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RESEARCH REPORT

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STOCHASTIC MODEL OF THIN MARKET OF NONDIVISIBLE COMMODITY

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Abstract

We assume a thin market with finite number of buyers and sellers, each agent having a single jump demand xor supply function. Further, we assume that number of each agent's arrival is a Poisson distributed random variable. We describe the joint distribution of the market price and of the traded volume. Further, we examine a model with infinite number of agents (which may serve as an approximation of the model with the finite number of agents). Again, we describe the joint distribution of the price and the volume.

Keywords: Thin market, market price, traded volume

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1 The model with finite number of agents

Assume m buyers and n sellers of a non-divisible commodity. Assume the buyers and the sellers to have jump (individual) demand function, supply function respectively - in particular, the i-th buyer is willing to buy the commodity for maximal price x_i while the j-th seller is willing to sell the commodity for minimal price y_j. Assume that the amount demanded by the i-th buyer - denoted by D_i - is random variable having Poisson distribution with parameter p_i while the amount offered by the j-th seller - denoted by S_j - is random variable having Poisson distribution with parameter q_j. Finally, suppose that all variables D_i and S_j are mutually independent.

Define

$$D(p) \triangleq \sum_{x_i \geq p} D_i, \quad \delta(p) \triangleq \sum_{x_i \geq p} p_i$$

and

$$S(p) \triangleq \sum_{y_i \leq p} S_j, \quad \sigma(p) \triangleq \sum_{y_i \leq p} q_j.$$

Lemma 1.1

$$D(x) \sim \mathrm{Po}\left(\delta(x)\right)$$

for each $-\infty \leq x < \infty$,

$$S(y) \sim \mathrm{Po}\left(\sigma(y)\right)$$

for each $-\infty < y \leq \infty$,

$$D_{[x_1,x_2)} \triangleq D(x_1) - D(x_2) \sim \mathrm{Po}\left(\delta(x_2) - \delta(x_2)\right)$$

for each $-\infty \leq x_1 < x_2 < \infty$*,*

$$S_{(y_1,y_2]} \triangleq S(y_2) - S(y_1) \sim \mathrm{Po}\,(\sigma(y_2) - \sigma(y_1))$$

for each $-\infty < y_1 < y_2 \leq \infty$*. Moreover,* $D_{[x_1,x_2)}$ *is independent on* $D(x_2)$ *for each* $-\infty \leq x_1 < x_2 < \infty$*, and* $S_{(y_1,y_2]}$ *is independent on* $S(y_1)$ *for each* $-\infty < y_1 < y_2 \leq \infty$*.*

Proof. It is well known that sum of independent Poisson variables is Poisson with parameter equal to sum of the parameters of the summed variables. Hence, the assertion follows directly from the definitions. □

Denote $X_{[k]}$ the k-th greatest maximal price having emerged on the market. Preciously, if $D(-\infty) \geq k$ then we define $X_{[k]}$ by relation

$$D(X_{[k]}) \geq k, \quad D(X_{[k]}^{+}) < k, \tag{1}$$

if $D(-\infty) < k$ then we put $X_{[k]} = -\infty$.

Analogously, denote $Y_{(k)}$ the k-th least minimal price having emerged on the market: If $S(\infty) \geq k$ then we define $Y_{(k)}$ by relation

$$S(Y_{(k)}^{-}) < k, \quad S(Y_{(k)}) \geq k, \tag{2}$$

if $S(\infty) < k$ then we put $Y_{(k)} = \infty$.

Assume that the market price is determined as the average price maximizing total traded volume. More preciously, define

$$Q = \max_{p \in \mathrm{R}} (D(p), S(p)).$$

and assume that the market price P is determined by relation

$$P = \begin{cases} (X_{[Q]} - Y_{(Q)})/2 & \text{if } Q > 0 \\ \text{undefined} & \text{if } Q = 0 \end{cases}.$$

Our goal is to determine the joint probability distribution of random vector[1] (Q, P), namely, we wish to evaluate the quantity $\mathcal{P}\{Q = k, P < p\}$.

If $k > 0$ then

$$\begin{aligned} (Q = k) \wedge (P < p) \;&\Leftrightarrow\; (X_{[k]} \geq Y_{(k)}) \wedge (X_{[k+1]} < Y_{(k+1)}) \wedge (X_{[k]} + Y_{(k)} < 2p) \\ &\Leftrightarrow\; (Y_{(k)} \leq X_{[k]} < 2p - Y_{(k)}) \wedge (X_{[k+1]} < Y_{(k+1)}) \end{aligned} \tag{3}$$

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[1] We regard P as random element on space $\mathrm{R} \cup \{\text{undefined}\}$ with sigma algebra generated by Borel sets on R and the set $\{\text{undefined}\}$.

so that

$$\begin{aligned}\mathcal{P}\{Q=k,P<p\} &= \int_{\{x_k\geq y_k, x_{k+1}<y_{k+1}, x_k+y_k<2p\},} d\left(\mu_k(x_k,x_{k+1})\otimes\nu_k(y_k,y_{k+1})\right)\\ &= \int_{\{y_k<p\}}\left(\int_{\{y_k\leq x_k<2p-y_k, x_{k+1}<y_{k+1}\}} d\mu_k(x_k,x_{k+1})\right) d\nu_k(y_k,y_{k+1})\end{aligned}$$

where μ_k is probability distribution of extended[2] random vector $(X_{[k]},X_{[k+1]})$ and ν_k is probability distribution of extended random vector $(Y_{(k)},Y_{(k+1)})$ (we have used the fact that $(X_{[k]},X_{[k+1]})$ and $(Y_{(k)},Y_{(k+1)})$ are independent). Since

$$Q=0\Leftrightarrow X_{[1]}<Y_{(1)}$$

it holds that

$$\mathcal{P}\{Q=0\}=\int_{\{x<y\}} d\left(\mu_k(x)\otimes\nu_k(y)\right)=\int\left(\int_{\{x<y\}} d\mu_k(x)\right)d\nu_k(y). \tag{4}$$

1.1 The distribution of the market price and the traded volume

Before we state a formula for distribution of (P,Q) we have to derive the distribution of vectors $(X_{[k]},X_{[k+1]})$ and $(Y_{(k)},Y_{(k+1)})$, $k\in\mathbb{N}$.

