A Simple Proof of a Theorem by Harris*

Guilherme Carmona

Faculdade de Economia, Universidade Nova de Lisboa

Campus de Campolide, 1099-032 Lisboa, Portugal

email: gcarmona@fe.unl.pt

telephone: (351) 21 380 1671; fax: (351) 21 388 6073

February 26, 2003

Abstract

We present a simple proof of existence of subgame perfect equilibria in games with perfect information.

Journal of Economic Literature Classification Numbers: C73.

*I wish to thank Kemal Badur, and Jan Werner for very helpful comments. Financial support from the Subprograma Ciência e Tecnologia do 2º Quadro Comunitário de Apoio is gratefully acknowledged.

1 Introduction

This note presents a simple proof of a theorem by Harris [4] on the existence of subgame perfect equilibria in games of perfect information. More generally, it illustrates a method for establishing existence of equilibria in such games, and allows for a new characterization of equilibrium outcomes that provides an algorithm for their computation.

The method we present consists of approximating the payoff function of each player by a sequence of simple functions, in a way that is standard in measure and integration theory (see, for example, Wheeden and Zygmund [5]). The sequence of approximating payoff functions for each player, in addition of being a sequence of simple functions, satisfies two other important properties: first, it converges uniformly to the payoff function of that player in the original game; and second, the approximation is such that outcomes that are dominated according to the some payoff function in the approximating sequence are also dominated (by the same outcome) according to the original payoff function, and to all the subsequent payoff functions in the sequence.

The above approximation of each player's payoff function induces a sequence of games that differ from the original game only on the payoff function. For this sequence of games, it is easy to establish that each of them has a nonempty, compact set of equilibrium outcomes, and that the sequence of those sets shrinks to the set of equilibrium outcomes of the original game — thus, the original game has a nonempty, compact set of equilibrium outcome. Furthermore, the set of equilibrium outcomes of the original game can be computed by intersecting the corresponding sets of the approximating games.

2 Games of Perfect Information

In order to present our argument in the simplest possible way, we will consider only the case of games of perfect information with two players. It should be noted that nothing in our argument depends on this assumption, and our proof could be easily extended to the case of an arbitrary finite number of players. Finally, using a "truncation argument" similar to the one used by Harris [4, section 4.4, page 624] (see also Börgers [1]), one could extent our result to the case of a denumerable number of players.

A *game of perfect information* (with two players) is

$$G = \langle S_i, A_2, P_i \rangle_{i=1,2} \,,$$

where (1) S_i is a nonempty set, for $i = 1, 2$, (2) A_2 is a nonempty valued correspondence from S_1 into S_2, and (3), $P_i : H \to \mathbb{R}$, for $i = 1, 2$, where $H = graph(A_2)$.

When player 2 moves at stage 2, it is with perfect information on s_1. Hence the set of *strategies of player* i, $i = 1, 2$, is $F_1 = S_1$ and

$$F_2 = \{f_2 : S_1 \to S_2 |\ f_t(s_1) \in A_2(s_1)\}.$$

A strategy vector (f_1, f_2) is a *subgame perfect equilibrium* of G if for all $s_1 \in S_1$,

$$f_2(s_1) \quad \text{maximizes} \quad P_2(s_1, \cdot) \quad \text{in} \quad A_2(s_1),$$

and

$$P_1(f_1, f_2(f_1)) \geq P_1(s_1, f_2(s_1)).$$

Given a game of perfect information G the vector $(f_1, f_2(f_1))$ in H determined uniquely by (f_1, f_2) is called an *equilibrium path* of G.[1] In what follows we show that the existence of an equilibrium is equivalent to the existence of an equilibrium path. Therefore, we can show that a subgame perfect equilibrium exists by showing that an equilibrium path exists, which is a simpler task. The reason is that, under the assumption we shall use, the

[1] More precisely, the set of equilibrium paths of a game of perfect information is $P = \{(x, y) \in H$: there exists an subgame perfect equilibrium (f_1, f_2) such that $f_1 = x$ and $f_2(f_1) = y\}$.

set of equilibrium paths is compact, while the set of equilibrium strategies typically is not (see Harris [3, proposition 4]).

