

Explicit Transitional Dynamics in Growth Models

Danyang Xie*

International Monetary Fund

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Abstract

Transitional dynamics in growth models have been subject to much attention recently. With a few exceptions, existing studies rely on computational techniques. This paper uses a set of examples to illustrate that qualitative insights on the transitional dynamics can be gained at the expense of using special utility-production pairs. In continuous time framework, necessary and sufficient conditions are established for a utility-production pair to yield explicit dynamics. These conditions are potentially useful for applications in other dynamic settings.

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1 Introduction

In growth models, the equilibrium allocation is characterized by a system of differential or difference equations in state and co-state variables constrained by a set of initial and transversality conditions. Furthermore, if the models are in infinite horizon, the transversality conditions are in limit form, which is hard to deal with when computing the transitional dynamics, namely, the transitional trajectory to a steady state or a balanced growth path..

If the modelled economies are shown to have a unique stable steady state, computation of the transitional path is reasonably straightforward. First, we solve for the steady state. Then the system of equations is linearized around the steady state. Finally, the linearized system can be solved using standard softwares. This routine shall provide decent approximation of the transitional dynamics when the initial state variables are close to the steady state. Multiple shooting and backward shooting methods are also used on some occasions.

If the modelled economies are shown to have a unique balanced growth path, Mulligan and Sala-i-Martin (1993) suggests that we transform the variables into “state-like” and “control-like” variables which converge to a steady state. The transformed system of equations can then be solved as described above.

The numerical procedures outlined above are powerful when conducting a simulation exercise, but they are not without limitations. Qualitative details are usually buried in the computational complexities.

In this paper, we highlight the alternative method in studying transitional

dynamics: namely the explicit dynamics method. This method makes use of utility-production pairs (henceforth U-P pairs) that allow for explicit transitional dynamics. Its limitation lies in that certain restrictions are put on the utility and production functions; these restrictions may not be consistent with the real economy and hence should be discarded when simulation is the purpose. The benefit of using the U-P pairs is that qualitative results are seen explicitly and therefore provide guidance for simulation and estimation.

Explicit dynamics method is employed in Long and Plosser (1983) and McCallum (1989) in stochastic discrete time framework to analyze real business cycles. The U-P pair used there is the now familiar log utility and Cobb-Douglas production function. Benhabib and Rustichini (1994) finds that the pair of log utility and Cobb-Douglas production function can be extended to a class of CES utility and CES production function with a common parameter. In deterministic continuous time framework, Xie (1991) uses a class of U-P pairs to show that with externality in production, growth rate can increase over time and approach an upper bound which depends on preference and technology parameters. With similar U-P pairs, Xie (1994) shows explicitly that the two-sector growth model of Lucas (1988) contains multiple equilibria, and the global transitional dynamics is found to display surprising features (See Figure 1 in Xie (1994)). In an independent study, Benhabib and Perli (1994) uses local approximation technique to uncover similar characteristics of the Lucas model.

Because the set of papers above uses explicit dynamics method in specific and complex context, the method itself does not receive enough exposure. This paper aims at putting explicit dynamics method into the hands of the

researcher who is interested in transitional dynamics.

The rest of the paper is organized as follows. Section 2 deals with discrete time framework. We present a one-sector RBC model and use the U-P pair à la Benhabib and Rustichini (1994) to illustrate the explicit dynamics. Also, a deterministic variant is used to show that chaos can easily arise in the presence of externality. Section 3 deals with continuous time framework. We present a one-sector growth model and demonstrate how a U-P pair leads to explicit solution. Section 4 studies the one-sector growth model more closely and delivers necessary and sufficient condition for a U-P pair to permit explicit dynamics. Applications of the necessary conditions are illustrated in examples. Section 5 examines continuous time stochastic models. Section 6 concludes with a discussion of future research.

2 U-P Pairs in Discrete Time Framework

This section contains two models. The first is a one-sector RBC model. The second is a one-sector growth model. In the first model, the U-P pair à la Benhabib and Rustichini (1994) is used to illustrate the explicit dynamics. In the second model, we use log utility and Cobb-Douglas production function with externality to show the possibility of chaos. We see that chaos can arise no matter what value the discount factor assumes.