Lemma 1.2 *Let $k>0$ and define*

$$\pi_1(k,x)\triangleq\mathcal{P}\left\{X_{[k]}=x\right\},\qquad \pi_2(k,x_1,x_2)\triangleq\mathcal{P}\left\{X_{[k]}=x_1,X_{[k+1]}=x_2\right\}$$

Then it holds that

$$\pi_1(k,x)=\sum_{i=0}^{k-1}e^{-\delta(x+)}\frac{\delta(x^+)^i}{i!}-\sum_{i=0}^{k-1}e^{-\delta(x)}\frac{\delta(x)^i}{i!} \tag{5}$$

for each $x>-\infty$ and

$$\pi_1(k,-\infty)=\sum_{i=0}^{k-1}e^{-\delta(-\infty)}\frac{\delta(-\infty)^i}{i!}. \tag{6}$$

Further,

$$\pi_2(k,x_1,x_2) = \left(\frac{\delta(x_1)^k}{k!}-\frac{\delta(x_1^+)^k}{k!}\right)\left(e^{-\delta(x_2^+)}-e^{-\delta(x_2)}\right) \tag{7}$$

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[2] According to the usual definition, extended random variable is random element that may take values from $\mathbb{R}\cup\{\infty\}$, see [2] for information how to handle extended random variables. We regard as extended also the elements taking values in $\mathbb{R}\cup\{-\infty\}$

for each $-\infty < x_2 < x_1$,

$$\pi_2(k,x,x) = \pi_1(k,x) - \left(\frac{\delta(x)^k}{k!} - \frac{\delta(x^+)^k}{k!}\right) e^{-\delta(x)}.$$

for each $-\infty < x$,

$$\pi_2(k,x,-\infty) = \left(\frac{\delta(x)^k}{k!} - \frac{\delta(x^+)^k}{k!}\right) e^{-\delta(-\infty)}$$

for each $-\infty < x$ *and*

$$\pi_2(k,x_1,x_2) = 0 \tag{8}$$

for each $-\infty < x_1 < x_2$.

Proof.

$$\begin{aligned}
\pi_1(k,x) &= \mathcal{P}\left\{D(x) \geq k, D(x^+) < k\right\} \\
&= \mathcal{P}\left\{D(x^+) < k\right\} - \mathcal{P}\left\{D(x^+) < k, D(x) < k\right\} \\
&= \mathcal{P}\left\{D(x^+) < k\right\} - \mathcal{P}\left\{D(x) < k\right\}
\end{aligned}$$

because $D(x) < k \Rightarrow D(x^+) < k$. Similarly,

$$\pi_1(k,-\infty) = \mathcal{P}\left\{D(-\infty) < k\right\}$$

Further, if we denote $D_A \overset{\triangle}{=} \sum_{x_i \in A} D_i$, we get

$$\begin{aligned}
\pi_2(k,x_2,x_1) &= \mathcal{P}\left\{D(x_1^+) < k, D(x_1) \geq k, D(x_2^+) < k+1, D(x_2) \geq k+1\right\} \\
&= \mathcal{P}\left\{D(x_1^+) < k, D(x_1) = k, D(x_2^+) = k, D(x_2) > k\right\} \\
&= \mathcal{P}\left\{D(x_1^+) < k, D(x_1) = k, D_{(x_2,x_1)} = 0, D_{\{x_2\}} > 0\right\} \\
&= \mathcal{P}\left\{D(x_1^+) < k, D(x_1) = k\right\} \mathcal{P}\left\{D_{(x_2,x_1)} = 0, D_{\{x_2\}} > 0\right\} \\
&= \left[\mathcal{P}\left\{D(x_1) = k\right\} - \mathcal{P}\left\{D(x_1) = k, D(x_1^+) = k\right\}\right] \\
&\qquad \cdot \mathcal{P}\left\{D_{(x_2,x_1)} = 0, D_{\{x_2\}} > 0\right\} \\
&= \left[\mathcal{P}\left\{D(x_1) = k\right\} - \mathcal{P}\left\{D_{\{x_1\}} = 0, D(x_1^+) = k\right\}\right] \\
&\qquad \cdot \mathcal{P}\left\{D_{(x_2,x_1)} = 0, D_{\{x_2\}} > 0\right\} \\
&= \left[\mathcal{P}\left\{D(x_1) = k\right\} - \mathcal{P}\left\{D(x_1^+) = k\right\} \mathcal{P}\left\{D_{\{x_1\}} = 0\right\}\right] \\
&\qquad \cdot \mathcal{P}\left\{D_{(x_2,x_1)} = 0\right\} \left[1 - \mathcal{P}\left\{D_{\{x_2\}} = 0\right\}\right] \\
&= \left(e^{-\delta(x_1)} \frac{\delta(x_1)^k}{k!} - e^{-(\delta(x_1) - \delta(x_1^+))} e^{-\delta(x_1^+)} \frac{\delta(x_1^+)^k}{k!}\right) \\
&\qquad \cdot e^{-(\delta(x_2^+) - \delta(x_1))} \left[1 - e^{-(\delta(x_2) - \delta(x_2^+))}\right]
\end{aligned}$$

which is nothing else but (7). Similarly,

$$\begin{aligned}
\pi_2(k,x,x) &= \mathcal{P}\left\{D(x) \geq k+1, D(x^+) < k\right\} \\
&= \mathcal{P}\left\{D(x) \geq k, D(x^+) < k\right\} - \mathcal{P}\left\{D(x) = k, D(x^+) < k\right\} \\
&= \pi_1(k,x) - \left[\mathcal{P}\left\{D(x) = k\right\} - \mathcal{P}\left\{D(x) = k, D(x^+) = k\right\}\right] \\
&= \pi_1(k,x) - \left[\mathcal{P}\left\{D(x) = k\right\} - \mathcal{P}\left\{D_{\{x\}} = 0\right\}\mathcal{P}\left\{D(x^+) = k\right\}\right]
\end{aligned}$$

(we have used fact that $D(\bullet)$ is non-increasing). Finally,

$$\begin{aligned}
\pi_2(k,x,-\infty) &= \mathcal{P}\left\{D(x^+) < k, D(x) = k, D(-\infty) < k+1\right\} \\
&= \mathcal{P}\left\{D(x) = k, D(-\infty) = k\right\} \\
&\qquad -\mathcal{P}\left\{D(x^+) \geq k, D(x) = k, D(-\infty) = k\right\} \\
&= \mathcal{P}\left\{D(x) = k, D_{(-\infty,x)} = 0\right\} - \mathcal{P}\left\{D(x^+) = k, D_{(-\infty,x)} = 0\right\} \\
&= \left[\mathcal{P}\left\{D(x) = k\right\} - \mathcal{P}\left\{D(x^+) = k\right\}\right]\mathcal{P}\left\{D_{(-\infty,x)} = 0\right\}.
\end{aligned}$$

The relation (8) is obvious. □

Lemma 1.3 *Let $k > 0$ and define*

$$\rho_1(k,y) \triangleq \mathcal{P}\left\{Y_{(k)} = y\right\}, \qquad \rho_2(k,y_1,y_2) \triangleq \mathcal{P}\left\{Y_{(k)} = y_1, Y_{(k+1)} = y_2\right\}$$

