Long Range Interaction Generating Fat-Tails in Finance

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Abstract

It's commonly known that the correlation between stocks increases during market turbulent periods. In this work we propose a modellization of this feature, viewed as a collective effect, rearranging a toy-model first proposed in 2001. Equities are modelled as quasi random walk variables, where the non-Brownian components of stocks movement are linked to the market trend via a long range interaction function. Our model generates fat tails for stock probability distributions and implied volatility surfaces analogous to real data, suggesting an unitary picture of long range interaction, fat tails and volatility smiles.

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1 Introduction

One of the most important goals in the application of Physics to Finance, is to model accurately financial time series and try to understand the rules that determines the dynamical evolution of financial markets.

At the beginning of the 20th century Bachelier showed [1] in his pioneering work that stock price fluctuations could be described as a Brownian motion. Indeed more accurate analyses of financial data have shown that the geometric Brownian motion is inadequate to correctly describe empirical distributions when large events occur and that real markets extreme events are more likely to happen than in a random walk generated with Gaussian probability distributions. In a Gaussian word the '29 crisis would have never occurred.

In order to account for fat tailed probability distribution in empirical data, Mandelbrot, in his 1963 paper [2], proposed to model financial time series using Lèvy's stable processes. More recently other models have been proposed to well fit the empirical stock probability distributions: truncated Lèvy flights, random volatility models, ARCH and GARCH processes and others [3]. These models concern the univariate case, where only one single financial time series is considered. This approach doesn't give any information about the market mechanism, and it can't describe the collective effects empirically observed on real markets.

In Risk Management most of the analyses are based on the so called multivariate Brownian motion, that describes a set of N assets by using a Brownian motion for each stock. The multivariate Brownian motion sets the cross-correlations between stocks with a covariance matrix, constant and time-independent. Moreover it has been demonstrated that a couple of correlated series, following a Brownian motion, are asymptotically independent, so that extreme events occur independently, and this approach fails to describe the correlation pattern of extreme returns.

Over the last 2 or 3 years some collective models for the market have been presented, trying to describe the features of the system (e.g. [5] to [10]). Most of them are agent-based minority game, so the idea is to gain the general behavior of the market by describing the microscopical mechanism that produces stock prices, trying to model traders'strategies (e.g [11][12][13][14]).

Our approach is quite different. One of the most interesting collective effects on real markets is the increase of cross-correlation during large market events. In fact empirical evidence has shown that correlations between stocks are strictly dependent on market volatility and increase sharply during turmoil (e.g. [15] to [22]).

In [23] a toy model was first proposed to describe this collective effect, with the aim to show the role of the market correlation structure in the generation of some of the most important features of financial time series. This model has the nice feature to reproduce some interesting behaviors of real markets, but it fails to generate the correct stock probability distributions. Starting from this observation, in the present paper we face the problem of correcting the model of [23] and building a new model that can give a consistent unitary picture of correlations, fat tails and volatility smiles.

The structure of the paper is the following. In section 2 we present the model that we developed, underlying the driving ideas that led us in its building and discussing the different possible regimes that can be described by this model. We also recover the corresponding expression for the probability distribution of the average market return. The numerical studies performed and the results obtained are presented and discussed in section 3. In section 4 we draw our final conclusions and discuss the possible developments of the analysis and refinements of the model.

2 The Model

As discussed in section 1, one of the most interesting characteristics of financial markets is the increase of cross-correlation during large market events. Basically there are empirical evidences that when market crisis or running days occur, most of the stocks follow the general trend. Our goal is to study the role of this correlation between stocks and the "average" market in the generation of fat-tailed probability distributions for stock returns.

In our model, every stock price follows the equation[1]:

$$A_i(t+\Delta t) = A_i(t) + A_i(t)\,\sigma\, s_i(t) \qquad i = 1 \dots N \tag{1}$$

where $A_i(t)$ is the stock value at time t, σ is the volatility for the step Δt and $s_i(t)$ is a random spin, so $s_i(t) = \pm 1$ following some hopping probability to be chosen (to simplify the formulas we also choose $\Delta t = 1$).

The equation (1) is the analogous of the discrete log-normal prices description. In fact, if $s_i(t)$ has an equal probability to be up or down, the continuous limit of the (1) generates a log-normal distribution. In our case we must choose an appropriate probability for $s_i(t)$ to get the right behavior during market turbulent periods.

