# Divergence in Economic Performance: Transitional Dynamics with Multiple Equilibria

Danyang Xie*
C.R.D.E. and Département de sciences économiques
Université de Montréal
Montréal, Québec, H3C 3J7
and
Department of Finance and Economics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

July, 1993

*I thank Leonard Dudley, Bentley MacLeod, and seminar participants at Queens University for useful comments. Financial support from the C.R.D.E. and from the FCAR is gratefully acknowledged. Forthcoming *Journal of Economic Theory*.

# 1 Introduction

The seminal paper by Lucas [8] "On the Mechanics of Economic Development" is one of the most stimulating papers in New Growth Theory. Among the three models he presents, the one that emphasizes human capital accumulation through schooling has received the greatest attention. In that model, he concludes that although the *growth rates* in different countries tend to converge to each other in the long run, their income *levels* can be permanently different. The country that has greater initial endowments of physical capital and human capital will be permanently richer than the one with lower initial endowments. This last statement, he admits, is only a conjecture because he has never worked out the transitional dynamics of the model. Since then, there have been attempts by several authors who try to look more closely into the transitional dynamics. Among them, Mulligan and Sala-i-Martin [9] is a notable example. Given the complexity of the model itself, the transitional dynamics and the stability analysis in their paper are not transparent. To my knowledge, Benhabib and Perli [3] contains the clearest *local* stability analysis of the Lucas model, which complements the *global* analysis in the present paper.

In this paper, I tackle the transitional dynamics simply by solving the equilibrium paths explicitly. The cost involved in this attempt is that a restriction on the parameters across the utility function and the production function has to be imposed. It is the same restriction that I once used in Xie [12], namely, the inverse of the intertemporal rate of substitution being equal to the (physical) capital income share. Expressed in Lucas's notations, it is simply: $\sigma = \beta$. With this restriction imposed, the whole dynamics becomes transparent. The drawback of imposing such a restriction is that the model is not suitable for simulation exercises of the transitional dynamics; for this purpose, a better alternative can be found in Mulligan and Sala-i-Martin [10]. In terms of searching for theoretical properties of the transitional dynamics, however, the approach in this paper has definite advantage. The following is a list of the conclusions that the reader may actually see from the explicit solution.

**(a)** When parameter $\gamma$ in Lucas's specification that captures the external effects of human capital in goods production is larger than parameter $\beta$, then,

- Not only there exist a continuum of balanced growth paths as noted by Lucas, but there exist also a continuum of equilibrium paths starting from the same initial endowments of physical capital and human capital. These equilibrium paths can be indexed by $u_0$, the fraction of non-leisure time that is devoted to goods production at time zero.
- A country that has lower endowments of physical capital and human capital can overtake a country that has greater endowments, provided that the former devotes considerably smaller initial fraction of non-leisure time in goods production than the latter.
- Sacrifices in initial income due to a low $u_0$ will not only generate high growth and permanently high future income, but they are also justified based on the consideration of the representative individual's life-time welfare.

**(b)** When $\gamma < \beta$, then

- For each economy, there is a unique equilibrium path that converges to a balanced growth path.
- No overtaking can happen.

The rest of the paper is organized as follows. For the paper to be self contained, Section 2 presents the Lucas model. In Section 3, the equilibrium paths are solved explicitly. For the sake of clarity, the process of obtaining the solution is given in Theorem-Proof format. In Section 4, a numerical example is given and the equilibrium paths are depicted in a diagram. The evolution pattern of the equilibrium paths in the diagram casts doubt on Lucas's conjecture of equilibrium trajectories. Section 5 considers welfare implication of schooling. Section 6 contains a discussion on methodology for studying transitional dynamics. Section 7 concludes.

## 2 Lucas Model

In Lucas [8], he considers a closed economy with competitive markets. The economy is populated with many identical, rational agents. The population at time $t$ is $N(t)$, which is assumed to grow at a constant rate, $\lambda$.

Let $c(t),\ t \geq 0$ be a stream of real, per-capita consumption of a single good. Preferences over (per-capita) consumption stream are given by

$$\int_0^\infty \frac{1}{1-\sigma}\left[c(t)^{1-\sigma} - 1\right] N(t)e^{-\rho t}\, dt \tag{1}$$

where $\rho$ is the rate of time preference, and $\sigma$ is the inverse of the intertemporal elasticity of substitution.

Let $h(t)$ denote the skill level (human capital level) of a typical worker. Let $u(t)$ be the fraction of non-leisure time devoted to goods production. Then $1 - u(t)$ is the effort devoted to the accumulation of human capital. It is assumed that the growth of human capital, $\dot{h}$, takes a simple form as follows:

$$\dot{h} = \delta(1-u)h, \tag{2}$$

where parameter $\delta$ is positive.