Then it holds that

$$\rho_1(k,y) = \sum_{i=0}^{k-1} e^{-\sigma(y^-)}\frac{\sigma(y^-)^i}{i!} - \sum_{i=0}^{k-1} e^{-\sigma(y)}\frac{\sigma(y)^i}{i!}$$

for each $y < \infty$ and

$$\rho_1(k,\infty) = \sum_{i=0}^{k-1} e^{-\sigma(\infty)}\frac{\sigma(\infty)^i}{i!}.$$

Further

$$\rho_2(k,y_1,y_2) = \left(\frac{\sigma(y_1)^k}{k!} - \frac{\sigma(y_1^-)^k}{k!}\right)\left(e^{-\sigma(y_2^-)} - e^{-\delta(y_2)}\right)$$

for $y_1 < y_2 < \infty$,

$$\rho_2(k,y,y) = \rho_1(k,y) - \left(\frac{\sigma(y)^k}{k!} - \frac{\sigma(y^-)^k}{k!}\right)e^{-\sigma(y)}$$

for $y < \infty$,

$$\rho_2(k,y,\infty) = \left(\frac{\sigma(y)^k}{k!} - \frac{\sigma(y^-)^k}{k!}\right)e^{-\sigma(\infty)}$$

for $y < \infty$ and

$$\rho_2(k,y_2,y_1) = 0$$

for $y_2 < y_1 < \infty$.

Proof. Put $Y'_{(k)} \overset{\triangle}{=} -Y_{(k)}$, $\sigma'(p) \overset{\triangle}{=} \sigma(-p)$, etc. and repeat the proof of the previous lemma for $Y'_{(k)}$ instead of $X_{[k]}$, for $\sigma'(p)$ instead of $\delta(p)$ etc. □

Lemma 1.4 *Denote* $x_1, x_2, \ldots, x_{m'}$ *all the discontinuities of* $\delta(\bullet)$ *and put* $x_0 = -\infty$. *Denote* $y_1, y_2, \ldots, y_{n'}$ *all the discontinuities of* $\sigma(\bullet)$ *and put* $y_{n'+1} = \infty$. *If* $k > 0$ *then it holds that*

$$\mathcal{P}\left\{Q = k, P < p\right\} = \sum_{i_1=0}^{m'} \sum_{i_2=0}^{m'} \sum_{j_1=1}^{n'+1} \sum_{j_2=1}^{n'+1} I_{i_1,i_2,j_1,j_2}(p,k)\pi_2(k, x_{i_1}, x_{i_2})\rho_2(k, y_{j_1}, y_{j_2}) \tag{9}$$

where

$$I_{i_1,i_2,j_1,j_2}(p,k) = \begin{cases} 1 & \textit{if } x_{i_1} \geq y_{j_1}, x_{i_2} < y_{j_2}\ x_{i_1} + y_{i_2} < 2p, \\ 0 & \textit{otherwise.} \end{cases}$$

and

$$\mathcal{P}\left\{Q = 0\right\} = \sum_{\{x_i < y_j : i=0,1,\ldots,m', j=1,2,\ldots,n'+1\}} \pi_1(1, x_i)\rho_1(1, y_j). \tag{10}$$

Proof. The formula (9) is an application of (3), the formula 10) is an application of (4). □

Remark. Note that the distribution does not depend directly on the number of agents and on intensities of theri arrival but it depends only on the functions $\delta(\bullet)$ and $\sigma(\bullet)$.

2 The model with continuous $\delta(\bullet)$ and $\sigma(\bullet)$

In accordance with the section 1, assume that the number of buyers willing to buy the commodity for at most x is a random variable defined by $D(x)$. Similarly, assume that the number of sellers who are willing to sell the commodity for at least y is a random variable defined by $S(x)$. Further, assume that the Lemma 1.1 holds for $D(\bullet)$ and $S(\bullet)$. Opposed to the section 1, assume $\delta(\bullet)$ is continuous non-increasing such that $\lim_{x\to\infty} \delta(x) = 0$ and $\sigma(\bullet)$ is continuous non-decreasing such that $\lim_{x\to-\infty} \sigma(x) = 0$.

The new model may be understood either as an approximation of the model of the section 1 or as a model with infinite number of agents.

Define $X_{[k]}$ and $Y_{(k)}$ the same way as in section 1. We derive their distribution first.

Lemma 2.1

$$\phi_k(x) \triangleq \mathcal{P}\left\{X_{[k]} < x\right\} = a_k(x) \tag{11}$$

$$\Phi_k(x_1,x_2) \triangleq \mathcal{P}\left\{X_{[k]} < x_1, X_{[k+1]} < x_2\right\} = \begin{cases} a_k(x_1) & x_1 < x_2 \\ a_{k+1}(x_2) - \alpha_k(x_2,x_1) & x_1 \geq x_2 \end{cases} \tag{12}$$

where

$$a_k(x) \triangleq e^{-\delta(x)} \sum_{i=0}^{k-1} \frac{\delta(x)^i}{i!}, \quad \alpha_k(x,x') \triangleq e^{-\delta(x)} \frac{\delta(x')^k}{k!}.$$

Proof. Ad. (11).

$$\mathcal{P}\left\{X_{[k]} < x\right\} = \mathcal{P}\left\{D(x) < k\right\} \overset{(1)}{=} \mathcal{P}\left\{\mathrm{Po}\left(\delta(x)\right) < k\right\} = a_k(x).$$

Ad. (12). If $x_1 \leq x_2$ then from $X_{[k]} < x_1$ it follows that $X_{[k+1]} \leq X_{[k]} < x_1 \leq x_2$ so that

$$\mathcal{P}\left\{X_{[k]} < x_1, X_{[k+1]} < x_2\right\} = \mathcal{P}\left\{X_{[k]} < x_1\right\} \overset{(11)}{=} a_k(x_1)$$

If $x_2 < x_1$ then

$$\begin{aligned} \mathcal{P}\left\{X_{[k]} < x_1, X_{[k+1]} < x_2\right\} &= \mathcal{P}\left\{D(x_1) < k, D(x_2) < k+1\right\} \\ &= \mathcal{P}\left\{D(x_1) < k+1, D(x_2) < k+1\right\} \\ &\quad -\mathcal{P}\left\{D(x_1) = k, D(x_2) < k+1\right\} \end{aligned}$$