Our probability has to meet two empirical requests:

- (A) during large market events stocks are closely related to the market trend
- (B) in market calm periods stocks behavior follows a log-normal distribution

Actually, to be closest to real behavior, classical financial models use a matrix to set linear cross correlations between couple of stocks. Obviously this kind of correlations are time independent and the matrix is symmetric. In this case it is possible to diagonalize the matrix choosing a stock portfolio in which stocks are independent. Since in our work we are not interested to examine the classic cross-correlation, during market calm periods our model chooses this ***ansatz***, therefore the second empirical request coincides to having independent spins when the market is not turbulent.

The form for the conditioned probability of our model is:

$$P\Big(s_i(t+1)\Big) = \frac{1}{2} + \frac{1}{2} s_i(t+1)\, g\Big(M(t)\Big), \tag{2}$$

where $P\Big(s_i(t)\Big)$ is the probability to have the value $s_i(t)$ at time t, and $M(t)$ is a macroscopic variable describing the global market behavior. Here we make the simplest possible choice to identify the variable $M(t)$ with the $s_i(t)$ average, in strict analogy with a "real" spin system, in which this is the expression for the global magnetization driving the system behavior. Hence we put:

$$M(t) = \sum_{i=1}^{N} \frac{s_i(t)}{N}, \tag{3}$$

with N equal to the number of stocks present in our market.

It is interesting to note that, as we are going to show, with this simple choice we are able to reproduce some peculiar aspects of financial systems. In any case we will discuss in the conclusions the possibility of making more realistic choices, which take into account the different relevance and role that different stocks can play in the evolution of a market.

[1] In this model we do not consider the free-risk interest rate. Actually it is simple to introduce a non zero rate by adding a term in equation (1), with negligibledifferences in the main results.

The function $g(M)$ appearing in eq. (2) must satisfy the requirement that $|g(M)| < 1$ and it controls the correlation between the single stock increase and the average normalized market return. In this view $M(t)$, the normalized market return, represents a general trend indicator. For $g(M) = 0$ the model in the continuum limit obviously reduces to N uncorrelated stocks evolving according to a log-normal process. The form of the function $g(M)$ must be chosen in a such a way to satisfy the requirements (A) and (B) corresponding to the two opposite market regimes.

To satisfy the first request, like in an Ising model, we could set

$$g(M) = \tanh(M) \tag{4}$$

This is a good choice for pushing stocks to follow the average market return in turbulent periods but it doesn't suit the second request. In fact in calm periods, when $M(t) \simeq 0$, we can set $\tanh(M) \simeq M$ and therefore the generated correlation between stocks and market is linear and the system is rather different from a log-normal market.

Our need is to identify a sort of threshold to start the correlation when large market events occur, therefore we need a "measure" for the market turmoil. The solution is almost simple: in the log-normal picture the probability distribution for $M(t)$ is a binomial distribution:

$$P_0(M) = \binom{N}{N(\frac{1+M}{2})} \cdot P_0(s_i = +1)^{N(\frac{1+M}{2})} \cdot P_0(s_i = -1)^{N-N(\frac{1+M}{2})} \tag{5}$$

where $P_0\Big(s_i(t) = \pm 1\Big) = \frac{1}{2}$ is the probability to have $s_i = \pm 1$ in a log-normal description. We can approximate the equation (5) by the normal distribution obtaining

$$P_0\Big(M(t)\Big) \simeq \frac{1}{\sqrt{2\pi\frac{N}{4}}} e^{-\frac{M^2}{\frac{2}{N}}} \tag{6}$$

The (6) provides a natural way to "measure" the turmoil of the market in terms of standard deviation of the normalized market return in a log-normal market.

So we correct equation (4) by setting

$$g(M) = \tanh\left[M\left(1 - e^{-\frac{M^2}{\frac{2}{N}\gamma}}\right)\right] \tag{7}$$

introducing a free parameter γ that determines the extension of the threshold for having a large market event. The equation (7) satisfies both the empirical requests (A) and (B). We choose the strange $\Big(1 - \exp(-M^2/\frac{2}{N}\gamma)\Big)$ factor to smoothen the gap between the two situations: turbulent or calm period. As we are going to show in the next sections, changing the value of the γ parameter, one can change the interaction, modulating the influence of the global market trend on the single stock behavior. In this way it is possible to study different regimes, simply adjusting the γ value.