The output, $Y$, in this economy depends on the capital stock, $K$, the effective work force, $uhN$, and the average skill level of workers, $h_a$ :

$$Y = AK^\beta \left[uhN\right]^{1-\beta} h_a^\gamma, \tag{3}$$

where parameter $\beta$ is the income share of physical capital; parameter $\gamma$ is positive, and is intended to capture the external effects of human capital. $A$ is a constant. The accumulation of physical capital is assumed to take a natural form:

$$\dot{K} = Y - Nc. \tag{4}$$

In equilibrium, $h_a = h$, because all workers are treated as being identical. If this substitution is made in (3) at this stage, what one obtains from maximizing (1) subject to (2), (3), and (4) would be the social optimum allocation. In order to calculate a competitive equilibrium for this economy however, one should, first of all, derive the first order conditions taking the whole path of

$\{h_a(t) : t \geq 0\}$ as given. Specifically, the current-value Hamiltonian should be written as

$$H(K, h, \theta_1, \theta_2, c, u;\ \sigma, \beta, \gamma, \delta, \{N(t),\ h_a(t) :\ t \geq 0\}) = \frac{1}{1-\sigma}\left[c^{1-\sigma} - 1\right] N \\ + \theta_1 \left[ AK^\beta \left[uhN\right]^{1-\beta} h_a^\gamma - Nc \right] + \theta_2 \delta (1-u) h, \tag{5}$$

where $\theta_1$ and $\theta_2$ are the co-state variables for $K$, and $h$, respectively. Things taken as given are put after the semicolon in the Hamiltonian. Having derived the first order conditions, one can then impose the equilibrium condition, $h_a = h$. This procedure yields the following equations:

$$c^{-\sigma} = \theta_1, \tag{6}$$

$$\theta_1 (1-\beta) AK^\beta (uNh)^{-\beta} Nh^{1+\gamma} = \theta_2 \delta h, \tag{7}$$

$$\dot{\theta}_1 = \rho\theta_1 - \theta_1 \beta AK^{\beta-1} (uNh)^{1-\beta} h^\gamma, \tag{8}$$

$$\dot{\theta}_2 = \rho\theta_2 - \theta_1 (1-\beta) AK^\beta (uN)^{1-\beta} h^{-\beta+\gamma} - \theta_2 \delta (1-u), \tag{9}$$

$$\dot{K} = AK^\beta (uNh)^{1-\beta} h^\gamma - Nc, \tag{10}$$

$$\dot{h} = \delta(1-u)h. \tag{11}$$

The boundary conditions include the initial conditions: $K(0) = K_{0,}$ $h(0) = h_0$, and the transversality conditions: $K\theta_1 e^{-\rho t} \to 0$ and $h\theta_2 e^{-\rho t} \to 0$, as $t \to \infty$.

This completes the Lucas model. In the following section, the restriction, $\beta = \sigma$, is imposed so that the solution to the equations above is explicit.

# 3 Explicit Dynamics

**Lemma 1** *Along any equilibrium path, $\theta_2$ always grows at a constant rate, $\rho - \delta$.*

**Proof.** Substituting equation (7) into equation (9), we obtain

$$\begin{aligned}\dot{\theta}_2 &= \rho\theta_2 - \theta_2\delta u - \theta_2\delta(1-u) \\ &= (\rho-\delta)\theta_2.\end{aligned} \tag{12}$$

Therefore, $\dot{\theta}_2/\theta_2$ is constant.

**Lemma 2** *When $\sigma = \beta$ is imposed in the Lucas model, the aggregate consumption, $Nc$, along any equilibrium path, is always proportional to the capital stock, $K$.*

**Proof.** Let $C$ denote the aggregate consumption, $Nc$. Then we have

$$\begin{aligned}\tfrac{\dot{C}}{C} - \tfrac{\dot{K}}{K} &= \tfrac{\dot{N}}{N} + \tfrac{\dot{c}}{c} - \tfrac{\dot{K}}{K} \\ &= \lambda - \left(\tfrac{1}{\sigma}\right)\tfrac{\dot{\theta}_1}{\theta_1} - \tfrac{\dot{K}}{K}, \text{ from equation (6)} \\ &= \lambda - \left(\tfrac{1}{\beta}\right)\rho + \tfrac{C}{K}, \text{ from equations (8), (10) and } \sigma = \beta.\end{aligned}$$

One solution to this differential equation is that $C = (\rho/\beta - \lambda)K$. We will verify later that this is the solution which satisfies the transversality conditions. The restriction $\sigma = \beta$ is the key to make the equilibrium solution so simple.

**Lemma 3** *The fraction of non-leisure time devoted to the production of goods, $u$, is governed in equilibrium by the following differential equation:*

$$\dot{u} = u\frac{1}{\beta}\left[\lambda + (1+\gamma-\beta)\delta - \rho\right] - \frac{(\gamma-\beta)\delta}{\beta}u^2. \tag{13}$$

*When $\gamma \neq \beta$, the equation above can be rewritten as:*

$$\dot{u} = \frac{(\gamma-\beta)}{\beta}\delta u(u^* - u), \tag{14}$$

*where $u^* = \left[\lambda + (1+\gamma-\beta)\delta - \rho\right] / \left[(\gamma-\beta)\delta\right]$ is the steady state value of $u$.*
**Proof.** See Appendix A.