Since $D(x_2) \geq D(x_1) \wedge D(x_2) < k+1 \Rightarrow D(x_1) < k+1$ we have

$$\begin{aligned} \mathcal{P}\left\{D(x_1) < k+1, D(x_2) < k+1\right\} &= \mathcal{P}\left\{D(x_2) < k+1\right\} = \mathcal{P}\left\{\mathrm{Po}\left(\delta(x_2)\right) < k+1\right\} \\ &= a_{k+1}(x_2) \end{aligned}$$

and, similarly,

$$\begin{aligned} \mathcal{P}\left\{D(x_1) = k, D(x_2) < k+1\right\} &= \mathcal{P}\left\{D(x_1) = k, D(x_2) = k\right\} \\ &= \mathcal{P}\left\{D(x_1) = k, D_{[x_2,x_1)} = 0\right\} \\ &= \mathcal{P}\left\{\mathrm{Po}\left(\delta(x_1)\right) = k\right\} \mathcal{P}\left\{\mathrm{Po}\left(\delta(x_2) - \delta(x_1)\right) = 0\right\} \\ &= e^{-\delta(x_1)} \frac{\delta(x_1)^k}{k!} e^{-(\delta(x_2)-\delta(x_1))} = e^{-\delta(x_2)} \frac{\delta(x_1)^k}{k!} \end{aligned}$$

□

Lemma 2.2

$$\psi_k(y_1) \triangleq \mathcal{P}\left\{Y_{(k)} < y\right\} = b_k(y) \tag{13}$$

$$\Psi_k(y_1,y_2) \triangleq \mathcal{P}\left\{Y_{(k)} < y_1, Y_{(k+1)} < y_2\right\} = \left\{\begin{array}{ll} b_k(y_2) & y_2 < y_1 \\ b_k(y_1) - \beta_k(y_2,y_1) & y_2 \geq y_1 \end{array}\right. \tag{14}$$

$$\Psi_k^{\infty}(y) \triangleq \mathcal{P}\left\{Y_{(k)} < y, Y_{(k+1)} = \infty\right\} = e^{-\sigma(\infty)}\frac{\sigma(y)^k}{k!} \tag{15}$$

where

$$b_k(y) \triangleq e^{-\sigma(y)}\sum_{j=k}^{\infty}\frac{\sigma(y)^j}{j!} = 1 - e^{-\sigma(y)}\sum_{j=0}^{k-1}\frac{\sigma(y)^j}{j!}, \quad \beta_k(y,y') \triangleq e^{-\sigma(y)}\frac{\sigma(y')^k}{k!}.$$

Proof. Ad. (13).

$$\begin{aligned} \mathcal{P}\left\{Y_{(k)} < y\right\} &\overset{(2)}{=} \mathcal{P}\left\{D(y^-) \geq k\right\} = \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y^-)\right) \geq k\right\} \\ &= \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y)\right) \geq k\right\} = b_k(y) \end{aligned}$$

Ad. (14). If $y_2 \leq y_1$ then $Y_{(k+1)} < y_2$ implies $Y_{(k)} \leq Y_{(k+1)} < y_2 \leq y_1$ so that

$$\begin{aligned} \mathcal{P}\left\{Y_{(k)} < y_1, Y_{(k+1)} < y_2\right\} &= \mathcal{P}\left\{Y_{(k+1)} < y_2\right\} = \mathcal{P}\left\{S(y_2^-) \geq k+1\right\} \\ &= \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_2)\right) \geq k+1\right\} \\ &= b_k(y_2). \end{aligned}$$

If $y_1 < y_2$ then

$$\begin{aligned} &\mathcal{P}\left\{Y_{(k)} < y_1, Y_{(k+1)} < y_2\right\} \\ =&\mathcal{P}\left\{S(y_1^-) \geq k, S(y_2^-) \geq k+1\right\} \\ =&\mathcal{P}\left\{S(y_1^-) \geq k+1, S(y_2^-) \geq k+1\right\} + \mathcal{P}\left\{S(y_1^-) = k, S(y_2^-) \geq k+1\right\} \\ =&\mathcal{P}\left\{S(y_1^-) \geq k+1\right\} + \mathcal{P}\left\{S(y_1^-) = k, S(y_2^-) - S(y_1^-) > 0\right\} \\ =&\mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_1)\right) \geq k+1\right\} + \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_1)\right) = k\right\}\mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_2) - \sigma(y_1)\right) > 0\right\} \\ =&\mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_1)\right) \geq k\right\} - \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_1)\right) = k\right\} \\ &\quad + \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_1)\right) = k\right\}\left[1 - \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_2) - \sigma(y_1)\right) = 0\right\}\right] \\ =&\mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_1)\right) \geq k\right\} \\ &\quad - \mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_1)\right) = k\right\}\mathcal{P}\left\{\mathrm{Po}\left(\sigma(y_2) - \sigma(y_1)\right) = 0\right\} \\ =&b_k(y_1) - e^{-(\sigma(y_2)-\sigma(y_1))}e^{-\sigma(y_1)}\frac{\sigma(y_1)^k}{k!} = b_k(y_1) - e^{-\sigma(y_2)}\frac{\sigma(y_1)^k}{k!}. \end{aligned}$$

Ad. (15).

$$\begin{aligned} &\mathcal{P}\left\{Y_{(k)} < y, Y_{(k+1)} = \infty\right\} \\ &\qquad =\mathcal{P}\left\{S(y^-) \geq k, S(\infty) < k+1\right\} = \mathcal{P}\left\{S(y^-) = k, S(\infty) = k\right\} \\ &\qquad =\mathcal{P}\left\{S(y) = k, S(\infty) - S(y) = 0\right\} = e^{-(\sigma(\infty)-\sigma(y))}e^{-\sigma(y)}\frac{\sigma(y)^k}{k!} \\ &\qquad =e^{-\sigma(\infty)}\frac{\sigma(y)^k}{k!}. \end{aligned}$$

□

Now we may determine the joint distribution of (Q, P).

Lemma 2.3 *If* $k > 0$ *then*

$$\begin{aligned}
&\mathcal{P}\{Q = k, P < p\} \\
&\quad = \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{i=0}^{k-1} \frac{\delta(2p-y)^i}{i!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y)dy \\
&\quad - \int^{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{k-1} \frac{\delta(y)^i}{i!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y)dy \\
&\quad + \int^{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{k} \frac{\delta(y)^i}{i!} \frac{\sigma(y)^{k}}{k!} s(y)dy \\
&\quad + \int_{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{k} \frac{\delta(y)^i}{i!} \frac{\sigma(2p-y)^{k}}{k!} s(y)dy \\
&\quad - \int^{p} \left(\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz \right) \frac{\delta(2p-y)^k}{k!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y)dy.
\end{aligned}$$

where $s(y) = \frac{\partial \sigma(y)}{\partial y}$ *in all points in which the derivative exists and* $s(y) = 0$ *in all the remaining points*

$$\mathcal{P}\{Q = 0\} = \int e^{-\sigma(y)-\delta(y)} s(y)dy + e^{-\sigma(\infty)}.$$

Proof. Assume that $y_k < p$ and defin

$$I_{k,p}(y_k, y_{k+1}) \quad \stackrel{\triangle}{=} \quad \int_{\{x_k \geq y_k, x_{k+1} < y_{k+1}, (x_k + y_k)/2 < p\}} d\mu_k(x_k, x_{k+1}).$$