2.1 Market return probability distribution

Now we can calculate the probability to have a market return $\widetilde{M}(t+1)$ at time $t+1$, depending on the value of the market return $M(t)$ at the previous step. It's a sort of transition matrix that allow to pass from a state with M to a state with $\widetilde{M}$. Thanks to some simple probability considerations we can write:

$$G(M,\widetilde{M}) = \binom{N}{N(\frac{1+\widetilde{M}}{2})} \left(\frac{1}{2}+\frac{1}{2}g(M)\right)^{N\frac{1+\widetilde{M}}{2}} \cdot \left(\frac{1}{2}-\frac{1}{2}g(M)\right)^{N\frac{1-\widetilde{M}}{2}} \tag{8}$$

that can be approximated by a normal distribution like that of equation (6):

$$G(M,\widetilde{M}) \simeq \frac{1}{\sqrt{2\pi\frac{N}{4}\left[1-g(M)^2\right]}} \cdot \exp\left[-\frac{\left[\widetilde{M}-g(M)\right]^2}{\frac{2\left(1-g(M)^2\right)}{N}}\right] \tag{9}$$

Therefore it's easy to write a recursive expression for the probability to have a state with average market return $\widetilde{M}$ at time $t+1$:

$$P_M\left(t+1,\widetilde{M}\right) = \sum_M G\left(M,\widetilde{M}\right) \cdot P_M\left(t,M\right) \tag{10}$$

¿From Frobenius-Perron theorem [24], it can be demonstrated that $P_M\left(t+1,\widetilde{M}\right)$ converges to a stable time independent probability distribution $P_M\left(M\right)$. Unfortunately, because of the lack of the detailed balance, we can't obtain an analytical expression for this distribution. This compels us to a numerical Monte Carlo approach for detecting the properties of our model.

3 Numerical Results

Our aim studying this model was to show the fundamental role of the correlation between each stock and the market generating some stylized facts of real markets.

The free parameter in the model is γ, that represents a sort of threshold for extreme events, linked to the standard deviation of the market average probability distribution for a system of N independent spins (see eq.(7) and the following discussion). In other words γ is linked to the pathological periods of the simulated market: small γ values generate a market that often falls in extreme periods, while big γ values generate a market that is very rarely turbulent.

We have studied the different behaviors of the model depending on the γ value, focusing on two main tasks:

a) the probability distributions for stock returns

b) the kurtosis of these probability distributions

The model wants to reproduce some real markets behaviors. So we choose for our numerical simulation a N value close to real number of stocks in a market. This number typically varies from some hundreds to some thousands. That's why our choice is $N = 1000$.

3.1 Probability Distributions

The behavior of our model remarkably changes if different γ value are used. Figure 1 shows the stock return probability distribution obtained with our model compared with a Gaussian fit for different γ values: $\gamma = 0.05$, $\gamma = 2$, $\gamma = 50$.