2.1 One-Sector RBC Model

Consider an economy with long-lived representative agent. The preferences of the agent are as follows:

$$E_t \sum_{\tau=0}^{\infty} \beta^{\tau} [\theta u(c_{t+\tau}) + (1 - \theta)v(1 - n_{t+\tau})] \quad (1)$$

where E_t is mathematical expectation conditional on time t information; $\beta \in (0, 1)$ is the discount factor; θ is the weight on the utility of the single consumption good produced in this economy; $1 - n_{t+\tau}$ is the leisure at time $t + \tau$ (the total time available to the representative individual is normalized at unity every period).

The production technology exhibits constant returns to scale in capital k and labor n :

$$y_t = A_t f(k_t, n_t) \quad (2)$$

where A_t is the technology shock and is assumed to be *i.i.d.*

Capital accumulation takes the natural form:

$$k_{t+1} = y_t + (1 - \delta)k_t - c_t \quad (3)$$

where δ is rate of capital depreciation.

If we let $u(c) = \ln c$, $v(1 - n) = \ln(1 - n)$, $f(k, n) = k^{\alpha}n^{1-\alpha}$ and $\delta = 1$, then this model is exactly the one-sector Long and Plosser model reported in McCallum (1989). The consumption, output, and capital are shown explicitly to display cyclical behavior in response to technology shock. One minor unsatisfactory point is that labor is irresponsive to the shock, which Long

and Plosser (1983) explained is due to the special combination of log utility, Cobb-Douglas production function and 100% depreciation.

Benhabib and Rustichini (1994) suggests that the pair of log utility and Cobb-Douglas production function is a special example of the following class of pairs which permits explicit dynamics with 100% depreciation:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, v(1-n) = \frac{(1-n)^{1-\sigma} - 1}{1-\sigma} \quad (4)$$

and

$$f(k, n) = [\alpha k^{1-\sigma} + (1-\alpha)n^{1-\sigma}]^{1/(1-\sigma)}. \quad (5)$$

It is easy to see that log utility and Cobb-Douglas production function is the case with $\sigma = 1$. The above class of U-P pairs can be indexed by the common parameter σ .

To see how the special class of U-P pairs allow for explicit dynamics, we write down the Lagrangian for the optimization problem:

$$\begin{aligned} \mathcal{L} = & E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left\{ \theta \frac{c_{t+\tau}^{1-\sigma} - 1}{1-\sigma} + (1-\theta) \frac{(1-n_{t+\tau})^{1-\sigma} - 1}{1-\sigma} \right. \\ & \left. + \lambda_{t+\tau} [A_{t+\tau} [\alpha k_{t+\tau}^{1-\sigma} + (1-\alpha)n_{t+\tau}^{1-\sigma}]^{1/(1-\sigma)} - c_{t+\tau} - k_{t+\tau+1}] \right\} \end{aligned}$$

The first order conditions are:

$$E_t \theta c_{t+\tau}^{-\sigma} = E_t \lambda_{t+\tau} \quad (6)$$

$$E_t (1-\theta) (1-n_{t+\tau})^{-\sigma} = E_t \lambda_{t+\tau} (1-\alpha) A_{t+\tau} [\alpha k_{t+\tau}^{1-\sigma} + (1-\alpha)n_{t+\tau}^{1-\sigma}]^{\sigma/(1-\sigma)} n_{t+\tau}^{-\sigma} \quad (7)$$

$$E_t \lambda_{t+\tau} = \beta E_t \left\{ \lambda_{t+\tau+1} \alpha A_{t+\tau+1} [\alpha k_{t+\tau}^{1-\sigma} + (1-\alpha)n_{t+\tau}^{1-\sigma}]^{\sigma/(1-\sigma)} k_{t+\tau+1}^{-\sigma} \right\}. \quad (8)$$

These equations hold for any $\tau \geq 0$.

The transversality condition is given by $E_t \beta^{t+\tau} \lambda_{t+\tau} k_{t+\tau+1} \rightarrow 0$ as $\tau \rightarrow \infty$.