Trivially

$$I_{k,p}(y_k, y_{k+1}) \quad = \quad \int_{\{x_k \geq y_k, x_{k+1} < y_{k+1}, x_k < 2p - y_k\}} d\mu_k(x_k, x_{k+1}).$$

Since $y_k < 2p - y_k$ under our assumption, we have

$$\begin{aligned}
&I_{k,p}(y_k, y_{k+1}) \\
&\quad = \int_{\{x_k < 2p - y_k, x_{k+1} < y_{k+1}\}} d\mu_k(x_k, x_{k+1}) - \int_{\{x_k < y_k, x_{k+1} < y_{k+1}\}} d\mu_k(x_k, x_{k+1}) \\
&\quad \overset{\text{Lemma 2.1}}{=} \begin{cases} \Phi_k(2p - y_k, y_{k+1}) - \Phi_k(y_k, y_{k+1}) & \text{if } y_{k+1} < \infty, \\ \phi_k(2p - y_k) - \phi_k(y_k) & \text{if } y_{k+1} = \infty. \end{cases}
\end{aligned} \tag{16}$$

Now, let us evaluate $\mathcal{P}\{Q=k, P<p\}$. According to the basic rules of probability calculus,

$$\begin{aligned}
&\mathcal{P}\{Q=k, P<p\} \\
&\qquad = \int_{\{x_k \geq y_k, x_{k+1} < y_{k+1}, (x_k+y_k)/2 < p\}} d(\mu_k(x_k, x_{k+1}) \otimes \nu_k(y_k, y_{k+1})) \\
&\qquad = \int \left(\int_{\{x_k \geq y_k, x_{k+1} < y_{k+1}, x_k < 2p - y_k\}} d\mu_k(x_k, x_{k+1}) \right) d\nu_k(y_k, y_{k+1}) \\
&\qquad = \int_{\{y_k < p\}} I_{k,p}(y_k, y_{k+1}) d\nu_k(y_k, y_{k+1})
\end{aligned}$$

because $x_k \geq y_k \wedge x_k < 2p - y_k$ implies $y_k < p$. Since $\mathcal{P}\left\{Y_{(k+1)} < Y_{(k)}\right\} = 0$, we may add inequality $y_{k+1} \geq y_k$ and obtain, using the calculus of the Stielties integrals,

$$\begin{aligned}
&\mathcal{P}\{Q=k, P<p\} \\
&\qquad = \int_{\{y_k < p, y_{k+1} \geq y_k\}} I_{k,p}(y_k, y_{k+1}) d\nu_k(y_k, y_{k+1}) \\
&\qquad = \int_{\{y_k < p, y_{k+1} \geq y_k, y_{k+1} < \infty\}} I_{k,p}(y_k, y_{k+1}) d\nu_k(y_k, y_{k+1}) \\
&\qquad \quad + \int_{\{y_k < p, y_{k+1} \geq y_k, y_{k+1} = \infty\}} I_{k,p}(y_k, y_{k+1}) d\nu_k(y_k, y_{k+1}) \\
&\qquad = \int_{\{y_k < p, y_{k+1} \geq y_k\}} I_{k,p}(y_k, y_{k+1}) d\Psi_k(y_k, y_{k+1}) \\
&\qquad \quad + \int_{\{y_k < p\}} I_{k,p}(y_k, \infty) d\Psi_k^{\infty}(y_k) \\
&\qquad \overset{\text{Lemma 2.2}}{=} \int_{\{y_k < p, y_{k+1} \geq y_k\}} I_{k,p}(y_k, y_{k+1}) d\left[b_k(y_k) - e^{-\sigma(y_{k+1})} \frac{\sigma(y_k)^k}{k!} \right] \\
&\qquad \quad + \int_{\{y_k < p\}} I_{k,p}(y_k, \infty) d e^{-\sigma(\infty)} \frac{\sigma(y_k)^k}{k!}
\end{aligned}$$

Since distribution function b_k defines zero measure on space R^2,

$$\int I_{k,p}(y_k, y_{k+1}) db_k(y_k) = 0$$

so that

$$\begin{aligned}\mathcal{P}\left\{Q=k,P<p\right\} &= \int_{\{y_k<p,y_{k+1}\geq y_k\}} I_{k,p}(y_k,y_{k+1})d\left(-e^{-\sigma(y_{k+1})}\frac{\sigma(y_k)^k}{k!}\right)\\ &\quad+\int_{\{y_k<p\}} I_{k,p}(y_k,\infty)de^{-\sigma(\infty)}\frac{\sigma(y_k)^k}{k!}\\ &\overset{(16)}{=} I_1+I_2\end{aligned}$$

where

$$\begin{aligned}I_2 &= \int_{\{y_k<p\}}[\phi_k(2p-y_k)-\phi_k(y_k)]de^{-\sigma(\infty)}\frac{\sigma(y_k)^k}{k!}\\ &= \int_{\{y_k<p\}}[a_k(2p-y_k)-a_k(y_k)]e^{-\sigma(\infty)}d\frac{\sigma(y_k)^k}{k!}\end{aligned}$$

and

$$\begin{aligned}I_1 &= \int_{\{y_k<p,y_{k+1}\geq y_k\}}\left[\Phi_k(2p-y_k,y_{k+1})-\Phi_k(y_k,y_{k+1})\right]d(-\beta(y_{k+1},y_k))\\ &= \int_{\{y_k<p,y_{k+1}\geq y_k\}}\Phi_k(2p-y_k,y_{k+1})d(-\beta(y_{k+1},y_k))\\ &\quad-\int_{\{y_k<p,y_{k+1}\geq y_k\}}\Phi_k(y_k,y_{k+1})d(-\beta(y_{k+1},y_k))\\ &\overset{\text{Lemma 2.1}}{=} \int_{\{y_k<p,y_{k+1}\geq y_k,2p-y_k<y_{k+1}\}} a_k(2p-y_k)d(-\beta(y_{k+1},y_k))\\ &\quad+\int_{\{y_k<p,y_{k+1}\geq y_k,2p-y_k\geq y_{k+1}\}} a_{k+1}(y_{k+1})d(-\beta(y_{k+1},y_k))\\ &\quad-\int_{\{y_k<p,y_{k+1}\geq y_k,2p-y_k\geq y_{k+1}\}} e^{-\delta(y_{k+1})}\frac{\delta(2p-y_k)^k}{k!}d(-\beta(y_{k+1},y_k))\\ &\quad-\int_{\{y_k<p,y_{k+1}\geq y_k\}} a_k(y_k)d(-\beta(y_{k+1},y_k)) = I_{1,1}+I_{1.2}-I_{1,3}-I_{1,4}\end{aligned}$$