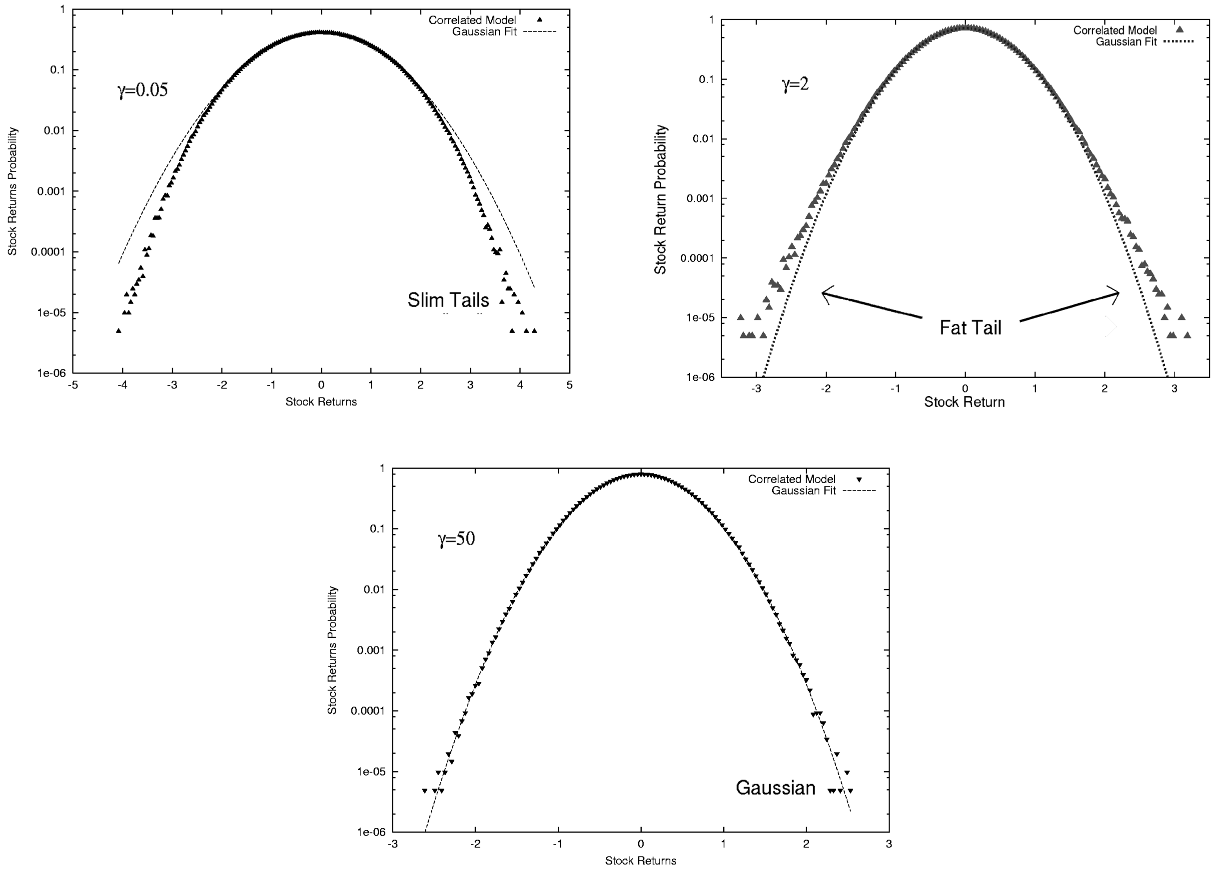

Figure 1: Comparison between the probability distribution for stock return of our model and a Gaussian fit for different γ values: $\gamma = 0.05$, $\gamma = 2$, $\gamma = 50$.(N=1000, Time=600 steps)

It's evident the presence of fat tails in the distribution generated with our model with $\gamma = 2$, while for $\gamma = 0.05$ there are slim tails and for $\gamma = 50$ the probability distribution is well fitted by the Gaussian. These three cases represent different behaviors of the correlation between every stock and the average market. According to what was said in section (2), we can think that for a small γ value the market is always turbulent, so the correlation is always present and stocks move all together causing the presence of slim tailed distribution.

The other extreme case, corresponding to a big γ value (like $\gamma = 50$ in our example), represents a market that is always in calm periods and, therefore, in which no correlations occur between stocks. The result is a ***"log-normal"*** market. In this picture $\gamma = 2$ corresponds to a right intermediate situation suitable to generate fat tailed distributions.

As discussed, our model can have three different regimes, depending on the γ value, that tunes the strength of the long range interaction near $M \simeq 0$. It's interesting to note that it's possible to find the threshold corresponding to the transition between the slim-tail regime and the fat-tail one, that is for $\gamma \simeq 0.90$, but there isn't a clear threshold between the fat-tail and the log-normal regime.

In spite of the construction of the expression (7) in terms of standard deviation of the market average of N log-normal stocks, trying to eliminate the dependence from the number of the spins, our model is N-value dependent. However our simulation proved that the qualitative behavior of the model is analogous for all the values of N.

3.2 Kurtosis

In order to better appreciate the characteristics of fat tails we present in figure 2 the kurtosis behavior as a function of time, for probability distributions generated with our model. On the left are visible the kurtosis trends corresponding to the three γ values considered in figure 1.