Similar to McCallum (1989), we guess that the solution takes the following form:

$$c_t = by_t \text{ and } k_{t+1} = (1 - b)y_t \text{ with } b \text{ constant for any } t.$$

When A_t is *i.i.d.*, so is $A_t^{1-\sigma}$. Let us denote the unconditional mean of $A_t^{1-\sigma}$ by π , i.e. $EA_t^{1-\sigma} = \pi$. Then we can easily verify that the above guess is correct when

$$b = 1 - (\alpha\beta\pi)^{1/\sigma}. \quad (9)$$

And we have:

$$k_{t+1} = (\alpha\beta\pi)^{1/\sigma} A_t [\alpha k_t^{1-\sigma} + (1 - \alpha)n_t^{1-\sigma}]^{1/(1-\sigma)} \quad (10)$$

$$c_t = [1 - (\alpha\beta\pi)^{1/\sigma}] A_t [\alpha k_t^{1-\sigma} + (1 - \alpha)n_t^{1-\sigma}]^{1/(1-\sigma)} \quad (11)$$

$$n_t = \frac{[\theta(1 - \alpha)]^{1/\sigma}}{[\theta(1 - \alpha)]^{1/\sigma} + (1 - \theta)^{1/\sigma} [1 - (\alpha\beta\pi)^{1/\sigma}] A_t^{-(1-\sigma)/\sigma}} \quad (12)$$

To check our calculation, set $\sigma = 1$. In this case, $\pi = 1$ and

$$n_t = \frac{\theta(1 - \alpha)}{\theta(1 - \alpha) + (1 - \theta)(1 - \alpha\beta)},$$

which is identical to (1.19) in McCallum (1989) after making the notations consistent.

We see from equation (10) to (12) that productivity shock, even if it is *i.i.d.*, can generate persistent business fluctuations. In particular, equation (12) says that the employment can be pro-cyclical if $\sigma < 1$.

Clearly, it may not be reasonable to assume that the utility and production function share a common parameter σ . But the literature in RBC seems

to have used the special pair of $\sigma = 1$ without realizing the fact. This exercise seems to suggest, in the light of equation (12), that the existing RBC simulation can be improved upon if we use a log utility function and a CES production function with $\sigma < 1$: the employment variation may be raised.

The class of U-P pairs can also be used in variants of the model to generate explicit dynamics. For instance, labor indivisibility described in Rogerson (1984) and used in Hansen (1985) for RBC simulation can be incorporated. These pairs can also be useful in multi-sector models to simplify the dynamics.

2.2 One-Sector Growth Model with Externality: Chaos

Consider a representative individual's utility maximization problem:

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$\text{subject to: } k_{t+1} = Ak_t^\alpha B(\hat{k}_t) - c_t$$

where $B(\hat{k}_t)$ captures the external effect with \hat{k}_t representing average stock of capital in this economy. This is a model that uses log utility and Cobb-Douglas production function. We will see that the presence of externality does not jeopardize the nature of explicit dynamics. Again, 100% depreciation is assumed.

The first order conditions can be written as follows:

$$\frac{1}{c_t} = \lambda_t$$

$$\lambda_t = \beta \lambda_{t+1} \alpha A k_{t+1}^{\alpha-1} B(\hat{k}_{t+1})$$

The transversality condition is:

$$\lambda_t k_{t+1} \beta^t \rightarrow 0 \text{ as } t \rightarrow \infty.$$

The solution in this model is very much similar to the one in previous model, namely we have:

$$c_t = (1 - \alpha\beta)y_t \tag{13}$$

$$k_{t+1} = \alpha\beta y_t. \tag{14}$$

Note that the above derivation does not require any specification for $B(\hat{k}_t)$. Suppose the external effects are two-fold, $B(\hat{k}_t) = \hat{k}_t^{1-\alpha}(1 - \hat{k}_t)$. The term $\hat{k}_t^{1-\alpha}$ captures the positive effect considered in endogenous growth literature such as Romer (1986) and Lucas (1988). The other term $(1 - \hat{k}_t)$ captures the negative effect such as pollution or congestion considered in Day (1982). Then equation (14) becomes:

$$k_{t+1} = \alpha\beta A k_t (1 - k_t) \tag{15}$$

where the equilibrium condition $\hat{k}_t = k_t$ has been substituted in.

