phi(t)^\theta . \tau(t)^\mu \tag{1}$$

Assumption 3. *Technology variables evolve in time according to the following differential equations:*

$\dot{T}(t) = [a(.) + \tau(t)].T(t), T(0) = T_0$ *given.*

$\dot{A}(t) = g(.).[T(t) - A(t)], g(.) > 0, A(0) = A_0$ *given.*

The first equation simply states that the growth rate of basic science is the sum of an exogenous variable that we consider not to evolve in time with the variable that we took as controllable by the social planner. The second equation translates the idea that the level of technology in practice will evolve faster when there is a large gap between technology possibilities and the stock of knowledge immediately available to produce. We can think about a convergence process: the lower the level of technology ready to use relatively to the benchmark level, the faster will grow the first.

As it is easy to understand, the time evolution of $A(t)$ cannot be dependent only upon the proposed gap, thus we add a $g(.)$ function that is a function of other economic variables like, for instance, human capital. The necessity of considering other variables besides the gap term becomes clear later when we realize that if only these two variables, $A(t)$ and $T(t)$, are considered, when in a long run situation we choose not to apply any technology to practical uses then we will have an infinite growth rate for $T(t)$, what has no economic meaning whatsoever. The influence of other factors, present in the g function, eliminates the possibility of an infinite technology growth rate in any situation. Nevertheless, since we are treating the relation between the two technology variables we assume, for simplifying purposes that all the rest is exogenous and thus we will treat $g(.)$ as a parameter of our setup. We just have to keep in mind that this is a set of other influences over the evolution of $A(t)$, that must eliminate any non plausible economic possibility.

Definition 3. State constraint. *Given the time evolution rules for technology in assumption 3 and recovering the definition of variable* $\phi(t)$ *in assumption 1, the state constraint of our model is:*

$$\dot{\phi}(t) = g(.).[1-\phi(t)] - [a(.)+\tau(t)].\phi(t), \quad \phi(0) = A_0 / T_0 \tag{2}$$

Definition 4. Technology choice optimal control problem. *Consider an infinite horizon. At each moment in time, the social planner intends to choose the value of the share* $\phi(t)$ *and the rate* $\tau(t)$ *that maximize the infinite stream of* $\upsilon(\mathbf{v})$ *functions, given the constraint in definition 3 and the initial values* A_0 *and* T_0*. The following optimization problem is considered:* $\max_{\tau(t)} \int_0^{+\infty} \upsilon(\mathbf{v}).dt$ *subject to* $\dot{\phi}(t) = g(.).[1-\phi(t)] - [a(.)+\tau(t)].\phi(t), \quad \phi(0) = A_0 / T_0$.

Assuming that the evolution of the technological capabilities can be partially controlled and that the level of technology in practice is a function of the gap between both concepts of technology, the economy's representative agent will solve an optimal control problem where two goals are simultaneously in order: to accelerate the pace of theoretical innovation and to reduce the gap between what is possible and what is effectively available.

Definition 5. Discounted problem. *The optimal control problem described in definition 4 may be understood under a discounted setup. Discounting future outcomes makes sense since present technology results are certainly more valued than future technology results. Thus, we may consider the optimization problem as* $\max_{\tau(t)} \int_0^{+\infty} \upsilon(\mathbf{v}).e^{-\rho.t}.dt$, *with* $\rho>0$.

In the steady state analysis and on the treatment of transitional dynamics of the next sections we will consider both cases: the non discounting and the discounting setups.

III. THE STEADY STATE

Given the several assumptions and definitions of the previous section, we present our first proposition. This applies to the non discounted case.

Proposition 1. *In the long run, the state constraint* $\tau(t) = g(.).\dfrac{1-\phi(t)}{\phi(t)} - a(.)$ *intersects the indifference curve* $\tau(t) = \dfrac{\mu}{\theta}.\left[\dfrac{\theta-\mu}{\theta}.\dfrac{1}{g(.)+a(.)}\right]^{\frac{\theta-\mu}{\mu}}.\left[\dfrac{g(.)}{\phi(t)}\right]^{\frac{\theta}{\mu}}$ *in the unique steady state point* $\{\bar{\phi},\bar{\tau}\} = \left\{\dfrac{\theta-\mu}{\theta}.\dfrac{g(.)}{g(.)+a(.)}, \dfrac{\mu}{\theta-\mu}.[g(.)+a(.)]\right\}$.