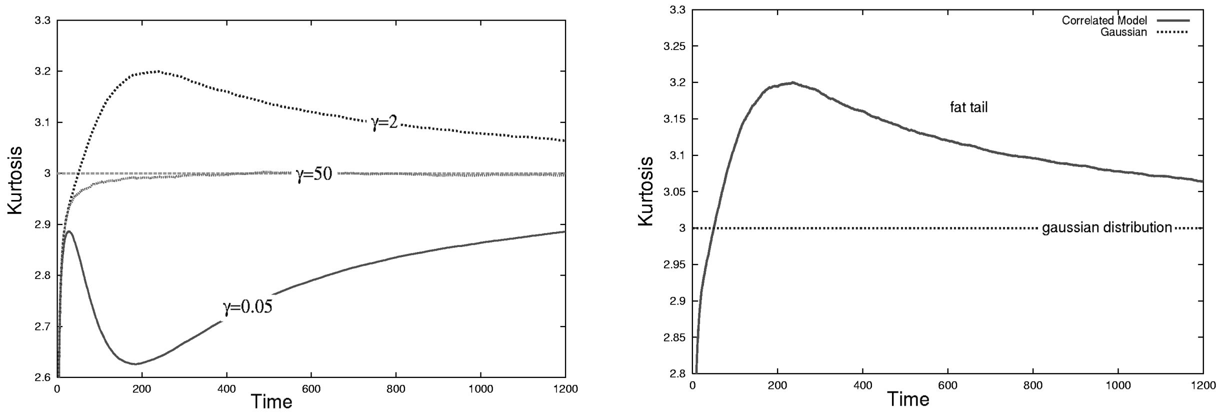

Figure 2: Left: kurtosis of the probability distribution for stock return of our model for different γ values: $\gamma = 0.05$, $\gamma = 2$, $\gamma = 50$. Right: Kurtosis with $\gamma = 2$. (N=1000, Time=1200 steps)

In the right picture of figure 2 we present the behavior of the kurtosis for the $\gamma = 2$ case alone. Apart from a negligible initial transient, due to the discrete nature of the model, it's interesting to note that the kurtosis is greater than 3 for each time. We must also remark the behavior of the curve, that presents a maximum after about 200 time steps and than decreases to the value 3, indicating that probability distributions in our model are not stable and for long times they converge to Gaussian distributions, with a trend analogous to real market distributions.

The kurtosis is the clearest index to appreciate the behavior of the system near the first threshold, between slim and fat tails (see section 3.1 and figure 3). For $\gamma = 0.90$ there is a mix: after the initial transient, there is a short period with a maximum of the kurtosis greater than 3, so there are fat tails, than the kurtosis decreases and there is a relative minimum under 3, so there are slim tails, finally the model converges to a Gaussian distribution and the kurtosis tends to 3.

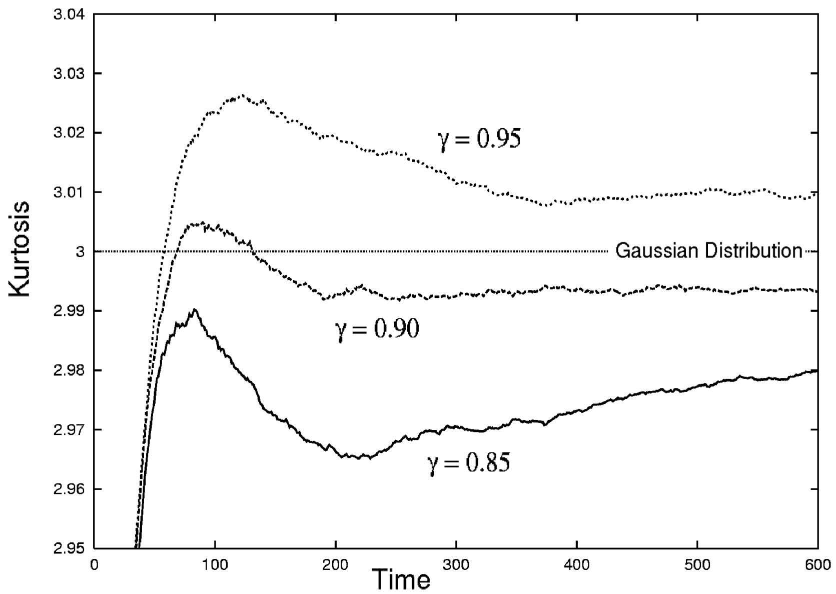

Figure 3: Kurtosis behavior near the threshold between slim and fat tails (N=1000, $\gamma = 0.85$, $\gamma = 0.90$, $\gamma = 0.95$)

3.3 Volatility Smiles

Another interesting check for our model is its capability to generate the well known phenomenon of implied volatility smiles with shape and time dependence analogous to that of real market smiles.