**Theorem 4** *In the Lucas model with $\sigma = \beta$, the following statements are true if and only if $\gamma > \beta$, and $\delta < \rho - \lambda < \delta(1+\gamma-\beta)$:*

- 1) *$u^*$ lies in the interval* $(0,1)$*;*
  2) *in equilibrium, for any* $u(0) \in (0,1]$*,* $u(t)$ *converges to* $u^*$*, as* $t$ *goes to infinity.*

**Proof.** The "if" part is straightforward. First of all, the parameter restrictions ensure $u^* \in (0,1)$, hence 1) is true. Also, equation (14) can be solved analytically:

$$u = \frac{u_0 \, u^* e^{u^* \frac{(\gamma-\beta)\delta}{\beta} t}}{u^* - u_0 + u_0 e^{u^* \frac{(\gamma-\beta)\delta}{\beta} t}}. \tag{15}$$

Since $\gamma > \beta$, statement 2) is true.

For the "only if" part, note that statement 1) and the definition of $u^*$ imply $\gamma \neq \beta$. Then statement 2) requires that $\gamma > \beta$. Finally, statement 1) implies that $\delta < \rho - \lambda < \delta(1+\gamma-\beta)$.

Remark: In Lucas's paper, a positive $\gamma$ is used to address the question why wealthier countries have higher wages than poorer ones for labor of any given skill. Theorem 1 implies that when $\gamma < \beta$, the equilibrium path must have $u_0 = u^*$. Once the readers finish reading the rest of the paper, they are invited to prove by themselves the results listed in the introduction under the item (b). The rest of the paper focuses on the case in which $\gamma > \beta$ because it allows for richer dynamics of the economies that we observe in the world.

Now, let $\nu = \delta(1-u^*)$, and $\kappa = \frac{(1+\gamma-\beta)}{1-\beta}\,\nu$. Following Lucas [8], we define,

$$Z_1 = e^{-(\kappa+\lambda)t} K,$$

$$Z_2 = e^{-\nu t} h.$$

Then, along any equilibrium path, $Z_1$ and $Z_2$ satisfy the following differential equations (assuming $N_0 = 1$):

$$\frac{\dot{Z}_1}{Z_1} = A Z_1^{\beta-1} Z_2^{(1+\gamma-\beta)} u^{1-\beta} - (\kappa + \frac{\rho}{\beta}), \tag{16}$$

$$\frac{\dot{Z}_2}{Z_2} = \delta(u^* - u), \tag{17}$$

$$Z_1(0) = K_0, \text{ and } Z_2(0) = h_0.$$

**Lemma 5** *$Z_2$ can be solved analytically:*

$$Z_2 = \left[ \frac{h_0}{u_0^{\frac{\beta}{\gamma-\beta}}} \right] u^{\frac{\beta}{\gamma-\beta}}, \tag{18}$$

*where $u$ has a closed form expression* (15).
**Proof.** From (14) and (17), we obtain that

$$\frac{\dot{Z}_2}{Z_2} = \frac{\beta}{(\gamma-\beta)} \frac{\dot{u}}{u}.$$

Therefore, $Z_2$ is a constant times $u^{\beta/(\gamma-\beta)}$. One can easily verify that the constant is $h_0/(u_0^{\beta/(\gamma-\beta)})$.

**Theorem 6** *Starting from $(K_0, h_0)$, an economy has a continuum of equilibria, indexed by $u_0$.*

**Proof.** For any $u_0 \in (0, 1]$, a path of $u$ can be calculated by equation (15). Once this is done, $Z_2$ can be calculated using formula (18). Then, the values for $u$ and $Z_2$ thus obtained can be substituted into equation (16). A numerical calculation of $Z_1$ is easy to perform because equation (16) then becomes a first order differential equation in $Z_1$ with an initial boundary condition. Finally, $Z_1$ and $Z_2$ can be transformed back to derive $K$ and $h$. Other variables such as $c, \theta_1$, and $\theta_2$, can be derived accordingly. To verify that the obtained path of $K$, $h$, $c, \theta_1$ and $\theta_2$ for any given $u_0$ is an equilibrium path, we only need to check the transversality conditions; the first order conditions are obviously satisfied.

Since $u$ converges to $u^*$, $Z_1$ and $Z_2$ will converge to a steady state $Z_1^*$ and $Z_2^*$. The steady state values of $Z_1$ and $Z_2$ depend only on $h_0$, and $u_0$, not on $K_0$:

$$Z_1^*(h_0, u_0) = \left[ \frac{A h_0^{1+\gamma-\beta} (u^*)^{\frac{\gamma}{\gamma-\beta}}}{(\kappa + \rho/\beta) u_0^{\frac{\beta}{\gamma-\beta}(1+\gamma-\beta)}} \right]^{\frac{1}{1-\beta}}, \tag{19}$$

$$Z_2^*(h_0, u_0) = h_0 \left[ \frac{u^*}{u_0} \right]^{\frac{\beta}{\gamma-\beta}}. \tag{20}$$

