

# A SHORT AND CONSTRUCTIVE PROOF OF TARSKI'S FIXED-POINT THEOREM

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ABSTRACT. I give short and constructive proofs of Tarski's fixed-point theorem, and of a much-used extension of Tarski's fixed-point theorem to set-valued maps.

## 1. INTRODUCTION

I give short and constructive proofs of two related fixed-point theorems. The first is Tarski's fixed-point theorem: If  $F$  is a monotone function on a non-empty complete lattice, the set of fixed points of  $F$  form a non-empty complete lattice. The second is a much-used extension of Tarski's fixed-point theorem to set-valued functions: If  $\varphi : X \rightarrow 2^X$  is monotone—when  $2^X$  is endowed with the induced set order—the set of fixed-points of  $\varphi$  form a non-empty complete lattice.

Tarski's [4] original proof is beautiful and elegant, but non-constructive and somewhat uninformative. Cousot and Cousot [1] give a constructive proof of Tarski's fixed-point theorem. Their proof is long and quite involved, though. The proof I present is much simpler, and fits in a napkin. On the other hand, [1] obtain certain sub-products from their approach that I do not obtain; I shall only be concerned with Tarski's fixed-point theorem, and its extension to set-valued functions.

The extension to set-valued functions was developed by Smith [3] and Zhou [7]. I give a constructive proof of Zhou's version of the result. Zhou's version of the result is important in game theory ([5],[6]). Smith has a weaker monotonicity requirement than Zhou, but Smith does not obtain a lattice structure on the set of fixed-points. In addition, Smith needs a continuity assumption.

## 2. DEFINITIONS

A set  $X$  endowed with a partial order  $\leq$  is denoted  $\langle X, \leq \rangle$ .  $\langle X, \leq \rangle$  is a *complete lattice* if, for all nonempty  $B \subseteq X$ ,  $\bigwedge B$  and  $\bigvee B$  exist in

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$X$ . If  $A \subseteq X$ , say that  $A$  is a *subcomplete sublattice* of  $\langle X, \leq \rangle$  if, for all nonempty  $B \subseteq A$ ,  $\bigwedge B \in A$ , and  $\bigvee B \in A$ .

Say that  $A \subseteq X$  is smaller than  $B \subseteq X$  in the *induced set order* (denoted  $A \trianglelefteq B$ ) if

$$(x \in A, y \in B) \Rightarrow (x \wedge y \in A, x \vee y \in B).$$

The induced set order is a partial order on the set of sublattices of  $X$ .

Denote by  $\leq$  the usual linear order on ordinal numbers.

Let  $\langle X, \leq \rangle$  be a lattice and  $\langle Y, \sqsupseteq \rangle$  be a partially ordered set. A function  $F : X \rightarrow Y$  is *monotone* if  $x \leq y$  implies  $F(x) \sqsupseteq F(y)$ . Say that a set-valued map  $\varphi : X \rightarrow 2^X$  is monotone if it is monotone when  $\varphi(X)$  is ordered by the induced set order.

### 3. THEOREMS

Let  $\langle X, \leq \rangle$  be a lattice and  $F : X \rightarrow X$  be monotone. The set of fixed points of  $F$  is  $\mathcal{E}(F) = \{x \in X : x = F(x)\}$ .

**Lemma 1.**  $\langle \mathcal{E}(F), \leq \rangle$  has a smallest element.

*Proof.* Let  $\eta$  be an ordinal number with cardinality greater than  $X$ , let  $\xi = \eta + 1$ . Define  $f : \xi \rightarrow X$  by transfinite recursion as  $f(0) = \bigwedge X$ , and

$$f(\beta) = \bigvee_{\alpha < \beta} F(f(\alpha))$$

for  $\beta > 0$ .