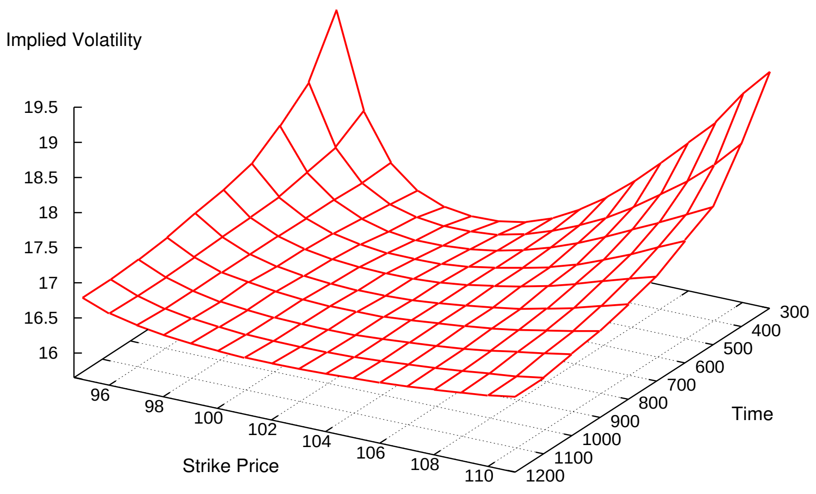

Figure 4: Implied volatility smiles on different time horizons (N=1000, γ=2)

We have constructed some European call options with different strike prices and expiry dates, by using a stock that follows our model with $\gamma = 2$ and choosing a zero free-risk interest rate. Then we have calculated the implied volatility of this stock by using the Black & Scholes formulas [25]. In figure 4 an implied volatility surface is presented. The surface is obtained by using calls on the same stock, on different time horizons. The shape of the surface is similar to real smiles and the time dependence of the concavity can be seen. In analogy with real smiles, the concavity decreases increasing the expiry date of the options ([26] to [30]).

Actually, regarding the kurtosis shape in the $\gamma = 2$ case, the correct time dependence of the concavity is a predictable result for our model.

4 Conclusions

The basis of this work is the attempt to recognize and to model an interesting collective effect in real markets viewed as complex systems.

Our approach to the market is aimed at obtaining some trends qualitatively analogous to real market behaviors by using a simplified "mean field Ising-like" model that describes the interaction between each stock and the market. The task is to show the role of this interaction structure in generating some of the most important features of financial time series. We have reached this purpose by introducing a suitable explicit analytic formula for the conditioned probability driving the evolution of the model. According to this formula the single stock is conditioned in its evolution at a certain time by the global trend of the market and this long range interaction is expressed by a function, appearing in the conditioned probability, that depends simply on the average variation of all the other stocks at the previous time instant.

In spite of the introduced simplifications, we gain an unified picture of three features not necessarily considered linked to each other in previous studies:

- enforcing of stock cross correlation structure during turmoils
- fat-tailed probability distributions for stock returns
- smiles of implied volatility.

In our approach the form of the interaction term between the single stock and the average market depends on the value of a single free parameter (γ) and it is possible to adjust this parameter in such a way to describe different regimes for the market evolution. The dependence of the probability distribution on γ has been studied.

In conclusion we have shown that the correlation between each stock and the average market return generates, in a natural way, fat tailed probability distributions for stock returns, analogous to real market probability distributions and at the same time it also generates volatility smiles with the right shape and time-dependence, for the same values of the parameter γ.

A detailed analysis of another aspect of the market behavior that is the average correlation between couples of stocks in our model has been performed [31]. The results obtained show that for the same values of the γ parameter giving fat-tailed distributions and correct implied volatility smiles, our model is able to reproduce also the correct behavior of the market, that is an increase of the correlation between stocks in the turmoil periods.

In this view, the modellization of a collective effect leads, in a natural way, to a possible unified explanation of different salient features of the market. This possibility suggests the opportunity of developing a more complex analysis of the interactions, that up to now we assumed, for simplicity, to be equal for every component of the system.

It is natural to think that in a more realistic description one takes into account the different "weight" that the various stocks can have in determining the evolution of the market and the fact that every stock is interacting with different correlated or anti-correlated clusters. The study of structural properties of the correlation network has recently received a lot of attention [32] and it is, at our opinion, worthwhile to analyze the embedding of our model in these scenarios and study the stability of the results. More generally we plan to consider models like spin glasses, in which the interactions between stocks in the market present phenomena of frustration and dilution.