Equation (15) shows that depending on the values of $\alpha\beta A$, the dynamics of capital can be simple or complex. For example, if $\alpha\beta A = 4$, chaos arises.

Day (1982) shows that chaos may arise in two variants of the Solow growth model. In the first variant, he lets the saving rate be constant but introduces a pollution effect. In the second variant, he leaves the production function in the Solow model untouched but allows variable saving rate. While he demonstrates chaotic possibilities, his two models do not involve any optimization decision. In contrast, our conclusion comes from individual's well-defined optimization problem which can be decentralized to a competitive equilibrium.

Another point is worth noting. In competitive chaos literature, for example Boldrin and Montrucchio (1984), Neumann et al (1988), a small discount factor β is usually needed for complex dynamics. In our simple model here, it is shown explicitly that for any β , chaos can arise if α and A assume appropriate values.

3 Continuous Time Framework

Now we turn to continuous time framework. Let us use the following standard one-sector growth model as illustration:

$$\begin{aligned} & \max \int_0^{\infty} u(c)e^{-\rho t} dt \\ & \text{subject to: } \dot{k} = f(k) - \delta k - c, \text{ with } k_0 \text{ given.} \end{aligned} \quad (16)$$

Write down the Hamiltonian,

$$\mathcal{H} = u(c) + \lambda(f(k) - \delta k - c)$$

The first order conditions are:

$$u'(c) = \lambda \quad (17)$$

$$\dot{\lambda} = \rho\lambda - \lambda(f'(k) - \delta) \quad (18)$$

The transversality condition is $\lambda k e^{-\rho t} \rightarrow 0$ as $t \rightarrow \infty$.

Let us use CES utility function and production function:

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \quad (19)$$

and

$$f(k) = Ak^\beta \text{ with } \beta \in (0, 1). \quad (20)$$

Then we find that

$$\begin{aligned} \frac{\dot{c}}{c} - \frac{\dot{k}}{k} &= -\left(\frac{1}{\sigma}\right) \frac{\dot{\lambda}}{\lambda} - \frac{\dot{k}}{k} \\ &= \frac{c}{k} - \frac{\rho + (1 - \sigma)\delta}{\sigma} + \left(\frac{\beta}{\sigma} - 1\right) f(k)/k \end{aligned} \quad (21)$$

Clearly, if $\sigma = \beta \in (0, 1)$, equation (21) is much simplified. Indeed in this case, we find the obvious solution:

$$\frac{c}{k} = \frac{\rho + (1 - \sigma)\delta}{\sigma}. \quad (22)$$

which is the only solution that satisfies the transversality condition. As a result, the evolution of capital becomes:

$$\dot{k} = Ak^\sigma - [\rho + (1 - \sigma)\delta]k/\sigma, \text{ with } k_0 \text{ given.} \quad (23)$$

Explicit solution for the above equation exists but is omitted here.

This class of U-P pairs is the same in spirit as the class in discrete time framework that the utility and production function share a common parameter. The difference in form should be noted however: the class in discrete time involves CES utility and CES production function whereas the class in continuous time involves CES utility function and Cobb-Douglas production function. Also, in continuous time, we do not need a 100% depreciation of capital to have explicit solution.

This class of U-P pairs still permits explicit dynamics when the production function is modified to include externality or other productive factors. As a result, it has a number of applications. In one of these applications,

Xie (1991) shows that in the presence of positive externality, growth rate of capital can increase over time and approach an upper bound that depends on preference and technological parameters in an intuitive way. In another application, Xie (1994) shows that the two-sector growth model of Lucas (1988) contains multiple equilibria; furthermore, it shows explicitly that the transitional dynamics conjectured by Lucas is incorrect.