Proof: We want to prove that the optimal control problem in definition 4 has a steady state, that this steady state is unique and that it is the point given in the proposition. The steady state corresponds to the point where the time derivative of the share variable equals zero: $\dot{\phi}(t) = 0$. In this point, the state constraint is the one in the proposition. Therefore, in the steady state, the optimal control problem is reduced to a static optimization problem: $\max \upsilon(\mathbf{v})$ subject to $\tau(t) = g(.).\frac{1-\phi(t)}{\phi(t)} - a(.)$. The solution of the maximization problem gives a second steady state relation between $\tau(t)$ and $\phi(t)$: $\tau(t) = \frac{\mu}{\theta}.g(.).\frac{1}{\phi(t)}$. Solving the system that includes this equation and the constraint of the problem we get the unique steady state point presented in proposition 1.

Replacing the steady state values in objective function (1), one obtains the following value for υ: $\upsilon^* = \bar{\phi}^{\phi}.\bar{\tau}^{\mu} = (\theta-\mu)^{\theta-\mu}.\frac{\mu^{\mu}}{\theta^{\theta}}.g.(.)^{\theta}.[g(.)+a(.)]^{\mu-\theta}$. The indifference curve that includes the steady state point is then given by $\phi(t)^{\theta}.\tau(t)^{\mu} = \upsilon^*$. Rearranging, we get the indifference curve in the form given in the proposition•

Relatively to the steady state point note that it has to obey to the boundary values set for $\phi(t)$, that is, we must guarantee that $\bar{\phi} \in (0,1)$. For such we just have to impose hereafter the following condition: $\theta > \mu$.

Taking the constraint and the indifference curve that passes through the steady state point we may present this unique point in graphical terms (see figure 1).

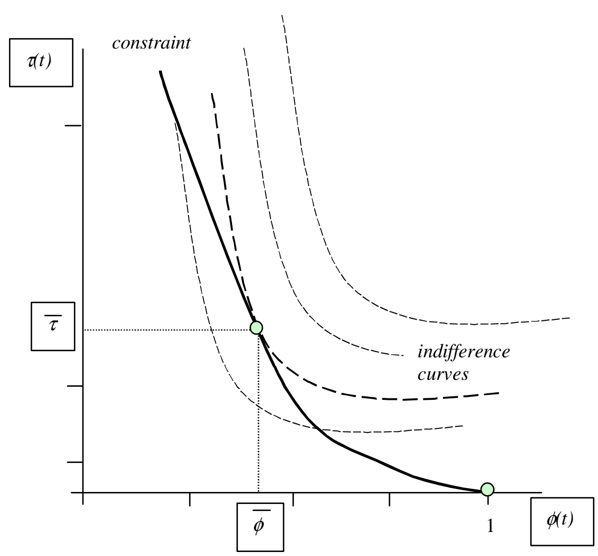

Figure 1: The steady state point.

Note, in figure 1, the counterfactual result $\lim_{\phi \to 0} \tau(t) = +\infty$. As we have stated, this is overcome by considering a function $g(.)$ that in the long run imposes a roof for the value that $\tau(t)$ may achieve. Nevertheless, even if we consider the hypothetical case in figure 1 where the resources of the economy are all concentrated on the growth of the technology frontier, the objective function imposes an equilibrium point where a finite $\tau(t)$ is always accomplished.

The discounted version of proposition 1 is:

Proposition 2. *Considering that the felicity function is discounted in time at a rate* $\rho > 0$ *, in the long run the state constraint* $\tau(t) = g(.).\frac{1-\phi(t)}{\phi(t)} - a(.)$ *intersects the indifference curve* $\tau(t) = \frac{\mu}{\theta}.\left(\frac{\theta-\mu}{\theta}\right)^{\frac{\theta-\mu}{\mu}}.\frac{\rho + g(.) + a(.)}{\left[\frac{\mu}{\theta}.\rho + g(.) + a(.)\right]^{\frac{\theta}{\mu}}}.\left[\frac{g(.)}{\phi(t)}\right]^{\frac{\theta}{\mu}}$ *in the unique steady state point* $\{\bar{\phi}, \bar{\tau}\} = \left\{\frac{(\theta-\mu).g(.)}{\mu.\rho + \theta.[g(.) + a(.)]}, \frac{\mu}{\theta-\mu}.[\rho + g(.) + a(.)]\right\}$.