Define a correspondence E_2 from S_1 into S_2 as follows:

$$E_2(s_1) = \{y \in A_2(s_1) : P_2(s_1, y) \geq U_2(s_1, s_2), \quad \text{for all} \quad s_2 \in A_2(s_1)\};$$

this correspondence gives the equilibrium paths of the one-player game $\langle A_2(s_1), P_2 \rangle$. The set

$$E_1 = \{(x, y) \in H \ : \ y \in E_2(x) \quad \text{and for all} \quad s_1 \in S_1 \quad \text{there exists} \quad s_2 \in E_2(s_1) \\ \text{such that} \quad P_1(x, y) \geq P_1(s_1, s_2)\}$$

is then easily seen to be the set of equilibrium paths of the game $G = \langle S_i, A_2, P_i \rangle_{i=1,2}$ (i.e., in the notation of footnote 1, $E_1 = P$.)

Lemma 1 *A game of perfect information has a subgame perfect equilibrium if and only if $E_1 \neq \emptyset$ and $E_2(s_1) \neq \emptyset$, for all $s_1 \in S_1$.*

Proof. Necessity follows from the fact that if (f_1, f_2) is a subgame perfect equilibrium then $(f_1, f_2(f_1)) \in E_1$ and $f_2(s_1) \in E_2(s_1)$, for all $s_1 \in S_1$.

For sufficiency, let $(x, y) \in E_1$. Define player 1 strategy by $f_1 = x$. For player 2, we define $f_2(f_1) = y$. For $s_1 \neq f_1$, because $(x, y) = (f_1, f_2(f_1)) \in E_1$, it follows that there exists $s_2 \in E_2(s_1)$ such that

$$P_1(f_1, f_2(f_1)) \geq P_1(s_1, s_2);$$

hence we define $f_2(s_1) = s_2$. ■

3 Approximation by Simple Functions and Harris' Theorem

Let $G = \langle S_i, A_2, P_i \rangle_{i=1,2}$ be a game of perfect information, and assume that for all $i = 1, 2$, $P_i(H) \subset [0, 1)$. For $i = 1, 2$ and $k \in \mathbb{N}$, let $P_i^k : H \to \mathbb{R}$ be defined by

$$P_i^k(h) = \frac{j-1}{2^k} \quad \text{if} \quad \frac{j-1}{2^k} \le P_i(h) < \frac{j}{2^k}, \tag{1}$$

for $j = 1, \dots, 2^k$. Define $G_k = \left\langle S_i, A_2, P_i^k \right\rangle_{i=1,2}$, for all $k \in \mathbb{N}$. We have that $\left|\left|P_i^k - P_i\right|\right|_\infty \le \frac{1}{2^k}$, and also that P_i^k is a simple function; this last fact allows us to solve the game G_k by backwards induction, and thus show that it has a subgame perfect equilibrium. The following lemma summarizes this fact.

Lemma 2 *For all $k \in \mathbb{N}$, G_k has a subgame perfect equilibrium.*

Proof. It follows immediately by backward induction:

Let $s_1 \in S_1$. Since P_2 is simple, then $\{P_2(s_1, s_2) : s_2 \in A_2(s_1)\}$ is finite, and so there exists s_2^* that maximizes $s_2 \mapsto P_2(s_1, s_2)$ in $A(s_1)$. Thus, define $f_2(s_1) = s_2^*$. This defines an optimal strategy $f_2 : S_1 \to S_2$ for player 2.

Similarly, since P_1 is simple, then $\{P_1(s_1, f_2(s_1)) : s_1 \in S_1\}$ is finite, and so there exists s_1^* that maximizes $s_1 \mapsto P_1(s_1, f_2(s_1))$ in S_1. Thus, define $f_1 = s_1^*$. This defines an optimal strategy $f_1 \in S_1$ for player 1. Clearly, $f = (f_1, f_2)$ is a subgame perfect equilibrium. ■

We will prove Harris' theorem by approximating a given game G by a sequence of games $\{G_k\}_{k=1}^{\infty}$ as defined above. In fact, this will be a consequence of Theorem 1 below, which shows that the set of equilibrium outcomes of G_k converges to the set of equilibrium outcomes of G.