4 More on Continuous Time Framework

In this section, we give a complete characterization of the U-P pairs in the continuous time one-sector growth model that allow for closed form solution. By closed form solution, we mean that the solution has the form $c = g(k)$, with $g(\cdot)$ as a known function. For convenience, in the following discussion, we assume zero percent depreciation $\delta = 0$.

4.1 Propositions

PROPOSITION 1: Let utility function be given by $u(c)$. There is a closed form solution $c = g(k)$ if and only if the production function belongs to the following class:

$$f(k) = g(k) + \frac{\rho \int u'(g(k)) dk - u(g(k))}{u'(g(k))} \quad (24)$$

and that the transversality condition is satisfied.

Proof. For the “necessary” part, suppose that there is closed form solution $c = g(k)$. The first order conditions (16)–(18) (note that we assume $\delta = 0$)

yield

$$\frac{u''(g(k))}{u'(g(k))}g'(k)[f(k) - g(k)] = \rho - f'(k)$$

Rearranging terms we obtain

$$u'(g(k))f'(k) + u''(g(k))g'(k)f(k) = \rho u'(g(k)) + u''(g(k))g'(k)g(k)$$

The left-hand side is the derivative of $u'(g(k))f(k)$ with respect to k . Thus, taking integral of both sides, we have:

$$\begin{aligned} u'(g(k))f(k) &= \rho \int u'(g(k))dk + \int u''(g(k))g'(k)g(k)dk \\ &= \rho \int u'(g(k))dk + u'(g(k))g(k) - \int u'(g(k))g'(k)dk \\ &= \rho \int u'(g(k))dk + u'(g(k))g(k) - u(g(k)) \end{aligned}$$

From this equation we get:

$$f(k) = g(k) + \frac{\rho \int u'(g(k))dk - u(g(k))}{u'(g(k))}$$

Furthermore, the transversality condition has to be satisfied.

For the “sufficient” part, it is easy to verify that if $f(k)$ can be written as in equation (24) with a known function $g(\cdot)$, then $c = g(k)$ coupled with $\dot{k} = f(k) - g(k)$ satisfies all the first order conditions. If furthermore, the transversality condition is satisfied, then $c = g(k)$ is the optimal consumption rule. Hence, the solution is in closed form. ■

PROPOSITION 2: Let production function be given by $f(k)$. There is a closed form solution $c = g(k)$ if and only if $u(c)$ belongs to the following class

$$u(c) = \int \exp \left[\int \frac{h'(c)[\rho - f'(h(c))]}{f(h(c)) - c} dc \right] dc, \quad (25)$$

and that the transversality condition is satisfied. In equation (25), $h(\cdot) = g^{-1}(\cdot)$.

Proof. For the “necessary part”, suppose that there is closed form solution $c = g(k)$. By the definition of $h(\cdot)$, we have $k = h(c)$. Thus, $\dot{k} = h'(c)\dot{c}$. Combining this with the first order conditions (16)–(18) yields

$$f(h(c)) - c = h'(c) \frac{u'(c)}{u''(c)} [\rho - f'(h(c))] \quad (26)$$

Rearranging terms we obtain:

$$\frac{u''(c)}{u'(c)} = \frac{h'(c) [\rho - f'(h(c))]}{f(h(c)) - c} \quad (27)$$

Taking an integral from both sides of equation (27) and further calculating, we arrive:

$$u'(c) = \exp \left[\int \frac{h'(c) [\rho - f'(h(c))]}{f(h(c)) - c} dc \right]. \quad (28)$$

Therefore, $u(c)$ must belong to the class given by equation (25). In order to be sure that $c = g(k)$ is the solution, the transversality condition must be verified.

For the “sufficient” part, suppose the utility function can be written as in (25) for some explicit function $h(c)$. Let $g(\cdot) = h^{-1}(\cdot)$. To show that $c = g(k)$ is the optimal consumption rule, we reverse the above process to verify that all the first order conditions are satisfied. Since the transversality condition is also satisfied, we have shown that $c = g(k)$ is optimal and in closed form.

■

Proposition 1 and 2 can be used to find U-P pairs that permit closed form solution. We shall use a few examples to demonstrate the usefulness of these results.