Proof: *We just have to follow the same steps as in proposition 1. Solving the static steady state optimization problem we arrive to the steady state values of both variables. Replacing these in the objective function we will be able to determine the expression of the indifference curve that represents the highest level of* υ *that is possible to accomplish given the state constraint*•

Regard the determinants of the steady state values. In the long run, ratio $\phi(t)$ is as higher as the higher is the value of function $g(.)$ and the lower is the value of $a(.)$. This gives the obvious result that the convergence between the level of technology in practice and the technological frontier is accelerated when there are factors contributing directly to such speed of convergence, translated in function $g(.)$, and suffers a slowdown when there are no controllable factors promoting the expansion of the technology possibilities frontier, what is translated by an increment in $a(.)$. Also, the elasticities of intertemporal substitution have an impact over the value of the ratio variable. In the steady state, the higher is θ and the lower is μ, the higher will be the value of $\phi(t)$. Discounting the objective function in time results in a larger distance between steady state values $\bar{A}$ and $\bar{T}$.

For variable $\tau(t)$ we can highlight the following: a lower value of elasticity θ and a higher value of elasticity μ lead to a higher steady state growth rate of $T(t)$. Similarly, $\tau(t)$ has to gain with larger values for parameter ρ and exogenous variables $g(.)$ and $a(.)$. Note that in the steady state variables $A(t)$ and $T(t)$ have to grow at exactly the same rate, given the state constraint. This rate will be $a(.)+\bar{\tau}=\frac{\mu}{\theta-\mu}\left[\rho+g(.)\right]+\frac{\theta}{\theta-\mu}.a(.)$. Looking at the technology growth rate we observe that the larger the value of θ relatively to μ, the more $a(.)$ has an impact over the technology growth and the less ρ and $g(.)$ will be relevant factors in terms of technological progress.

IV. Some Remarks about Transitional Dynamics

We proceed to the characterization of transitional dynamics through the presentation of proposition 3.

Proposition 3. *Given the condition* $0<\mu<\theta<1$, *the technology choice model exhibits saddle-path stability and the stable and unstable trajectories are both negatively sloped:*

$$\tau(t)=\frac{\frac{\theta}{\mu}.\bar{\tau}^2-\lambda_1.\bar{\tau}+\frac{1-\theta}{1-\mu}.\frac{g(.)}{\bar{\phi}}.\bar{\tau}}{\frac{\theta}{\mu}.\bar{\tau}-\lambda_1}-\frac{\frac{1-\theta}{1-\mu}.\frac{g(.)}{\bar{\phi}^2}.\bar{\tau}}{\frac{\theta}{\mu}.\bar{\tau}-\lambda_1}.\phi(t) \qquad (S)$$

$$\tau(t)=\frac{g(.)+(\bar{\tau}+\lambda_2).\bar{\phi}}{\bar{\phi}}-\frac{\frac{g}{\bar{\phi}}+\lambda_2}{\bar{\phi}}.\phi(t) \qquad (U)$$

where $\lambda_1<0$ *and* $\lambda_2>0$ *are the eigenvalues of the Jacobian matrix that is derived from the linearization of the Hamiltonian system concerning our dynamic problem.*

Proof: *Using the tools of optimal control analysis, we can present a Hamiltonian function, where* $p(t)$ *is a co-state variable associated to* $\phi(t)$*,*

$$\aleph[\phi(t),\tau(t),p(t)] = \upsilon(\mathbf{v}) + p(t).\{g(.).[1-\phi(t)] - [a(.)+\tau(t)].\phi(t)\} \tag{3}$$