Theorem 1 *Let $G = \langle S_i, A_2, P_i \rangle_{i=1,2}$, be a game of perfect information.*

Suppose that

1. *for all $i = 1, 2$, S_i is a compact topological space;*
2. *H is a closed subset of $S_1 \times S_2$;*
3. *for all $i = 1, 2$, P_i is continuous;*
4. *$A_2 : S_1 \rightarrow\rightarrow S_2$ is lower hemicontinuous.*

Then, $E_1 = \cap_{k=1}^{\infty} E_1^k = \cap_{k=1}^{\infty} \overline{E_1^k}$ and $E_2(s_1) = \cap_{k=1}^{\infty} E_2^k(s_1) = \cap_{k=1}^{\infty} \overline{E_2^k(s_1)}$, for all $s_1 \in S_1$.

Proof. We show first that $E_1 \subseteq \cap_{k=1}^{\infty} E_1^k$ and $E_2(s_1) \subseteq \cap_{k=1}^{\infty} E_2^k(s_1)$, for all $s_1 \in S_1$. This follows from the fact that for all $i = 1, 2$ and $h, l \in H$, $P_i(h) \geq P_i(l)$ implies $P_i^k(h) \geq P_i^k(l)$.

We show next that $\cap_{k=1}^{\infty} \overline{E_1^k} \subseteq E_1$. Let $(x, y) \in \cap_{k=1}^{\infty} \overline{E_1^k}$. Then, for each $k \in \mathbb{N}$, there a net $\{(x_j^k, y_j^k)\}_{j \in J_k} \subseteq E_1^k$ converging to (x, y).

First, we will show that $y \in E_2(x)$. Let $s_2 \in A_2(x)$, $k \in \mathbb{N}$, and $\{s_j\}_{j \in J_k}$ be such that $s_j \in A(x_j^k)$ for all j, and $s_j \to s_2$ (such net exist since A is lower hemi-continuous). Since for all $j \in J_k$ we have that $y_j^k \in E_2^k(x_j^k)$, then

$$P_2^k(x_j^k, y_j^k) \geq P_2^k(x_j^k, s_j),$$

for all $j \in J_k$. Since $\left|\left|P_2^k - P_2\right|\right|_{\infty} \leq \frac{1}{2^k}$, we obtain

$$P_2(x_j^k, y_j^k) \geq P_2(x_j^k, s_j) - \frac{2}{2^k},$$

and so,

$$P_2(x, y) \geq P_2(x, s_2) - \frac{2}{2^k},$$

since P_2 is continuous. Finally, letting $k \to \infty$, it follows that

$$P_2(x, y) \geq P_2(x, s_2).$$

Hence, it follows that $y \in E_2(x)$.

We are left to show that for all $s_1 \in S_1$ there exists $s_2 \in E_2(s_1)$ such that $P_1(x, y) \geq P_1(s_1, s_2)$. Let $s_1 \in S_1$ and $k \in \mathbb{N}$. For each $j \in J_k$, let $w_j^k \in E_2^k(s_1)$ be such that

$$P_1(x_j^k, y_j^k) \geq P_1(s_1, w_j^k).$$

Since $\{w_j^k\}_{j \in J_k} \subseteq A_2(s_1)$, and $A_2(s_1)$ is compact, we may assume that $\{w_j^k\}_{j \in J_k}$ converges; let w^k be such that $w_j^k \to w^k$. The sequence $\{w^k\}_{k \in \mathbb{N}}$ lies also on $A_2(s_1)$ and so we may again assume that it converges; let w be such that $w^k \to w$. An argument parallel to the one used above establishes that

$$P_1(x, y) \geq P_1(s_1, w).$$

Hence, it is enough to show that $w \in E_2(s_1)$, which can again be done with an argument similar to the one used to show that $y \in E_2(x)$.