4.2 Examples

EXAMPLE 1: Suppose $u(c) = [c^{1-\sigma} - 1]/(1 - \sigma)$ and suppose we want a closed form solution $c = \gamma k$ with γ constant. In the language of Proposition 1, $g(k) = \gamma k$. The appropriate production function is given by equation (24):

$$\begin{aligned}
 f(k) &= g(k) + \frac{\rho \int u'(g(k)) dk - u(g(k))}{u'(g(k))} \\
 &= \gamma k + \frac{\rho \int (\gamma k)^{-\sigma} dk - [(\gamma k)^{1-\sigma} - 1]/(1-\sigma)}{(\gamma k)^{-\sigma}} \\
 &= \gamma k + \frac{\rho \gamma^{-\sigma} k^{1-\sigma}/(1-\sigma) + J - [(\gamma k)^{1-\sigma} - 1]/(1-\sigma)}{(\gamma k)^{-\sigma}} \\
 &= \gamma k [1 + (\rho/\gamma - 1)/(1 - \sigma)] + (J + 1/(1 - \sigma))(\gamma k)^\sigma
 \end{aligned}$$

where J is any constant. Note that when $\gamma = [\rho + (1 - \sigma)\delta]/\sigma$, the formula above yields $f(k) = Ak^\sigma - \delta k$, which is exactly the case we discussed in detail in Section 3. The transversality condition is easily verified for $0 < \sigma < 1$ because the capital k approaches a steady state and so does the consumption c .

EXAMPLE 1': Suppose $f(k) = Ak^\sigma$ ($\sigma \in (0, 1)$), and suppose we want a closed form solution $c = (\rho/\sigma)k$. Then Proposition 2 can be used to find out the appropriate utility functions. The calculation is a bit difficult because we have to do integral twice. Nonetheless, the answer is that $u(c) = J_1 c^{1-\sigma} + J_2$ with constant J_1 and J_2 (J_1 positive). In general, Proposition 1 is easier to use than Proposition 2.

EXAMPLE 2: Suppose $u(c) = -e^{-c}$, and $g(k) = 2(\rho k)^{1/2}$. Equation (24)

implies that the production function must be of the following form:

$$\begin{aligned}
f(k) &= g(k) + \frac{\rho \int u'(g(k)) dk - u(g(k))}{u'(g(k))} \\
&= 2(\rho k)^{1/2} + \frac{\rho \int e^{-2(\rho k)^{1/2}} dk + e^{-2(\rho k)^{1/2}}}{e^{-2(\rho k)^{1/2}}} \\
&= 1/2 + (\rho k)^{1/2} + J e^{2(\rho k)^{1/2}}
\end{aligned}$$

where J is any constant. The simplest member in the above class is the one with $J = 0$. In this case,

$$f(k) = 1/2 + (\rho k)^{1/2},$$

which behaves nicely as a production function except that $f(0) \neq 0$. To make sure that $c = 2(\rho k)^{1/2}$ is the solution, we need to check the transversality conditions. Note that $\dot{k} = f(k) - c = 1/2 - (\rho k)^{1/2}$. k converges to a steady state and obviously so does c . Hence the transversality condition is satisfied.

EXAMPLE 3: Suppose $u(c) = -e^{-c}$, and $g(k) = \gamma \ln k$. The equation (24) says that we need the following function to make $c = g(k)$ optimal:

$$f(k) = \gamma \ln k + Jk^\gamma + \rho k / (1 - \gamma) + 1,$$

with J an arbitrary constant. For instance, when we set $J = 0$, we get a rather simple concave production function $f(k) = \gamma \ln k + \rho k / (1 - \gamma) + 1$ provided that γ is in the interval $(0, 1)$. To verify that $c = \gamma \ln k$ is indeed optimal in this case, again we need to check the transversality condition. In fact, from the following first order conditions

$$e^{-c} = \lambda, \tag{29}$$

$$\dot{k} = \gamma \ln k + \rho k / (1 - \gamma) + 1 - c, \quad (30)$$

and

$$\dot{\lambda} = \rho \lambda - \lambda [\gamma / k + \rho / (1 - \gamma)], \quad (31)$$