The first order optimality conditions are

$$\aleph_{\tau} = 0 \Rightarrow \upsilon_{\tau} = p(t).\phi(t) \tag{4}$$

and

$$\aleph_{\phi} = \rho.p(t) - \dot{p}(t) \Rightarrow \dot{p}(t) = [\rho + g(.) + a(.) + \tau(t)].p(t) - \upsilon_{\phi} \tag{5}$$

The transversality condition $\lim_{t\to+\infty} p(t).e^{-\rho.t}.\phi(t) = 0$ also applies. Equation (5) may be rearranged given the relation in equation (4). The growth rate of the co-state variable comes:

$$\dot{p}(t)/p(t) = \rho + g(.) + a(.) - \frac{\theta-\mu}{\mu}.\tau(t) \tag{6}$$

From the previous conditions we derive the equation of motion of the controllable innovation rate:

$$\dot{\tau}(t) = \left\{\frac{\theta}{\mu}.\tau(t) - \frac{1-\theta}{1-\mu}.g(.).\frac{1}{\phi(t)} - \frac{\theta}{1-\mu}.[g(.)+a(.)] - \frac{1}{1-\mu}.\rho\right\}.\tau(t) \tag{7}$$

Solving the system $[\dot{\phi}(t)\ \dot{\tau}(t)]' = 0$, we arrive to the same steady state result as in proposition 2, or the steady state in proposition 1 if ρ=0. The motion properties of the system in the steady state vicinity may be analyzed through the linearization of the model in the steady state neighborhood. The linearized system is:

$$\begin{bmatrix}\dot{\phi}(t)\\ \dot{\tau}(t)\end{bmatrix} = J.\begin{bmatrix}\phi(t)-\bar{\phi}\\ \tau(t)-\bar{\tau}\end{bmatrix} \text{ with } J = \begin{bmatrix}\left.\frac{\partial\dot{\phi}(t)}{\partial\phi(t)}\right|_{(\bar{\phi},\bar{\tau})} & \left.\frac{\partial\dot{\phi}(t)}{\partial\tau(t)}\right|_{(\bar{\phi},\bar{\tau})}\\ \left.\frac{\partial\dot{\tau}(t)}{\partial\phi(t)}\right|_{(\bar{\phi},\bar{\tau})} & \left.\frac{\partial\dot{\tau}(t)}{\partial\tau(t)}\right|_{(\bar{\phi},\bar{\tau})}\end{bmatrix} = \begin{bmatrix}-\frac{g(.)}{\bar{\phi}} & -\bar{\phi}\\ \frac{1-\theta}{1-\mu}.\frac{g(.)}{\bar{\phi}^2}.\bar{\tau} & \frac{\theta}{\mu}.\bar{\tau}\end{bmatrix}$$

Concerning matrix J, we find that $Tr(J)=\rho$, as it would be expected under an optimal control problem Jacobian matrix. Also, we arrive to the result $|J| = -\frac{\theta-\mu}{\mu.(1-\mu)}.\frac{g(.)}{\bar{\phi}}.\bar{\tau}$; since we have imposed the condition $\theta>\mu$, then the determinant of the Jacobian matrix corresponds to a negative quantity. Given the trace and determinant signs, we must conclude that the order two J matrix has two eigenvalues with opposite signs what implies saddle-path stability.[3] The eigenvalues are $\{\lambda_1, \lambda_2\} = \left\{\frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 - |J|}, \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}\right)^2 - |J|}\right\}$. It is straightforward that for $|J|<0$ we have $\lambda_1 < 0$ and $\lambda_2 > 0$ both real. Then, we confirm the saddle-path stability result.

With an equilibrium that is saddle-path stable one can find the stable and unstable arms that define the system dynamics. For that, it is necessary to compute the eigenvector matrix, P, associated to matrix J. The first column of matrix P corresponds to eigenvalue λ_1 and the second to λ_2, $P = \begin{bmatrix} 1 & 1 \\ -\frac{\frac{1-\theta}{1-\mu}.\frac{g(.)}{\bar{\phi}^2}.\bar{\tau}}{\frac{\theta}{\mu}.\bar{\tau} - \lambda_1} & -\frac{\frac{g}{\bar{\phi}} + \lambda_2}{\bar{\phi}} \end{bmatrix}$. Given that the second line of matrix J respects to the control variable the same is true for matrix P. In this way, we can identify the elements of this second line as being the slopes of the stable and of the unstable paths, respectively. We confirm that they are both negative. To find the intercept terms, one has just to regard that both trajectories pass through the steady state point and that they have the slopes indicated in the P matrix. Then, it is rather straightforward to reach to the analytical expressions in the proposition•

Note the economic interpretation of the accomplished result: saddle path stability implies that, once reached the negatively sloped stable trajectory, the variables evolve to the steady state following opposite directions - an increasing growth rate for technology implies a slower technology gap straightening and vice-versa, as the system adjusts to the long run locus.