5 Acknowledgments

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A Reduction to single stock evolution

To simplify the numerical simulation we have to notice that every spin "sees" all the others only by the average M. We can consider M as a random variable following the conditioned probability given by equation (8) and so it's possible to set each spin as an independent random variable, following a probability conditioned by an other random variable, distributed by the expression (8). Basically to get $s_i(t+1)$ knowing $M(t-1)$ we can proceed extracting a value $M(t)$ using distribution (8) and then extracting the spin using the probability (7). With the initial condition $M(t=0)=0$ it's possible to generate a random walk for the single spin to a generic time t.

Actually things are not so obvious. We must remark that the probability for the spin $s_i(t)$ at time t also depends, via M, on itself at the previous step. In rigorous terms we can't consider M as an independent random variable, but its value is related to $s_i(t-1)$.

A simple example is given in case $N=2$. Setting $s_1(t_0)=1$ and $M(t_0-1)=0$ it's obvious that $M(t_0)$ can never be -1, but from equation (8), the probability to have $M(t_0)=-1$ is $1/4$. There's something wrong! To have the right behavior we have to rewrite $G(M(t), \widetilde{M}(t+1))$ extracting the dependence from the i-th spin value at time $t+1$. Remembering that $M(t)=\frac{\sum_i s_i(t)}{N}$ from equation (8) we can write:

$$G(M,\widetilde{M}) = P(s_i=+1)\binom{N-1}{N(\frac{1+\widetilde{M}}{2})-1}\left(\frac{1}{2}+\frac{1}{2}g(M)\right)^{N(\frac{1+\widetilde{M}}{2})-1}\left(\frac{1}{2}-\frac{1}{2}g(M)\right)^{N\frac{1-\widetilde{M}}{2}} +$$

$$+ \quad P(s_i=-1)\binom{N-1}{N\frac{1+\widetilde{M}}{2}}\left(\frac{1}{2}+\frac{1}{2}g(M)\right)^{N\frac{1+\widetilde{M}}{2}}\left(\frac{1}{2}-\frac{1}{2}g(M(t))\right)^{N(\frac{1-\widetilde{M}}{2})-1} \tag{11}$$

with the **assumptions**:

$$\begin{array}{lll} -1 < M \leq 1 & for & s_i = +1 \\ -1 \leq M < 1 & for & s_i = -1 \end{array} \tag{12}$$

Simplifying we obtain

$$\begin{aligned} G(M,\widetilde{M}) &= \sum_{s_i=\pm 1}\frac{1+s_i\,g(M)}{2}\cdot\binom{N-1}{N(\frac{1+\widetilde{M}}{2})-\frac{1+s_i}{2}}\cdot \\ &\cdot\left(\frac{1}{2}+\frac{1}{2}g(M)\right)^{N(\frac{1+\widetilde{M}}{2})-\frac{1+s_i}{2}}\left(\frac{1}{2}-\frac{1}{2}g(M(t))\right)^{N(\frac{1-\widetilde{M}}{2})-\frac{1-s_i}{2}} \end{aligned} \tag{13}$$

and then we can write the requested expression:

$$G(M,\widetilde{M},s_i)=\binom{N-1}{N(\frac{1+\widetilde{M}}{2})-\frac{1+s_i}{2}}\cdot\left(\frac{1}{2}+\frac{1}{2}g(M)\right)^{N(\frac{1+\widetilde{M}}{2})-\frac{1+s_i}{2}}\left(\frac{1}{2}-\frac{1}{2}g(M)\right)^{N(\frac{1-\widetilde{M}}{2})-\frac{1-s_i}{2}} \tag{14}$$

The main structure of the expression (14) has a known binomial form, so it's possible to approximate it with a normal distribution:

$$G(M,\widetilde{M},s_i)\simeq\frac{1}{\sqrt{2\pi\frac{N-1}{4}\left[1-g(M)^2\right]}}\exp\left[-\frac{\left[\widetilde{M}-\left(\frac{(N-1)g(M)+s_i}{N}\right)\right]^2}{2\frac{N-1}{N^2}\left[1-g(M)^2\right]}\right] \tag{15}$$

under the additional request $-1 < M < 1$, negligible for large N. We have to notice that the assumptions (12) are taken for probabilistic reasons, since, as we have seen, if $s_i = +1$ we can never get $M = -1$ and vice-versa.

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