where

$$\begin{aligned}I_{1,4} &= \int^p\left[a_k(y_k)\int_{y_k} d\left(-e^{-\sigma(y_{k+1})}\right)\right]d\frac{\sigma(y_k)^k}{k!}\\ &= \int^p a_k(y_k)\left[e^{-\sigma(y_k)}-e^{-\sigma(\infty)}\right]d\frac{\sigma(y_k)^k}{k!},\end{aligned}$$

$$\begin{aligned}I_{1,1} &= \int^p\left[a_k(2p-y_k)\int_{2p-y_k} d\left(-e^{-\sigma(y_{k+1})}\right)\right]d\frac{\sigma(y_k)^k}{k!}\\ &= \int^p a_k(2p-y_k)\left[e^{-\sigma(2p-y_k)}-e^{-\sigma(\infty)}\right]d\frac{\sigma(y_k)^k}{k!},\end{aligned}$$

$$\begin{aligned} I_{1,2} &= \int \left(a_{k+1}(y_{k+1}) \int^{p\wedge 2p-y_{k+1}\wedge y_{k+1}} d\frac{\sigma(y_k)^k}{k!} \right) d\left(-e^{-\sigma(y_{k+1})}\right) \\ &= \int a_{k+1}(y_{k+1}) \frac{\sigma(2p-y_{k+1}\wedge y_{k+1})^k}{k!} d\left(-e^{-\sigma(y_{k+1})}\right) \end{aligned}$$

and

$$I_{1,3} = \int^p \left(\frac{\delta(2p-y_k)^k}{k!} \int_{y_k}^{2p-y_k} e^{-\delta(y_{k+1})} d\left(-e^{-\sigma(y_{k+1})}\right) \right) d\frac{\sigma(y_k)^k}{k!}$$

Put all together,

$$\begin{aligned} &\mathcal{P}\left\{Q=k, P<p\right\} \\ &= I_{1,1} - I_{1,4} + I_2 - I_{1,3} + I_{1.2} \\ &= \int^p \Big(a_k(2p-y_k)\left[e^{-\sigma(2p-y_k)} - e^{-\sigma(\infty)}\right] - a_k(y_k)\left[e^{-\sigma(y_k)} - e^{-\sigma(\infty)}\right] \\ &\quad + [a_k(2p-y_k) - a_k(y_k)]e^{-\sigma(\infty)} \Big) d\frac{\sigma(y_k)^k}{k!} \\ &\quad - \int^p \frac{\delta(2p-y_k)^k}{k!} \left(\int_{y_k}^{2p-y_k} e^{-\delta(y_{k+1})} d\left(-e^{-\sigma(y_{k+1})}\right) \right) d\frac{\sigma(y_k)^k}{k!} \\ &\quad + \int a_{k+1}(y_{k+1}) \frac{\sigma(2p-y_{k+1}\wedge y_{k+1})^k}{k!} d\left(-e^{-\sigma(y_{k+1})}\right) \\ &= \int^p \left(a_k(2p-y_k)e^{-\sigma(2p-y_k)} - a_k(y_k)e^{-\sigma(y_k)} \right) d\frac{\sigma(y_k)^k}{k!} \\ &\quad - \int^p \left(\frac{\delta(2p-y_k)^k}{k!} \int_{y_k}^{2p-y_k} e^{-\delta(y_{k+1})} d\left(-e^{-\sigma(y_{k+1})}\right) d\frac{\sigma(y_k)^k}{k!} \right) \\ &\quad + \int^p a_{k+1}(y_{k+1}) \frac{\sigma(y_{k+1})^k}{k!} d\left(-e^{-\sigma(y_{k+1})}\right) \\ &\quad + \int_p a_{k+1}(y_{k+1}) \frac{\sigma(2p-y_{k+1})^k}{k!} d\left(-e^{-\sigma(y_{k+1})}\right) \\ &= \int^p \left(e^{-\sigma(2p-y_k)-\delta(2p-y_k)} \sum_{i=0}^{k-1} \frac{\delta(2p-y_k)^i}{i!} - e^{-\sigma(y_k)-\delta(y_k)} \sum_{i=0}^{k-1} \frac{\delta(y)^i}{i!} \right) d\frac{\sigma(y_k)^k}{k!} \\ &\quad - \int^p \left(\frac{\delta(2p-y_k)^k}{k!} \int_{y_k}^{2p-y_k} e^{-\delta(y_{k+1})} d\left(-e^{-\sigma(y_{k+1})}\right) d\frac{\sigma(y_k)^k}{k!} \right) \\ &\quad + \int^p \frac{\sigma(y_{k+1})^k}{k!} e^{-\delta(y_{k+1})} \sum_{i=0}^{k} \frac{\delta(y_{k+1})^i}{i!} d\left(-e^{-\sigma(y_{k+1})}\right) \\ &\quad + \int_p \frac{\sigma(2p-y_{k+1})^k}{k!} e^{-\delta(y_{k+1})} \sum_{i=0}^{k} \frac{\delta(y_{k+1})^i}{i!} d\left(-e^{-\sigma(y_{k+1})}\right). \end{aligned}$$

Since $\sigma(\bullet)$ is continuous non-decreasing, it has a derivative almost everywhere. Hence, all the distribution functions have a derivative almost everywhere so that

$$
\begin{aligned}
&\mathcal{P}\{Q = k, P < p\} \\
&\qquad = \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{i=0}^{k-1} \frac{\delta(2p-y)^i}{i!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y)dy \\
&\qquad - \int^{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{k-1} \frac{\delta(y)^i}{i!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y)dy \\
&\qquad + \int^{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{k} \frac{\delta(y)^i}{i!} \frac{\sigma(y)^{k}}{k!} s(y)dy \\
&\qquad + \int_{p} e^{-\sigma(y)-\delta(y)} \sum_{i=0}^{k} \frac{\delta(y)^i}{i!} \frac{\sigma(2p-y)^{k}}{k!} s(y)dy \\
&\qquad - \int^{p} \left(\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz \right) \frac{\delta(2p-y)^k}{k!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y).
\end{aligned}
$$

□

Before we proceed, we handle an important special case.