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[3] Recall that for any two eigenvalues $\lambda_1$and λ_2 of an order two square matrix is true that $Tr(J)= \lambda_1+\lambda_2$ and $|J|= \lambda_1.\lambda_2$.

Simpler expressions for the stable and unstable trajectories in proposition 3 may be written for the limit case where no discounting is considered. For ρ=0 the eigenvalue expressions come considerably simplified: $\{\lambda_1,\lambda_2\}=\left\{-\sqrt{\frac{\theta}{(\theta-\mu).(1-\mu)}}.[g(.)+a(.)],\ \sqrt{\frac{\theta}{(\theta-\mu).(1-\mu)}}.[g(.)+a(.)]\right\}$. The stable and the unstable trajectories can be now presented using solely parameter values:

$$\tau(t)=\left\{\frac{\mu}{\theta-\mu}-\left(\frac{\theta}{\theta-\mu}\right).\left[1-\sqrt{\frac{\theta-\mu}{\theta.(1-\mu)}}\right]\right\}.[g(.)+a(.)]-\\ -\left(\frac{\theta}{\theta-\mu}\right)^2.\left[1-\sqrt{\frac{\theta-\mu}{\theta.(1-\mu)}}\right].\frac{[g(.)+a(.)]^2}{g(.)}.\phi(t) \qquad (S')$$

$$\tau(t)=\left\{\frac{\mu}{\theta-\mu}-\left(\frac{\theta}{\theta-\mu}\right).\left[1+\sqrt{\frac{\theta-\mu}{\theta.(1-\mu)}}\right]\right\}.[g(.)+a(.)]-\\ -\left(\frac{\theta}{\theta-\mu}\right)^2.\left[1+\sqrt{\frac{\theta-\mu}{\theta.(1-\mu)}}\right].\frac{[g(.)+a(.)]^2}{g(.)}.\phi(t) \qquad (U')$$

Lines (*S´*) and (*U'*), being particular cases of (*S*) and (*U*), display as well negatively sloped transition trajectories. Furthermore, one is now able to observe clearly that the slope of the stable trajectory is less accentuated than the slope of the unstable trajectory, in absolute value; this implies that (*U'*) is steeper than (*S'*). Figure 2 presents a sketch of the saddle-path dynamics of the model.

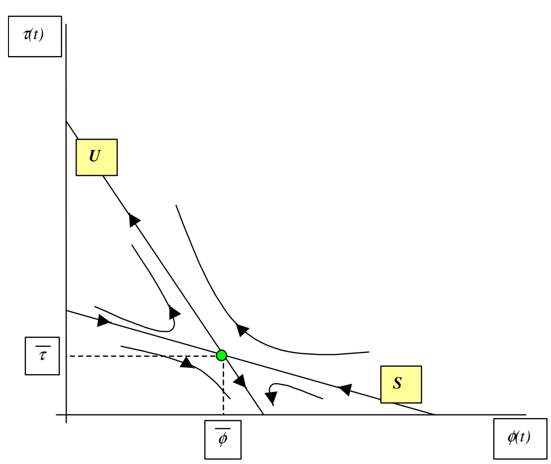

Figure 2: Transitional dynamics on the technology choices model.