we see that $c = \gamma \ln k$ implies $\dot{k} = \rho k / (1 - \gamma) + 1$. Also, equation (29) says that $\lambda = e^{-\gamma \ln k} = k^{-\gamma}$. Thus

$$\lambda k e^{-\rho t} = k^{1-\gamma} e^{-\rho t},$$

which converges to $[k_0 + (1 - \gamma) / \rho]^{\rho / (1 - \gamma)}$ and therefore *the transversality condition is violated*. As a result $c = \gamma \ln k$ is NOT the optimal consumption rule for the case $u(c) = -e^{-c}$ and $f(k) = \gamma \ln k + \rho k / (1 - \gamma) + 1$.

WARNING: Example 3 shows that after deriving the production function, we need to verify the transversality condition to make sure that the consumption rule is indeed optimal for the utility and production pair.

REMARK: In Example 3, we can verify that for any arbitrary constant J , the utility-production pair will violate the transversality condition if $c = \gamma \ln k$. Therefore, we can conclude that when $u(c) = -e^{-c}$, there exists no production function such that the optimal consumption rule has the form: $c = \gamma \ln k$.

5 Continuous Time Stochastic Models

In Rebelo and Xie (1998), we gave examples of explicit solutions in stochastic monetary models with continuous time. Again, the examples made use of the U-P pairs described in the last section. We showed that with inelastic

labor supply, a constant nominal interest rate (it does not necessarily have to be zero) is optimal in a monetary model.

In this section, I would like to point out the same U-P pairs can also yield explicit dynamics in a model with capitalist spirit (see discrete time model of Zou (1994) and continuous time AK model of Gong and Zou (1998)).

5.1 Additively Separable Utility Function

$$\max \int_0^\infty \left[\frac{c^{1-\sigma}}{1-\sigma} + \theta \frac{k^{1-\sigma}}{1-\sigma} \right] e^{-\rho t} dt$$

$$\text{subject to : } dk = (Ak^\alpha - c) dt + \varepsilon k dz$$

$$k_0 \text{ given}$$

where dz is an increment of standard Wiener process. Note that k is included in the objective function to capture the idea of capitalist spirit. We show below that when $\alpha = \sigma \in (0, 1)$, there will be an explicit solution and we can then draw some qualitative conclusions for this model.

The Hamilton-Jacobi-Bellman equation for this problem is,

$$\begin{aligned} 0 = \max & \frac{c^{1-\sigma}}{1-\sigma} + \theta \frac{k^{1-\sigma}}{1-\sigma} - \rho J(k) \\ & + J'(k) (Ak^\sigma - c) + \frac{1}{2} J''(k) \varepsilon^2 k^2 \end{aligned}$$

We conjecture that the value function takes the following form:

$$J(k) = a + \frac{b^{-\sigma} k^{1-\sigma}}{(1-\sigma)}.$$

The optimum condition for c implies:

$$c^{-\sigma} = J'(k) = b^{-\sigma} k^{-\sigma}$$

Namely, $c = bk$. Putting this back into the Hamilton-Jacobi-Bellman equation, we obtain,

$$0 = \frac{b^{1-\sigma} k^{1-\sigma}}{1-\sigma} + \theta \frac{k^{1-\sigma}}{1-\sigma} - \rho \left[a + \frac{b^{-\sigma} k^{1-\sigma}}{(1-\sigma)} \right] + b^{-\sigma} A - b^{1-\sigma} k^{1-\sigma} - \sigma \frac{1}{2} b^{-\sigma} \varepsilon^2 k^{1-\sigma}$$

Comparing coefficients, we find that

$$0 = \sigma b + \theta b^\sigma - \left[\rho + \sigma(1-\sigma) \frac{1}{2} \varepsilon^2 \right]$$

$$a = \frac{A}{\rho b^\sigma}$$

An increase in θ , namely a stronger capitalist spirit, leads to a lower b . Thus, consumption will be a smaller proportion of capital. Growth rate is higher.