Consequently, $K$ eventually grows at the rate of $\kappa + \lambda$, and $h$ eventually grows at the rate of $\nu$. Hence, asymptotically, we have

$$\begin{aligned}\tfrac{\dot{\theta}_1}{\theta_1} + \tfrac{\dot{K}}{K} &= \beta(\lambda - \tfrac{\dot{K}}{K}) + \tfrac{\dot{K}}{K} = \beta\lambda + (1-\beta)(\kappa+\lambda) = \lambda + (1+\gamma-\beta)\delta(1-u^*) \\ &= \rho - (1+\gamma-\beta)\delta + (\gamma-\beta)\delta u^* + (1+\gamma-\beta)\delta(1-u^*) \\ &= \rho - \delta u^* < \rho,\end{aligned}$$

and

$$\frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{h}}{h} = \rho - \delta + \nu = \rho - \delta + \delta(1-u^*) = \rho - \delta u^* < \rho.$$

Therefore, the transversality conditions, $\theta_1 K e^{-\rho t} \to 0$ and $\theta_2 h e^{-\rho t} \to 0$, are satisfied.

**Theorem 7** *A country that has lower endowments of physical capital and human capital can overtake a country that has greater endowments, provided that the former devotes considerably smaller initial fraction of non-leisure time in goods production than the latter.*

**Proof.** We know that the two countries will asymptotically grow at the same rate. According to equations (19) and (20), however, the poorly endowed country can have a higher steady state value of $Z_1$ and $Z_2$ as long as it has a sufficiently smaller $u_0$. This means that the poorer country can overtake the initially richer country at some point in time. Also, unless the initially richer country does something that shifts its economy to a faster growing path, it will start to fall behind permanently.

# 4 A Numerical Example

Wherever possible, the parameter values used in this section are taken from Lucas's calibration of the U.S. economy. Specifically, they are: $\lambda = 0.013$, $\beta = 0.25$, $\gamma = 0.417$, $\delta = 0.05$, $A = 1$. In Lucas's paper, $\rho$ and $\sigma$ can not separately be identified. Note that in order to solve the dynamics explicitly, I have to impose the condition, $\sigma = \beta$. Thus, $\sigma = 0.25$. As a result, $\rho$ has to be set at 0.064 for $\rho + \sigma\kappa = 0.0675$ to hold. These parameter values imply that the steady state value of the fraction of time devoted to goods production,

$u^*$, is 0.82; the steady state growth rate of per capita capital, $\kappa$, 0.014; the steady state growth rate of average human capital level, $\nu$, 0.009. It is easy to verify that the parameter values satisfy the restrictions in Theorem 1 for the stability of the dynamic system.

Let the United States be called country 1. Suppose that in 1960 (t=0), the capital stock in the U.S. is normalized at 10, and the average human capital level is normalized at 1. As stated in Theorem 2, there is a continuum of equilibria indexed by $\{u_0 : \ u_0 \in (0, 1]\ \}$. For illustration, only three equilibrium paths are depicted in Figure 1. When $u_0 = u^* = 0.82$, $Z_2$ will be constant along the equilibrium path and $Z_1$ will decrease and converge to $Z_1(h_0, u_0) = Z_1(1, 0.82) = 4.699$. This equilibrium path is the vertical one in the diagram. When $u_0 = 0.75 < u^*, Z_2$ increases along the equilibrium path and converges to $Z_2(h_0, u_0) = Z_2(1, 0.75) = 1.132$. $Z_1$ decreases initially, passes the steady state locus, and then increases and converges to $Z_1(1, 0.75) = 5.645$. The behavior of this equilibrium path may appear surprising at the first glance, but it will become clear later when I explain in detail why the equilibrium path in this case has to pass through the steady state locus. For the moment, I only want to remind the reader that our dynamic system has three variables, $Z_1$, $Z_2$, and $u$, other than two variables that we normally encounter. When $u_0 = 0.90 > u^*$, both $Z_1$ and $Z_2$ will decrease and converge to $Z_1(1, 0.90) = 3.872$ and $Z_2(1, 0.90) = 0.878$, respectively. This equilibrium path is the one moving towards south-west. Remember that $Z_1$ and $Z_2$ are the transformed variables of $K$ and $h$. A decrease in $Z_1$, for example, only means that $K$ grows at less than its long run growth rate, $\kappa + \lambda$; it does not necessarily mean that $K$ itself decreases (although it certainly may). Similar statements apply to $Z_2$ and $h$.

Now, consider country 2 that has the same parameter values as those in the U.S.. The only difference of country 2 from the U.S. is that it has $(K_0, h_0) = (1, 1)$ in 1960, and hence was poorer than the U.S. initially. In Figure 1, five equilibrium paths for country 2 are drawn. I purposely set $h_0$ at the same level in the two countries because this setting eases upcoming discussions and generates a beautiful graph. The same analysis can be done when $h_0$ are different in the two countries.