That  $(\beta < \alpha) \Rightarrow (f(\beta) \leq f(\alpha))$  is immediate from the definition of  $f$ . Then, for any  $\alpha \in \eta$ ,  $f(\alpha + 1) = F(f(\alpha))$ , as  $f(\beta) \leq f(\alpha)$ , for all  $\beta < \alpha$ , and  $F$  is monotone. Since  $\eta$  has cardinality greater than  $X$ , there is  $\gamma \in \eta$  such that  $f(\gamma) = f(\gamma + 1)$ . Let  $\underline{e} = f(\gamma)$ , then  $\underline{e} = F(\underline{e})$ . So  $\underline{e} \in \mathcal{E}(F)$ .

I shall prove that  $\underline{e}$  is the smallest element in  $\langle \mathcal{E}(F), \leq \rangle$ . Let  $e \in \mathcal{E}(F)$ , and consider the proposition  $P_\alpha : f(\alpha) \leq e$ .  $P_\alpha$  implies that  $f(\alpha + 1) = F(f(\alpha)) \leq F(e) = e$ . By transfinite induction, then,  $\underline{e} \leq e$ .  $\square$

A version of Lemma 1 is also crucial in [1]'s proof of Tarki's fixed-point theorem. It was apparently first proven in [2]; my proof is more direct than the one in [2].

**Theorem 2.**  $\langle \mathcal{E}(F), \leq \rangle$  is a non-empty complete lattice.

*Proof.* By Lemma 1,  $\mathcal{E}(F)$  is nonempty. Let  $E \subseteq \mathcal{E}(F)$  be nonempty. I shall find  $\bigvee_{\mathcal{E}(F)} E$ .

Let  $x = \bigvee_X E$ , and let  $Y = \{z \in X : x \leq z\}$  be the set of upper bounds on  $E$ . If  $z \in Y$ , then, for any  $e \in E$ ,  $e \leq F(z)$ , as  $e = F(e) \leq F(z)$ . Thus  $F(Y) \subseteq Y$ . Let  $G = F|_Y$ ; so  $G$  maps  $Y$  into  $Y$ , and  $G$  is monotone.

By Lemma 1,  $\langle \mathcal{E}(G), \leq \rangle$  has a smallest element. By definition of  $G$ , this smallest element is  $\bigvee_{\mathcal{E}(F)} E$ .

The construction of  $\bigwedge_{\mathcal{E}(F)} E$  is symmetric.  $\square$

Let  $\varphi : X \rightarrow 2^X$  be a set-valued map such that, for all  $x \in X$ ,  $\varphi(x)$  is a non-empty subcomplete sublattice of  $X$ . Suppose that  $\varphi$  is monotone. The set of fixed points of  $\varphi$  is  $\mathcal{E}(\varphi) = \{x \in X : x \in \varphi(x)\}$ .

**Lemma 3.**  $\langle \mathcal{E}(\varphi), \leq \rangle$  has a smallest element.

*Proof.* Let  $F(x) = \bigwedge_X \varphi(x)$ . Note that,  $\forall x$ ,  $F(x) \in \varphi(x)$ , and that  $F$  is monotone. By Lemma 1, there is a smallest element, say  $\underline{e}$  of  $\langle \mathcal{E}(F), \leq \rangle$ . Note that  $\underline{e} = F(\underline{e}) \in \varphi(\underline{e})$ , so  $\underline{e} \in \mathcal{E}(\varphi)$ .

I shall prove that  $\underline{e}$  is the smallest element in  $\langle \mathcal{E}(\varphi), \leq \rangle$ . Let  $e \in \mathcal{E}(\varphi)$ . Let  $f$  be as in the proof of Lemma 1. Consider the proposition  $P_\alpha : f(\alpha) \leq e$ .  $P_\alpha$  implies that  $f(\alpha + 1) = F(f(\alpha)) \leq F(e) \leq e$ , as  $F(e) = \bigwedge_X \varphi(e)$ , and  $e \in \varphi(e)$ . By transfinite induction, then,  $\underline{e} \leq e$ .  $\square$

**Theorem 4.**  $\langle \mathcal{E}(\varphi), \leq \rangle$  is a non-empty complete lattice.

*Proof.* Lemma 3 implies that  $\mathcal{E}(\varphi)$  is nonempty. I shall prove that it is a complete lattice.