Lemma 2.4 *If* $\sigma(L) = 0$ *for some* $L \in \mathrm{R}$ *and* $\delta(H) = 0$ *for some* $H > L$ *then*

$$\mathcal{P}\{Q = k, P < p\} = 0$$

for each $p \leq L$,

$$\mathcal{P}\{Q = k, P < p\} = \mathcal{P}\{Q = k, p < H\}$$

for each $p > H$ *and it holds that and*

$$\mathcal{P}\{Q = 0\} = \int_{L}^{H} e^{-\sigma(y)-\delta(y)} s(y)dy + e^{-\sigma(H)}.$$

Proof. Since, according to (1), it has to be $P \leq X_{[Q]}$ and $\mathcal{P}\left\{X_{[k]} \geq H\right\} = 0$ for each $k > 0$. Therefore, $P \leq H$ almost sure so that $\mathcal{P}\{Q = k, P < p\} = \mathcal{P}\{Q = k, P < H\}$ for each $p \geq H$. Similarly we get that, for each $p \leq L$, it holds that $\mathcal{P}\{Q = k, P < p\} \leq \mathcal{P}\{Q = k, P < L\} = 0$. Finally, since $s(y) = 0$ for each $y < H$ and since $\delta(y) = 0$ for each $y \geq H$ under the assumptions of the present Lemma, we have

$$\begin{aligned}
\mathcal{P}\{Q=0\} &\overset{\text{Lemma 2.3}}{=} \int_L e^{-\sigma(y)-\delta(y)} s(y)dy + e^{-\sigma(\infty)} \\
&= \int_L^H e^{-\sigma(y)-\delta(y)} s(y)dy + \int_H e^{-\sigma(y)} s(y)dy + e^{-\sigma(\infty)} \\
&= \int_L^H e^{-\sigma(y)-\delta(y)} s(y)dy + [e^{-\sigma(H)} - e^{-\sigma(\infty)}] + e^{-\sigma(\infty)} \\
&= \int_L^H e^{-\sigma(y)-\delta(y)} s(y)dy + e^{-\sigma(H)}.
\end{aligned}$$

□

Our next goal is to determine the marginal distribution of the price. We do it in two steps.

Lemma 2.5

$$\begin{aligned}
&\mathcal{P}\{1 \le Q \le K, P < p\} \\
&= \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{K-1} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(y)dy \\
&\quad + \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{K-1} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(2p-y)dy \\
&\quad + \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \frac{\sigma(y)^K}{K!} \sum_{i=0}^{K} \frac{\delta(2p-y)^i}{i!} s(2p-y)dy \qquad (17) \\
&\quad + \int^p e^{-\sigma(y)-\delta(y)} \frac{\sigma(y)^K}{K!} \sum_{i=0}^{K} \frac{\delta(y)^k}{k!} s(y)dy \\
&\quad + \int^p \left[e^{-\sigma(2p-y)-\delta(2p-y)} - e^{-\sigma(y)-\delta(y)}\right] s(y)dy \\
&\quad - \int^p \left[\int_y^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz\right] \sum_{k=0}^{K-1} \left(\frac{\delta(2p-y)^{k+1}}{(k+1)!} \frac{\sigma(y)^k}{k!} s(y)\right) dy
\end{aligned}$$

Proof. According to Lemma 2.3,

$$\mathcal{P}\{1 \le Q \le K, P < p\} = I_1 + I_2 - I_3$$

where

$$I_3 = \int^p \left[\int_y^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz\right] \sum_{k=1}^{K} \left(\frac{\delta(2p-y)^k}{k!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y)\right) dy,$$

$$
\begin{aligned}
I_2 &= \int^p e^{-\sigma(y)} \left(\sum_{k=1}^{K} a_{k+1}(y) \frac{\sigma(y)^k}{k!} - \sum_{k=1}^{K} a_k(y) \frac{\sigma(y)^{k-1}}{(k-1)!} \right) s(y)dy \\
&= \int^p e^{-\sigma(y)} \left(\sum_{k=2}^{K+1} a_k(y) \frac{\sigma(y)^{k-1}}{(k-1)!} - \sum_{k=1}^{K} a_k(y) \frac{\sigma(y)^{k-1}}{(k-1)!} \right) s(y)dy \\
&= \int^p e^{-\sigma(y)} a_{K+1}(y) \frac{\sigma(y)^K}{K!} s(y)dy - \int^p e^{-\sigma(y)} e^{-\delta(y)} s(y)dy \\
&= \int^p e^{-\sigma(y)-\delta(y)} \frac{\sigma(y)^K}{K!} \sum_{i=0}^{K} \frac{\delta(y)^k}{k!} s(y)dy - \int^p e^{-\sigma(y)-\delta(y)} s(y)dy
\end{aligned}
$$

and

$$
\begin{aligned}
I_1 &= \int^p e^{-\sigma(2p-y)} \sum_{k=1}^{K} \frac{\sigma(y)^{k-1}}{(k-1)!} a_k(2p-y) s(y)dy \\
&\quad + \int_p e^{-\sigma(y)} \sum_{k=1}^{K} \frac{\sigma(2p-y)^k}{k!} a_{k+1}(y) s(y)dy \\
&= \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{K} \frac{\sigma(y)^{k-1}}{(k-1)!} \sum_{i=0}^{k-1} \frac{\delta(2p-y)^i}{i!} s(y)dy \\
&\quad + \int^p e^{-\sigma(2p-y)} \sum_{k=1}^{K} \frac{\sigma(y)^i}{i!} a_{k+1}(2p-y) s(2p-y)dy \\
&= \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=0}^{K-1} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(y)dy \\
&\quad + \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{K} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(2p-y)dy \\
&= \int^p e^{-\sigma(2p-y)-\delta(2p-y)} s(y)dy \\
&\quad + \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{K-1} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(y)dy \\
&\quad + \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{K-1} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(2p-y)dy \\
&\quad + \int^p e^{-\sigma(2p-y)-\delta(2p-y)} \frac{\sigma(y)^K}{K!} \sum_{i=0}^{K} \frac{\delta(2p-y)^i}{i!} s(2p-y)dy
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\mathcal{P}\left\{1 \leq Q \leq K, P < p\right\} \\
&\quad = \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} s(y)dy \\
&\qquad + \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{K-1} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} [s(y) + s(2p-y)]dy \\
&\qquad + \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \frac{\sigma(y)^K}{K!} \sum_{i=0}^{K} \frac{\delta(2p-y)^i}{i!} s(2p-y)dy \\
&\qquad + \int^{p} e^{-\sigma(y)-\delta(y)} \frac{\sigma(y)^K}{K!} \sum_{i=0}^{K} \frac{\delta(y)^k}{k!} s(y)dy - \int^{p} e^{-\sigma(y)-\delta(y)} s(y)dy \\
&\qquad - \int^{p} \left[\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz\right] \sum_{k=1}^{K} \left(\frac{\delta(2p-y)^k}{k!} \frac{\sigma(y)^{k-1}}{(k-1)!} s(y)\right) dy
\end{aligned}
$$

□

Finally, we are getting to the marginal distribution of the price.