Among the five equilibrium paths in country 2, three of them have $u_0 =$ 0.82, 0.75, and 0.90, respectively. Since $h_0$ are assumed to be the same in the two countries, equations (19) and (20) show that each of these three equilibrium paths in country 2 should converge to its corresponding equilibrium path in country 1. Figure 1 describes the convergence. The other two equilibrium paths of the five are associated with $u_0 = 0.7$, and $u_0 = 0.95$. The diagram indicates that, depending on whether country 2 follows the path with $u_0 = 0.7$, or $u_0 = 0.95$, its economy can permanently surpass or lag behind the economy in country 1 if the latter follows any of the three equilibrium paths. For example, If country 2 follows the path with $u_0 = 0.7$, and country 1 follows the path with $u_0 = 0.75$, then our calculation shows that country 2 will be able to overtake country 1 in both $Z_1$ and $Z_2$ (or equivalently in $K$ and $h$) in twenty four years. The success of the country 2 with $u_0 = 0.7$ suggests that countries such as Japan, Korea, Singapore, etc. may have followed similar high growing paths.

Figure 1 also shows that the equilibrium path of country 2 with $u_0 = 0.95$ intersects that of country 1 with $u_0 = 0.9$. This, however, does not mean that country 2 has caught up with country 1 at some point in time; in fact, it never has. The likely confusion results from the lack of time dimension in Figure 1.

Now, it is time for the explanation why some equilibrium paths pass through the steady state locus. Again, the explanation is put in a theorem.

**Theorem 8** *Let the steady state locus in $(Z_1, Z_2)$-plane be defined by:*

$$AZ_1^{\beta-1} Z_2^{(1+\gamma-\beta)} u^{*(1-\beta)} - (\kappa + \frac{\rho}{\beta}) = 0. \tag{21}$$

*Then, starting from $(Z_1(0), Z_2(0))$ above the steady state locus, any equilibrium path with $u_0 < u^*$ will necessarily pass through the steady state locus and head back later towards the steady state locus. Starting from $(Z_1(0), Z_2(0))$ below the steady state locus, any equilibrium path with $u_0 > u^*$ will necessarily pass through the steady state locus and head back later towards the steady state locus.*

**Proof.** Suppose $(Z_1(0), Z_2(0))$ lies above the steady state locus. For any equilibrium path with $u_0 < u^*$, equations (16) and (17) say that $Z_1$ must decrease initially, and that $Z_2$ increases. Hence, the equilibrium path moves toward south-east at first.

Also, we know that for any $t$, $u(t)$ is less than $u^*$. Because of this, when the path hits the steady state locus, equations (16) and (17) dictate that $\dot{Z}_1$ will still be negative, and that $\dot{Z}_2$ will still be positive. Therefore, the equilibrium path must pass through the steady state locus. Since $u(t)$ monotonically approaches $u^*$, $\dot{Z}_1$ will equal zero at some point in time and turn positive and finally converge to zero. The fact that $\dot{Z}_2$ is positive means that the equilibrium path heads back towards the steady state locus (see Figure 1).

Similar argument applies to the second half of the theorem.

Remark: Theorem 4 casts doubt on Figure 1 in Lucas's paper. Although when $\sigma \neq \beta$, things can become more complicated, the equilibrium paths conjectured in Lucas [8] do not seem very probable.

# 5 Welfare Consideration

The theoretical findings in Section 3 and Figure 1 in Section 4 both indicate that a country can achieve higher economic status in the future if more time is spent initially in schooling. On the other hand, less time spent in goods production means a sacrifice in current income. A natural question to ask is thus the following. In terms of his or her lifetime welfare, is the sacrifice in current income to the individual's best interest?

**Theorem 9** *Among all the equilibrium paths starting from* $(K_0, h_0)$ *and indexed by* $\{u_0 : \; u_0 \in (0,1] \}$*, the one associated with a lower* $u_0$ *always gives a better lifetime welfare.*

**Proof.** Let $V(u_0)$ be the value of the individual's lifetime welfare when the equilibrium path has $u(0) = u_0$. Because the explicit dynamics of the equilibrium path is known, $V(u_0)$ can be calculated as follows:

$$V(u_0) = \int_0^\infty \frac{N}{1-\beta}[c^{1-\beta} - 1]e^{-\rho t} dt,$$

$$\text{where} \quad c = (\rho/\beta - \lambda) Z_1 e^{\kappa t},$$

$$\dot{Z}_1 = A Z_1^\beta \left[ \frac{h_0}{u_0^{\frac{\beta}{\gamma-\beta}}} \right]^{1+\gamma-\beta} u^{\frac{\gamma}{\gamma-\beta}} - (\kappa + \rho/\beta)\, Z_1, \text{ with } Z_1(0) = K_0,$$

$$u = \frac{u_0 u^* e^{u^*(\gamma-\beta)\delta t/\beta}}{u^* - u_0 + u_0 e^{u^*(\gamma-\beta)\delta t/\beta}}.$$

It is shown in Appendix B that

$$V(u_0) = I_1 + I_2 \frac{u_0^{-\beta}}{\delta},$$

where $I_1$ and $I_2$ are constant and independent of $u_0$. Also, $I_2$ is positive. Therefore, the lower the $u_0$, the higher the welfare.