Let  $E \subseteq \mathcal{E}(\varphi)$  be nonempty. I shall prove that  $\bigvee_{\mathcal{E}(\varphi)} E$  exists.

Let  $x = \bigvee_X E$ , and let  $Y = \{z \in X : x \leq z\}$ . Define  $\psi : Y \rightarrow 2^Y$  by  $\psi(z) = Y \cap \varphi(z)$ .

First, I show  $\psi(z) \neq \emptyset$ . Note that  $x \leq z$  implies that, for any  $e \in E$ , there is  $\hat{z}_e \in \varphi(z)$  with  $e \leq \hat{z}_e$ , as  $e \in \varphi(e)$  and  $\varphi$  is monotone. But  $\varphi(z)$  is subcomplete, and thus

$$x \leq \bigvee_{e \in E} \hat{z}_e \in \varphi(z),$$

so  $\psi(z) \neq \emptyset$ .

Second, I show that  $\psi$  is monotone. Let  $z \leq z'$ , and fix  $y \in \psi(z)$  and  $y' \in \psi(z')$ .  $\varphi$  is monotone, so  $y \wedge y' \in \varphi(z)$  and  $y \vee y' \in \varphi(z')$ . Let  $e \in E$ ,  $e \in \varphi(e)$  implies  $e \vee (y \wedge y') \in \varphi(z)$ . But  $y \wedge y' \in Y$  so  $\varphi(z) \ni e \vee (y \wedge y') = y \wedge y'$ . Similarly,  $e \vee (y \vee y') \in \varphi(z')$  and  $y \vee y' \in Y$  implies  $y \vee y' \in \psi(z')$ . Thus  $y \wedge y' \in \psi(z)$  and  $y \vee y' \in \psi(z')$ .

Third,  $\psi(z)$  is a subcomplete sublattice because  $\psi(z) = Y \cap \varphi(z)$  and  $\varphi(z)$  is a subcomplete sublattice.

Thus,  $\psi$  satisfies the hypothesis of Lemma 3. Let  $e^* \in \mathcal{E}(\psi)$  be the smallest  $\psi$ -fixed point. If  $e \in \mathcal{E}(\varphi)$  is an upper bound on  $E$ , then  $e \in Y$  and thus  $e \in \mathcal{E}(\psi)$ . We have then  $e^* \leq e$ . But  $e^* \in \mathcal{E}(\varphi)$ , so  $e^* = \bigvee_{\mathcal{E}(\varphi)} E$ .

The proof that  $\bigwedge_{\mathcal{E}(\varphi)} E$  exists is symmetric.  $\square$

#### 4. CONTINUOUS $F$

The proof of Tarski's Theorem is elementary when  $F$  is order-continuous, in addition to monotone.

First, the proof of Lemma 1 goes: Let  $\underline{x} = \bigwedge X$ , the smallest point in  $X$ , and let  $\{x_n\}$  be the sequence of  $F$ -iterates from  $\underline{x}$ ; so  $x_n = F(x_{n-1})$  and  $x_0 = \underline{x}$ . Since  $F$  is monotone,  $\{x_n\}$  is a monotone sequence, and thus converges to a point  $\underline{e}$ . The continuity of  $F$  implies that  $\underline{e}$  is a fixed point, as  $x_{2n+1} = F(x_{2n})$ , and both  $\{x_{2n}\}$  and  $\{x_{2n+1}\}$  converge to  $\underline{e}$ . Further, if  $e$  is any other fixed point of  $F$ ,  $\underline{x} \leq e$ , and  $x_n \leq e$  implies  $x_{n+1} = F(x_n) \leq F(e) = e$ . By induction,  $\underline{e}$  is the smallest fixed point.

Second, Lemma 1 is used to prove Tarski's Theorem as in the proof of Theorem 2 above.

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