A higher ε , namely a greater uncertainty, leads to a higher b . The consumption will be a higher proportion of capital stock. Hence saving rate is lower. To understand this, note that an increase in uncertainty has two effects. The first relates to the precautionary motive to save and implies that saving should be higher. The second relates to the fact that higher uncertainty in the returns reduces incentive to invest. In the model specified here when $\alpha = \sigma < 1$, the second effect dominates and in equilibrium, saving is lower. Whether the result holds in general when $\alpha < 1$ but $\sigma > 1$ remains unanswered.

5.2 Multiplicatively Separable Utility Function

Let us take an alternative specification of the objective function.

$$\begin{aligned} \max \int_0^\infty \frac{c^{1-\sigma} k^\gamma}{1-\sigma} e^{-\rho t} dt \\ \text{subject to : } dk = (Ak^\alpha - c) dt + \varepsilon k dz \\ k_0 \text{ given} \end{aligned}$$

where $\gamma \in (0, \sigma)$. To be consistent with the idea of capitalist spirit, we again need to constrain σ to be less than 1. The Hamilton-Jacobi-Bellman equation for this problem is,

$$\begin{aligned} 0 = \max \frac{c^{1-\sigma} k^\gamma}{1-\sigma} - \rho J(k) \\ + J'(k) (Ak^\alpha - c) + \frac{1}{2} J''(k) \varepsilon^2 k^2 \end{aligned}$$

where $J(\cdot)$ is the value function. We conjecture that the value function takes the following form:

$$J(k) = a + \frac{b^{-\sigma} k^{\gamma+1-\sigma}}{(\gamma+1-\sigma)}$$

The optimum condition for c implies:

$$c^{-\sigma} k^\gamma = J'(k) = b^{-\sigma} k^{\gamma-\sigma},$$

namely, $c = bk$. Putting this back into the Hamilton-Jacobi-Bellman equation, we obtain,

$$\begin{aligned} 0 = \frac{b^{1-\sigma} k^{1-\sigma+\gamma}}{1-\sigma} - \rho \left[a + \frac{b^{-\sigma} k^{1-\sigma+\gamma}}{(\gamma+1-\sigma)} \right] \\ + b^{-\sigma} k^{\gamma-\sigma} [Ak^\alpha - bk] + (\gamma - \sigma) \frac{1}{2} b^{-\sigma} \varepsilon^2 k^{1-\sigma+\gamma} \end{aligned}$$

If $\alpha = \sigma - \gamma$, then our conjecture is confirmed if the coefficients satisfy:

$$b = \frac{\rho^{\frac{(1-\sigma)}{(\gamma+1-\sigma)}} + (1-\sigma)(\sigma-\gamma)\frac{1}{2}\varepsilon^2}{\sigma}$$

$$a = \frac{A}{\rho b^\sigma}$$

An increase in γ leads to a lower b . Thus, consumption will be a lower proportion of capital. Growth rate is higher. This is consistent with the result in the case with additively separable utility function.

An increase in ε increases b and lowers savings. This simply says that with multiplicatively separable utility function, the precautionary saving motive is stronger than the disincentive to invest. Again, this qualitative result has to be attached with a qualifier: $\alpha = \sigma - \gamma$.

6 Conclusion

In this paper, we highlight the U-P pairs that permit explicit dynamics in growth models. Both the discrete time case and the continuous time case are treated. We point out the similarities and differences in the two frameworks. Although the study is conducted in one-sector model, similar usage of the U-P pairs is possible in multi-sector model. See for example Long and Plosser (1983) and Devarajan et. al (1998) for discrete time case; and Xie (1994) and Mino (2001) for continuous time case. In continuous time case, the two propositions in Section 4 give researchers plenty choices over the U-P pairs that lead to closed form solutions. Corresponding propositions for the discrete time case have yet to be found. The paper also shows that

closed form solutions for continuous time, stochastic one-sector model found in Chang (1988) and Rebelo and Xie (1999) can be extended to models with capitalist spirit.

We hope that the results presented here about the U-P pairs lead to their more frequent application in a variety of context to obtain qualitative insights that provide guidance for simulation and estimation.

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