# 6 Discussions on the Methodology

In this section, I would like to discuss four different ways to study transitional dynamics, namely, phase diagram approach (e.g. Romer [11]), local stability analysis (e.g. Benhabib and Perli [3]), numerical methods (Mulligan and Sala-i-Martin [9], [10]), and the method used in this paper and in Xie [12] which can be labeled explicit dynamics method.

Among the four approaches, phase diagram approach puts the least restrictions on the functional forms of preferences and technology. Often, the restrictions only require the first and second derivatives of the utility and production functions to have desired signs and limiting behavior. Local stability analysis sometimes requires further specifications of the funtional forms. Numerical methods definitely need the functional forms to be explicit. Usually, the specification is identified with a set of parameters such as the intertemporal rate of substitution, the rate of time preference, the capital income share, etc.. The numerical methods involve solving differential equations with both initial conditions and transversality conditions (sometimes replaced by a convergence to steady states), which is potentially difficult. The explicit dynamics method needs to put some further restrictions on the set of parameters (such as $\sigma = \beta$ in this paper) in order to reduce the optimization problems to a set of differential equations with initial constraints only, which are easy to compute.

If we are interested in simulating an economy, numerical methods are the obvious choice. But even then, theoretical preparations may prove useful for guidance. To obtain theoretical results, we usually start with a phase diagram provided that the model is not complicated. When a complicated model is involved (e.g. the Lucas model), a phase diagram is too difficult to draw. In this case, local stability analysis used to be the only choice. This paper and Xie [12] show that an alternative method, namely the explicit dynamics method, can often work wonders: it can produce crystal clear transitional dynamics that allows us to draw qualitative conclusions. What needs to be done is to determine which qualitative results are likely to carry through under less restrictive assumptions on the parameters. For example, I have the confidence that the following results obtained under the assumption $\sigma = \beta$ hold when $\sigma \neq \beta$: (a) possibility of multiple equilibria when externality is strong; (b) divergence in income can occur; (c) more time initially in education leads to higher welfare. On the other hand, there are results that I do not expect to hold in general: (i) aggregate consumption is linear in capital stock; (ii) steady state values of $(Z_1, Z_2)$ do not depend on initial capital stock, $K_0$. Also, I was not sure whether the condition for the existence of multiple equilibria would still have the simple form, $\gamma > \beta$ when $\sigma \neq \beta$. I originally thought that the condition may generally take the form, $\gamma > \Gamma(\sigma, \beta, \rho, K_0, h_0)$, which reduces to $\gamma > \beta$ when $\sigma = \beta$. This caution is proven unnecessary when I found recently that Benhabib and Perli [3] had obtained the same condition $\gamma > \beta$ for the existence of multiple equilibria when $\sigma \neq \beta$.

All in all, the explicit dynamics method, if used with judgment, can be highly productive.

# 7 Conclusion

This paper takes it further the work by Robert E. Lucas Jr. on the mechanics of economic development. With the aid of explicit dynamics of the equilibrium paths, the paper uncovers several additional features of the Lucas model: A continuum of equilibria exist when the external effect of human capital in goods production is sufficiently large. The process of lagging behind, catching up with, and overtaking that we often observe in the world economy can be explained. Also, the paper shows that some equilibrium paths have such complicated patterns that the dynamics conjectured in Lucas [8] are too simple to be correct.

The issue of multiple equilibria warrants discussion. Do multiple equilibria exist in the real world? If they do, how a particular equilibrium is selected by an economy? Once an equilibrium path is selected , will it be possible for the economy to shift to another path in the future?

In OLG settings, Balasko and Shell [1] demonstrates that multiple equilibria can exist in a pure exchange economy with money; Galor [5] shows that indeterminacy can arise in a two-sector production economy without money. In models with long-lived representative individual, multiple equilibria are likely to exist when the aggregate production function exhibits increasing returns. Besides this paper and Benhabib and Perli [3], other examples include Benhabib and Farmer [2], Boldrin and Rustichini [4], and Chamley [**?**]. In these papers, the conditions for indeterminacy are summarised by a few inequalities in key parameters, for example $\gamma > \beta$ in the present paper. Although Lucas's calibration of the U.S. economy indicates that $\gamma > \beta$ is indeed satisfied, the issue of indeterminacy can only be settled by further empirical estimations of the parameters using data set of other developed countries.

Which equilibrium path actually prevails may depend on many factors. Cultural tradition and work ethics are only two of them. As to whether shifts in path are possible, my answer is yes, but only with the help of institutional reforms, outside intervention, etc. Let's take China for example. Comparing the period before 1979 when intellectuals were not respected and the period after the reform when they are, one can see a dramatic increase of the time and the effort devoted by Chinese to schooling in the latter period.

The normative implication of this paper is that a government should do whatever is necessary to improve the return to human capital. This effort by the government will induce people to spend more time in schooling and therefore will raise the productivity of physical capital and of labor. Also, the effort improves the society's welfare. This call for government intervention is similar in spirit with that in Cass and Shell [6], with different orientation: Their call for government action is to stabilize *fluctuations* in the presence of multiple equilibria due to "sunspots activities", whereas ours is to enhance *growth* in the presence of multiple equilibria due to "increasing returns".