V. Closing the Model with a Physical Capital Accumulation Constraint

Consider a Solow (1956)-like capital accumulation constraint,

$$\dot{K}(t) = s.f\left[K(t), L(t), A(t)\right] - \delta.K(t),\ K(0) = K_0,\ L(0) = L_0 \text{ given} \tag{8}$$

where $K(t) \geq 0$ is physical capital, $L(t) \geq 0$ a labor / population variable,[4] $A(t)$ is the already defined ready-to-use in production technology variable, $s \in (0,1)$ is the saving rate and $\delta \in (0,1)$ is the physical capital depreciation rate. Output, $Y(t)$, corresponds to the production function outcome and this is a labor-augmenting production function where technical progress is Harrod-neutral. Function f is a neoclassical production function with constant returns to scale and thus we may rewrite constraint (8) under an intensive form,

$$\dot{k}(t) = s.f\left[k(t), A(t)\right] - (n+\delta).k(t) \tag{9}$$

where $k(t) \equiv K(t)/L(t)$ is the stock of physical capital per unit of labor (or *per capita*). Output *per capita*, $y(t)$, corresponds to the new production function outcome. The neoclassical nature of function f means that the marginal returns of each input are positive and diminishing: $f_k > 0$, $f_{kk} < 0$, $f_A > 0$, $f_{AA} < 0$; the Inada conditions, that state that the marginal product of each input approaches infinity as the quantity of the input goes to zero and approaches zero as such quantity goes to infinity, apply as well.
Defining $\varphi(t) \equiv k(t)/T(t)$, we rewrite equation (9):

$$\dot{\varphi}(t) = s.f\left[\varphi(t), \phi(t)\right] - \left[n + \delta + a(.) + \tau(t)\right]\varphi(t),\quad \varphi(0) = k_0 / T_0 \text{ given} \tag{10}$$

Definition 6. Growth model. *A growth model is a maximization problem where a physical capital accumulation constraint and a constraint that describes the time evolution of the engine of growth are present. In this sense we have a growth model where the constraint in definition 3 and equation (10) are the constraints of the two sector optimization problem in which the intertemporal maximization of* $\upsilon(\mathbf{v})$ *is*

[4] That we assume to grow at an exogenously given rate: $n \geq 0$.

intended. Variables $\varphi(t)$ *and* $\phi(t)$ *are the state variables while* $\tau(t)$ *is the single control variable.*

Three propositions will be presented along this section. The first gives the steady state growth rate result. The other two are concerned with the stability of the model and with transitional dynamics.

Proposition 4. *In a growth model where technological choices are allowed, all the relevant economic variables grow in the long run at asymptotically the same rate. Per capita output, amount of capital per unit of labor, theoretical and practical technology indexes, they all grow in the steady state at the rate of technological progress that the economy is able to partially control. This rate is* $a(.)+\overline{\tau}$.

Proof: The condition $\dot{\varphi}(t)=0$ implies that $f\left(\overline{\phi}/\overline{\varphi}\right)=\left[n+\delta+\frac{\theta}{\theta-\mu}.a(.)+\frac{\mu}{\theta-\mu}.g(.)\right]/s$; thus, $\overline{\varphi}$ is a constant value in the steady state. Regarding the definition of $\varphi(t)$, the capital *per capita* variable must grow in the long run at the same rate as $T(t)$, that is, $a(.)+\overline{\tau}$. Recall that this is also the rate at which $A(t)$ grows in the long run given that $\overline{\phi}$ is constant.

Relatively to per capita output we know that $\overline{y}/\overline{T}=f(\overline{\varphi},\overline{\phi})$ and thus $\overline{y}=\left[\left(n+\delta_k+a(.)+\overline{\tau}\right)/s\right]\overline{\varphi}.T_0.e^{[a(.)+\overline{\tau}].t}$. In the steady state, *per capita* output grows at the same rate as the technology variables and the capital *per* labor unit input•

Proposition 5. *A model that puts together the optimal technological diffusion framework and a Solow-like capital accumulation constraint will exhibit saddle-path stability. The system is three dimensional, the unstable trajectory is one-dimensional and the stable trajectory is a two dimensional space.*