Lemma 2.6

$$
\begin{aligned}
&\mathcal{P}\left\{Q \geq 1, P < p\right\} \\
&\quad = \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{\infty} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(y)dy \\
&\qquad + \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=1}^{\infty} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} s(2p-y)dy \\
&\qquad + \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)} s(y)dy - \int^{p} e^{-\sigma(y)-\delta(y)} s(y)dy \\
&\qquad - \int^{p} \left[\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz\right] \sum_{k=0}^{\infty} \left(\frac{\delta(2p-y)^{k+1}}{(k+1)!} \frac{\sigma(y)^k}{k!} s(y)\right) dy.
\end{aligned}
$$

Proof. According to the basic properties of probability,

$$\mathcal{P}\left\{Q \geq 1, P < p\right\} \quad = \quad \lim_{K \to \infty} \mathcal{P}\left\{1 \leq Q \leq K\right\}$$

hence, we are seeking limit of (17) as $K \to \infty$. Obviously, the limit equals to sum of limits of the six summands of (17). Since

$$\begin{aligned}
&\lim_{K\to\infty}\int^{p} e^{-\sigma(2p-y)-\delta(2p-y)}\frac{\sigma(y)^K}{K!}\sum_{i=0}^{K}\frac{\delta(2p-y)^i}{i!}s(2p-y)dy\\
&\quad\leq \lim_{K\to\infty}\int^{p} e^{-\sigma(2p-y)-\delta(2p-y)}\frac{\sigma(y)^K}{K!}e^{\delta(2p-y)}s(2p-y)dy\\
&\quad= \lim_{K\to\infty}\int^{p} e^{-\sigma(2p-y)}\frac{\sigma(y)^K}{K!}s(2p-y)dy\\
&\quad\leq \lim_{K\to\infty}\frac{\sigma(p)^K}{K!}\int^{p} e^{-\sigma(2p-y)}s(2p-y)dy = \lim_{K\to\infty}\frac{\sigma(p)^K}{K!}\left[e^{-\sigma(2p-y)}\right]_{-\infty}^{p}dy\\
&\quad= \lim_{K\to\infty}\frac{\sigma(p)^K}{K!}\left[e^{-\sigma(p)}-e^{-\sigma(\infty)}\right]dy = 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{K\to\infty}\int^{p} e^{-\sigma(y)-\delta(y)}\frac{\sigma(y)^K}{K!}\sum_{i=0}^{K}\frac{\delta(y)^k}{k!}s(y)dy \leq &\quad \lim_{K\to\infty}\frac{\sigma(p)^K}{K!}\int^{p} e^{-\sigma(y)}s(y)\\
= &\quad \lim_{K\to\infty}\frac{\sigma(p)^K}{K!}[-e^{-\sigma(y)}]_{-\infty}^{p}\\
= &\quad \lim_{K\to\infty}\frac{\sigma(p)^K}{K!}[1-e^{-\sigma(p)}] = 0
\end{aligned}$$

we have that

$$\lim_{K\to\infty}\mathcal{P}\left\{1\leq Q\leq K\right\} = J_1 - J_2 - J_3$$

where

$$\begin{aligned}
J_1 &= \lim_{K\to\infty}\sum_{k=0}^{K}\int^{p} e^{-\sigma(2p-y)-\delta(2p-y)}\frac{\sigma(y)^k}{k!}\sum_{i=0}^{k}\frac{\delta(2p-y)^i}{i!}[s(y)+s(2p-y)]dy\\
&= \sum_{k=0}^{\infty}\int^{p} e^{-\sigma(2p-y)-\delta(2p-y)}\frac{\sigma(y)^k}{k!}\sum_{i=0}^{k}\frac{\delta(2p-y)^i}{i!}[s(y)+s(2p-y)]dy\\
&\overset{\text{Levi}}{=} \int^{p} e^{-\sigma(2p-y)-\delta(2p-y)}\sum_{k=0}^{\infty}\frac{\sigma(y)^k}{k!}\sum_{i=0}^{k}\frac{\delta(2p-y)^i}{i!}[s(y)+s(2p-y)]dy
\end{aligned}$$

(we have used Levi's theorem for series (see [1], 8.12.)),

$$J_2 = \int e^{-\sigma(y)-\delta(y)}s(y)dy,$$

and

$$J_3 = \int^{p}\left[\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)}s(z)dz\right]\sum_{k=0}^{\infty}\left(\frac{\delta(2p-y)^{k+1}}{(k+1)!}\frac{\sigma(y)^k}{k!}s(y)\right)dy$$

also by Levi's Theorem. □

Remark. The result of the last Lemma may be rewritten using Bessel functions. We recall that the modified Bessel function of the first kind and of order $k \in \mathrm{N}$ is defined by formula

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!(\nu+k+1)!}.$$

Particularly, the quantities J_1 and J_3 from the previous proof may be reformulated. Since it holds that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\sigma(y)^k}{k!} \sum_{i=0}^{k} \frac{\delta(2p-y)^i}{i!} \\
&\qquad = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\sigma(y)^{k+j}\delta(2p-y)^j}{(k+j)!j!} = \sum_{k=0}^{\infty} \sigma(y)^k \sum_{j=0}^{\infty} \frac{[\sigma(y)\delta(2p-y)]^j}{(k+j)!j!} \\
&\qquad = \sum_{k=0}^{\infty} \frac{\delta(2p-y)^{k/2}}{\sigma(y)^{k/2}} \left(\sigma(y)\delta(2p-y)\right)^{k/2} \sum_{j=0}^{\infty} \frac{[\sigma(y)\delta(2p-y)]^j}{(k+j)!j!} \\
&\qquad = \sum_{k=0}^{\infty} \left(\frac{\delta(2p-y)}{\sigma(y)}\right)^{k/2} I_k\left(2\sqrt{\sigma(y)\delta(2p-y)}\right)
\end{aligned}$$

for each y s.t. $\delta(2p-y) > 0$, we have

$$\begin{aligned}
J_1 &= \int_{2p-H}^{p} e^{-\sigma(2p-y)-\delta(2p-y)} \sum_{k=0}^{\infty} \left(\frac{\delta(2p-y)}{\sigma(y)}\right)^{k/2} I_k\left(2\sqrt{\sigma(y)\delta(2p-y)}\right) \\
&\qquad\qquad [s(y)+s(2p-y)]dy
\end{aligned}$$

where $H = \sup\{y, \delta(y) > 0\}$. Similarly

$$\begin{aligned}
J_3 &= \int^{p} \left[\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz\right] \delta(2p-y) \sum_{k=0}^{\infty} \frac{(\delta(2p-y)\sigma(y))^k}{(k+1)!k!} s(y)dy \\
&= \int_{L}^{p} \left[\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz\right] \frac{\sqrt{\delta(2p-y)}}{\sqrt{\sigma(y)}} \sqrt{\delta(2p-y)\sigma(y)} \\
&\qquad\qquad \sum_{k=0}^{\infty} \frac{(\delta(2p-y)\sigma(y))^k}{(k+1)!k!} s(y)dy \\
&= \int_{L}^{p} \left[\int_{y}^{2p-y} e^{-\sigma(z)-\delta(z)} s(z)dz\right] \sqrt{\frac{\delta(2p-y)}{\sigma(y)}} I_1\left(2\sqrt{\delta(2p-y)\sigma(y)}\right) s(y)dy
\end{aligned}$$

where $L = \inf\{y, \sigma(y) > 0\}$.

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