The positive implication of this paper is the following: given the spread of initial endowments across countries, given that multiple equilibria may exist, it is no wonder that we do not observe a clear convergence among the economies in the world. Also, it is understandable that some countries have had a great performance due to growth enhancing policies, some others have lagged behind permanently due to short-sighted behavior.

# A Proof of Lemma 3

Taking logs of both sides of equation (7) and then differentiating with respect to time, we have

$$\left[\frac{\dot{\theta}_1}{\theta_1} + \beta\frac{\dot{K}}{K}\right] - \beta\frac{\dot{u}}{u} + (1-\beta)\lambda + (1+\gamma-\beta)\frac{\dot{h}}{h} = \frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{h}}{h}.$$

Using Lemma 1, Lemma 2, and equation (6), we can simplify the equation above as,

$$\beta\lambda - \beta\frac{\dot{u}}{u} + (1-\beta)\lambda + (\gamma-\beta)\delta(1-u) = (\rho-\delta).$$

Or,

$$\dot{u} = u\frac{1}{\beta}\left[\lambda + (1+\gamma-\beta)\delta - \rho\right] - \frac{(\gamma-\beta)\delta}{\beta}u^2.$$

When $\gamma \neq \beta$, we can define $u^* = \left[\lambda + (1+\gamma-\beta)\delta - \rho\right] / \left[(\gamma-\beta)\delta\right]$. And we have,

$$\dot{u} = \frac{(\gamma-\beta)\delta}{\beta}u(u^* - u).$$

# B Calculation of $V(u_0)$

$$V(u_0) = \int_0^\infty \frac{N}{1-\beta}[c^{1-\beta} - 1]e^{-\rho t}dt,$$

$$\text{where} \quad c = (\rho/\beta - \lambda)Z_1 e^{\kappa t},$$

$$\dot{Z}_1 = AZ_1^\beta \left[\frac{h_0}{u_0^{\frac{\beta}{\gamma-\beta}}}\right]^{1+\gamma-\beta} u^{\frac{\gamma}{\gamma-\beta}} - (\kappa + \rho/\beta)\, Z_1, \text{ with } Z_1(0) = K_0,$$

$$u=\frac{u_0u^*e^{u^*(\gamma-\beta)\delta t/\beta}}{u^*-u_0+u_0e^{u^*(\gamma-\beta)\delta t/\beta}}.$$

Substituting $c$ by its expression in $Z_1$, we obtain

$$\begin{aligned}V(u_0)&=\int_0^\infty\tfrac{1}{1-\beta}\left[(\rho/\beta-\lambda)Z_1e^{\kappa t}\right]^{1-\beta}e^{-(\rho-\lambda)t}dt+\text{constant}\\&=\int_0^\infty\tfrac{1}{1-\beta}\left[(\rho/\beta-\lambda)Z_1\right]^{1-\beta}e^{-(\rho-\lambda-(1-\beta)\kappa)t}dt+\text{constant}.\end{aligned}$$

Note that $\rho-\lambda-(1-\beta)\kappa=\delta u^*>0$. We can use integration by parts and obtain,

$$\begin{aligned}V(u_0)&=\text{constant}+\tfrac{1}{\delta u^*}(1-\beta)\int_0^\infty\tfrac{1}{1-\beta}(\rho/\beta-\lambda)^{1-\beta}Z_1^{-\beta}\dot{Z}_1e^{-\delta u^*t}dt\\&=\text{constant}+\tfrac{1}{\delta u^*}(1-\beta)\int_0^\infty\tfrac{1}{1-\beta}(\rho/\beta-\lambda)^{1-\beta}Z_1^{-\beta}\\&\quad\times\left[AZ_1^\beta h_0^{1-\gamma-\beta}u_0^{-(1+\gamma-\beta)\beta/(\gamma-\beta)}u^{\gamma/(\gamma-\beta)}-(\kappa+\rho/\beta)Z_1\right]e^{-\delta u^*t}dt\\&=\text{constant}+J_1\int_0^\infty u_0^{-(1+\gamma-\beta)\beta/(\gamma-\beta)}u^{\gamma/(\gamma-\beta)}e^{-\delta u^*t}dt-J_2V(u_0),\end{aligned}$$

where constants $J_1$ and $J_2$ are obviously defined. They are independent of $u_0$. And they are seen to be unambiguously positive. In the derivation above, the fact that $Z_1\to Z_1^*$ is used implicitly.