Proof: The linearized three dimensional system is $\begin{bmatrix}\dot{\varphi}(t)\\ \dot{\phi}(t)\\ \dot{\tau}(t)\end{bmatrix}=\Im.\begin{bmatrix}\varphi(t)-\overline{\varphi}\\ \phi(t)-\overline{\phi}\\ \tau(t)-\overline{\tau}\end{bmatrix}$, with $\Im=\left[\begin{array}{c|cc}\lambda_3 & s.f_{\overline{\phi}} & -\overline{\varphi}\\ \hline 0 & \multicolumn{2}{c}{J}\end{array}\right]$. This system is the result of the steady state evaluation of the two state constraints plus the control variable differential equation derived from the optimization problem. Matrix *J* was presented earlier, $\overline{0}$ is a two elements zero column vector and $\lambda_3 = s.\left[f_{\overline{\varphi}} - f(\overline{\varphi},\overline{\phi})/\overline{\varphi}\right]$ is the third eigenvalue that is computed from the matrix.[5] Under the diminishing marginal returns assumption, $\lambda_3 < 0$. With two negative and one positive eigenvalues we guarantee the existence of a saddle-path stable equilibrium. The negative eigenvalues correspond to the stable path (two-dimensional) and the positive eigenvalue to the unstable path (therefore of dimension one)•

Proposition 6. *For the growth model, along the stable path the growth rate of technology variable evolves in the opposite direction of the theoretical technology level / level of technology in practice ratio and it suffers no change when the capital / technology ratio is modified.*

Proof: The proof of this proposition follows directly from the computation of the eigenvectors associated to the negative eigenvalues of matrix *J*. Let $P_1=(p_{11},p_{21},p_{31})'$ be the eigenvector associated to λ_1 and $P_3=(p_{13},p_{23},p_{33})'$ the eigenvector associated to λ_3. The slopes of the relation between each of the variables $\phi(t)$ *and* $\varphi(t)$ and the control variable $\tau(t)$ is given by the product $\begin{bmatrix} p_{11} & p_{13} \\ p_{21} & p_{23} \end{bmatrix} . \begin{bmatrix} p_{31} \\ p_{33} \end{bmatrix}$ what leads to the following stable plan:

$$\tau(t) - \overline{\tau} = -\frac{\frac{1-\theta}{1-\mu}.\frac{g(.)}{\overline{\phi}^2}.\overline{\tau}}{\frac{\theta}{\mu}.\overline{\tau} - \lambda_1}.\left[\phi(t) - \overline{\phi}\right] + 0.\left[\varphi(t) - \overline{\varphi}\right] \tag{11}$$

According to (11), the convergence to the steady state relation between $\phi(t)$ and $\tau(t)$ is of opposite sign, while there is no relation between the movement of variable $\varphi(t)$ and the movement of $\tau(t)$, as the proposition states•

We now draw the stable trajectory in a three dimensional diagram (figure 3).

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[5] The other two are the ones computed before for matrix *J*.