Similarly, one can show that $\cap_{k=1}^{\infty} \overline{E_2^k(s_1)} \subseteq E_2(s_1)$, for all $s_1 \in S_1$. This completes the proof. ■

Theorem 1 gives a characterization of subgame perfect equilibrium paths of G in terms of the equilibrium paths of the approximating games, and provides an algorithm for their computation. Also, it follows immediately from Theorem 1 that E_1 and $E_2(s_1)$, for all $s_1 \in S_1$, are compact. Also, Harris' Theorem follows easily from Lemma 2, and Theorem 1.

Theorem 2 (Harris) *Let G be a game of perfect information. Under the same assumptions of Theorem 1, G has a subgame perfect equilibrium.*

Proof. First note that because H is compact and P_i is continuous for $i = 1, 2$, there is no loss in generality by assuming that $P_i(H) \subset [0, 1)$, for all $i = 1, 2$.

For $k \in \mathbb{N}$, let G_k be as defined above. By Lemma 2 it follows that $E_2^k(s_1)$ is nonempty, for all $s_1 \in S_1$, and that E_1^k is nonempty. Since, for all $k \in \mathbb{N}$, $i = 1, 2$ and $h, l \in H$, $P_i^{k+1}(h) \geq P_i^{k+1}(l)$ implies $P_i^k(h) \geq P_i^k(l)$, it follows that $E_2^{k+1}(s_1) \subseteq E_2^k(s_1)$, for all $s_1 \in S_1$ and that $E_1^{k+1} \subseteq E_1^k$.

Claim 1 *For all $s_1 \in S_1$, $A_2(s_1)$ is closed.*

Proof. Let $\{y_j\}_{j \in J}$ be a convergent net in $A_2(s_1)$ and let $y \in S_2$ be such that $y_j \to y$. Since $(s_1, y_j) \in H$ for all $j \in J$, and H is closed, it follow that $(s_1, y) \in H$. Hence, $y \in A_2(s_1)$. ■

Since for all $s_1 \in S_1$, $A_2(s_1)$ is closed it follows that $\overline{E_2^k(s_1)} \subseteq C_2(s_1)$, for all $k \in \mathbb{N}$; furthermore, for all $s_1 \in S_1$ and $k \in \mathbb{N}$, $\overline{E_2^k(s_1)}$ is a nonempty, closed subset of a compact space and $\overline{E_2^{k+1}(s_1)} \subseteq \overline{E_2^k(s_1)}$. Hence, for all $s_1 \in S_1$, $\cap_{k=1}^{\infty} \overline{E_2^k(s_1)}$ is a nonempty subset of $A_2(s_1)$ (see Kelley [2, theorem 1, page 136]). Similarly, we can conclude that $\cap_{k=1}^{\infty} \overline{E_1^k}$ is a nonempty subset of H. Hence, it follows from Theorem 1 that $E_2(s_1)$ is nonempty, for all $s_1 \in S_1$, and that E_1 is nonempty, which completes the proof. ■

Remark 1 *Note that the above Theorem dispenses with the assumption used by Harris [4] that* S_i, $i = 1, 2$, *is Hausdorff.*

References

[1] Börgers, T.: Upper Hemicontinuity of the Correspondence of Subgame-Perfect Equilibrium Outcomes. Journal of Mathematical Economics **20**, 89-106 (1991)

[2] Kelley, J.: General Topology. New York: Springer (1955)

[3] Harris, C.: A Characterizarion of the Perfect Equilibria of Infinite Horizon Games. Journal of Economic Theory **37**, 99-125 (1985)

[4] Harris, C.: Existence and Characterization of Perfect Equilibrium in Games of Perfect Information. Econometrica **53**, 613-628 (1985)

[5] Wheeden, R. and A. Zygmund: Measure and Integral: an Introduction to Real Analysis. New York: Dekker (1977)