Substituting $u$ by its explicit expression and rearranging terms, we obtain,

$$V(u_0)=I_1+I_2\int_0^\infty\frac{u_0^{1-\beta}{u^*}^{\gamma/(\gamma-\beta)}\left(e^{u^*(\gamma-\beta)\delta t/\beta}\right)^{\gamma/(\gamma-\beta)}}{\left(u^*-u_0+u_0e^{u^*(\gamma-\beta)\delta t/\beta}\right)^{\gamma/(\gamma-\beta)}}e^{-\delta u^*t}dt,$$

where $I_1$ and $I_2$ are constant and independent of $u_0$; $I_2=J_1/(1+J_2)$ is positive. To further calculate the integral in the expression above, let's do a change of variable, $x=e^{u^*(\gamma-\beta)\delta t/\beta}$. Then $dx=[xu^*(\gamma-\beta)\delta/\beta]dt$. The integral is simplified as,

$$\int_1^\infty\left[\frac{\beta}{(\gamma-\beta)\delta}\right]\frac{u_0^{1-\beta}{u^*}^{\beta/(\gamma-\beta)}}{(u^*-u_0+u_0x)^{\gamma/(\gamma-\beta)}}dx=\frac{u_0^{-\beta}}{\delta}.$$

Therefore, we have finally established that

$$V(u_0)=I_1+I_2\frac{u_0^{-\beta}}{\delta},\ \text{for any } u_0\in(0,1].$$

## References

[1] Y. Balasko, and K. Shell, The Overlapping-Generations Model. II: The Case of Pure Exchange with Money, *J. Econ. Theory* **24** (1981), 112–142.

[2] J. Benhabib and R. Farmer, Indeterminacy and Increasing Returns, *J. Econ. Theory*, this issue.

[3] J. Benhabib and R. Perli, Uniqueness and Indeterminacy: Transitional Dynamics in a Model of Endogenous Growth, *J. Econ. Theory*, this issue.

[4] M. Boldrin and A. Rustichini, "Growth and Indeterminacy in Dynamic Models with Externalities," Working Paper, Northwestern University, January, 1993.

[5] O. Galor , A Two-Sector Overlapping-Generations Model: A Global Characterization of the Dynamical System, *Econometrica* **60** (1992), 1351–1386.

[6] D. Cass and K. Shell, Do Sunspots Matter? *J. Polit. Econ.* **91** (1983), 193–227.

[7] C. Chamley, Externalities and Dynamics in Models of 'Learning or Doing', *Int. Econ. Rev.* forthcoming.

[8] R. E. Lucas Jr., On the Mechanics of Economic Development, *J. Monet. Econ.* **22** (1988), 3–42.

[9] C. Mulligan and X. Sala-i-Martin, "Transitional Dynamics in Models of Endogenous Growth," Manuscript, September, 1991.

[10] C. Mulligan and X. Sala-i-Martin, "A Note on the Time-Elimination Method for Solving Recursive dynamic Economic Models," NBER Technical Working Paper No. 116, 1991.

[11] P. M. Romer, Increasing Returns and Long-Run Growth, *J. Polit. Econ.* **94** (1986), 1002-1073.

[12] D. Xie, Increasing Returns and Increasing Rates of Growth, *J. Polit. Econ.* **99** (1991), 429–435.

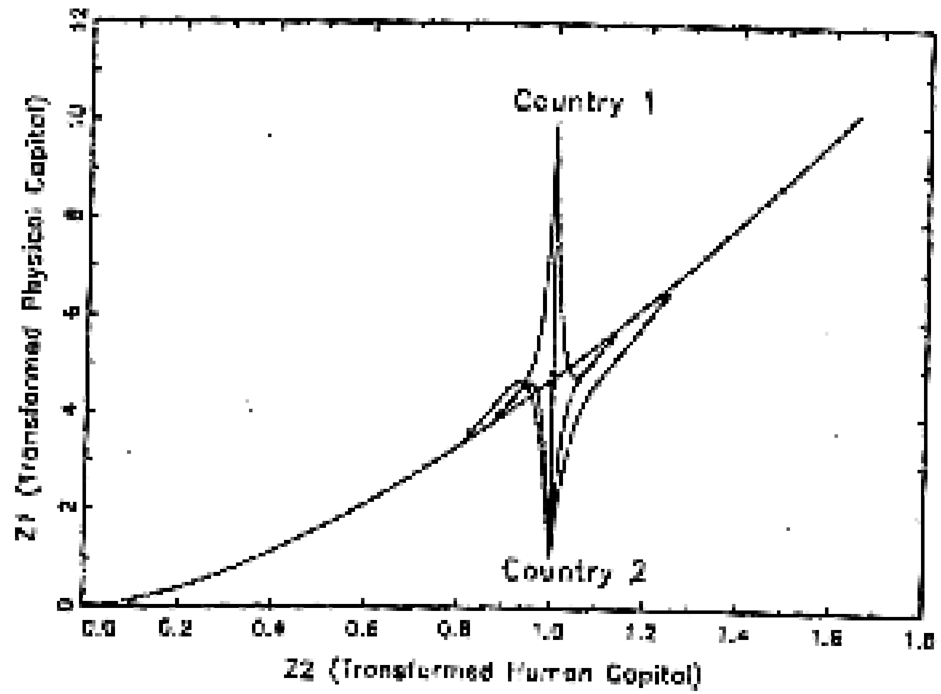


FIG. 1. Multiple equilibrium paths in country 1 and country 2.