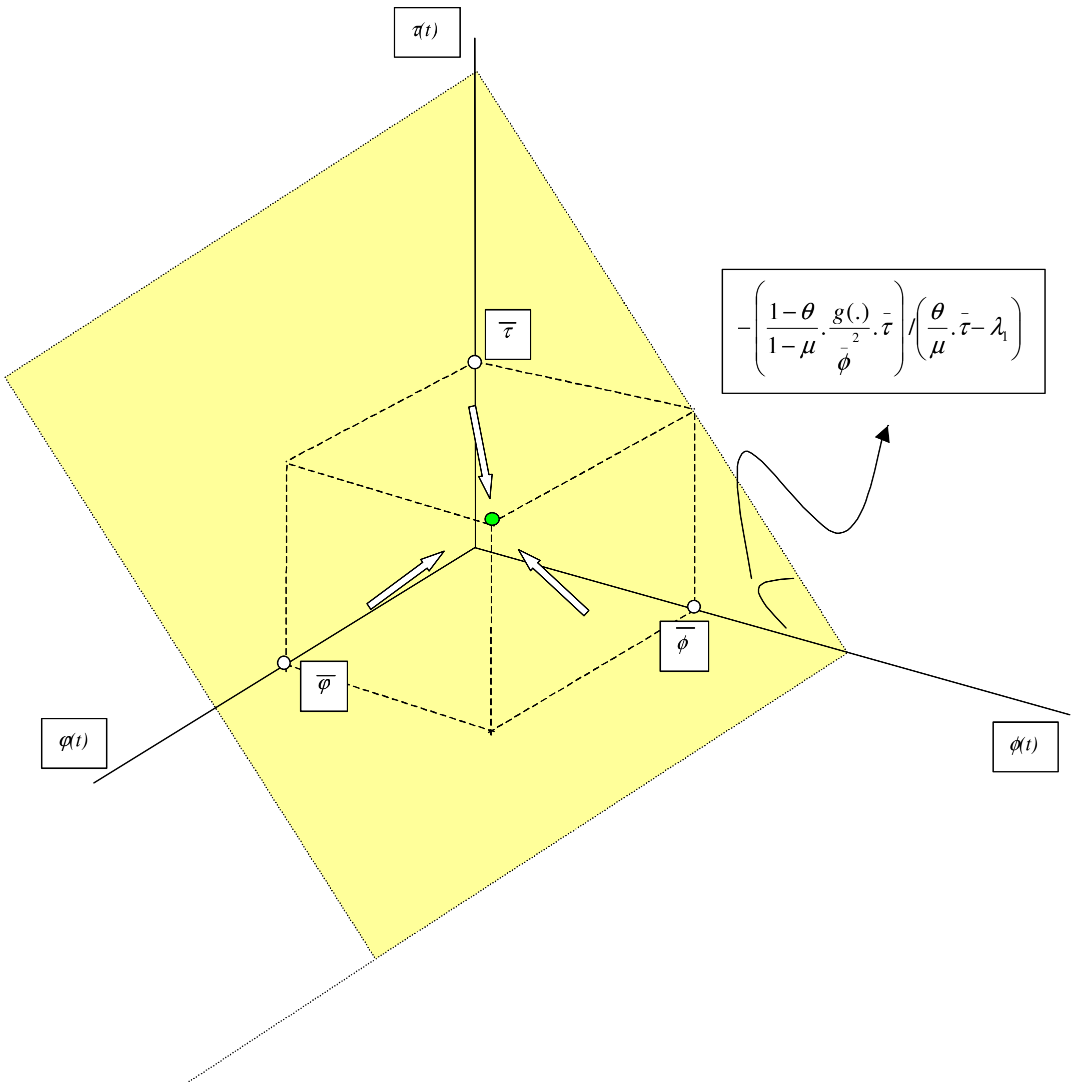

Figure 3: Saddle-path stable plan in the growth model

Figure 3 indicates that on the stable path our three variables converge to the steady state, independently of their initial values. Note that such a stable area respects the two conditions imposed by (11): we have an area where any set of values $[\varphi(t), \phi(t)]$ is possible and where the relation between $\varphi(t)$ and $\tau(t)$ is always translated by a horizontal line for whatever $\phi(t)$ (the slope is equal to zero) and the relation between $\phi(t)$ and $\tau(t)$ is given by a negative slope that is valid for all $\varphi(t)$.

VI. CONCLUSIONS

The crucial point of our analysis is the recognition of a trade-off between producing technology and applying technology. The objective of the economic system is twofold: first, we want to expand the technological capabilities of the economy and, second, we want to make this technology ready to use in the productive process. In this way, it is rather straightforward to construct an objective function and to use this function in an optimal control problem. The steady state results highlight the trade-off. In the steady state, the value of one of the variables included in the objective function will rise only if the value of the other falls. Transitional dynamics display a same kind of behavior: on the adjustment to the steady state, the growth rate of technology rises only in the presence of an increasing gap between technology in practice and the theoretical technological possibilities frontier.

The developed framework can be, as the consumption utility framework, applied to many fields of the economic thought. We have given an example of such a field: economic growth. Introducing a technology choices setup into a Solow growth model with an exogenous saving rate we were able to change the nature of the growth process. Growth becomes endogenous in the sense we include technology options in the economy's plans. If technology is an engine for growth and we model this variable then our model will be an endogenous growth model where, in the steady state, the growth rate of the economy does not depend only upon state parameters (g and a) but upon parameters that characterize the choices of the economy (in the case choices respecting to technology creation and diffusion over time: θ and μ).

This model intends to be a general framework. It may serve different purposes and may be applied in different areas to regain new insights about old problems. In the growth model case, we use the new framework to develop a new notion of endogenous growth - endogenous growth through technology decisions over time.

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