EXISTENCE OF EQUILIBRIUM IN INCOMPLETE MARKETS WITH NON-ORDERED PREFERENCES

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Abstract. In this paper we extend the results of recent studies on the existence of equilibrium in finite dimensional asset markets for both bounded and unbounded economies. We do not assume that the individual's preferences are complete or transitive. Our existence theorems for asset markets allow for short selling. We shall also show that the equilibrium achieves a constrained core within the same framework.

Date: March 2002.
JEL Classification. C62, D52, D81, G12.
Keywords. Equilibrium Existence, Incomplete Preferences, Incomplete Markets, Constrained Core.
During the preparation of this paper I have been influenced by helpful comments with many persons. Among those who deserve special mention are Simon Grant, David Kelsey, Frank Milne, Shasikanta Nandeibam and Peter Wagstaff.

1. Introduction

The original motivation for this paper was some recent work in the theory of finance. Recent studies on this topic take the extension of general equilibrium theory as its starting point, which is due to Debreu [3]. The first equilibrium existence result when consumption sets are unbounded below was proven by Hart [7] under the assumption that consumers' utility functions were Von Neumann-Morgenstern and that their directions of improvement were positively semi-independent. Later Werner [18] gave an existence result under the assumption that there exists at least one price for which there exist no-arbitrage opportunities for all consumers. Making fairly weak assumptions on preferences, Nielsen [13] obtained a very general result under the assumption that consumers' directions of improvement were positively semi-independent.

Recently, Bettzüge [1] conducted another study which deals with the issue of uniqueness of a general equilibrium in incomplete markets (GEI) . Giving a sufficient condition on the joint distribution of asset payoffs and individual endowments in a one physical commodity GEI model, he generalizes the Mitjushin and Polterovich's Theorem [1] to the case where the endowments might not be spanned by the assets' payoff vectors. Furthermore, he establishes that collinearity of the endowments suffices to translate this condition into a sufficient condition for uniqueness of the GEI equilibrium.

The aim of the paper is to generalize the previous literature in a number of directions. First, we shall show the existence of a competitive equilibrium in incomplete markets. It is well known that even in the simplest case of an economy with incomplete markets, where there is only one physical commodity[2], equilibrium will, in general, fail to exist. Giving sufficient conditions on asset payoffs, preferences, and endowments, this paper provides for uniqueness of equilibrium of the underlying economy when there is only one physical commodity.

Second, we shall not assume that the preferences are complete or transitive. For instance, most investors in financial markets are not single investors but rather corporate bodies. Therefore, most investment decisions are collective decisions. If markets are complete, then all group members would have the same preferences over investments. If markets are incomplete, then it is not possible to evaluate market values of all feasible investment decisions from available price system. As a result, even if the competitive conditions prevail, generically, investors will not be unanimous over the choice of corporate investment plans, see for instance Duffie and

[1]For further details see Mitjushin and Polterovich [12].

[2]It is fairly standard to make such an assumption in the finance literature, see for instance Lintner [9], Sharpe [17], Milne [11], Kelsey and Milne [8], and Bettzüge [1].

Shafer [5] and Haller [6]. Likewise, different investors will have different preferences over the corporate investment plans. In such cases, a corporate's investment decision will be the outcome of a collective decision process. Social choice theory implies that the outcome of such collective decision processes may incomplete or intransitive, if the processes are non-dictatorial.

Third, under weak conditions on the *strict preference relations*, the existence result will be extended to economies in which unrestricted short selling of assets is allowed and hence the portfolio space is not necessarily bounded below, see for instance Milne [10], Werner [18] , and Page and Wooders [14]. Thus, in our paper, existence is not standard since the asset consumption set A^h is potentially unbounded. Moreover, previous proofs allow either incompleteness or unboundedness, but not both. Our proof of existence allows both at the same time.

Finally, we present the first fundamental theorem of welfare economics in such a framework. We shall prove that if the portfolio space of an asset exchange economy is finite dimensional and the aggregate endowment is strictly positive, then the allocation is in the *Constrained Core* whenever the allocation is supported by the price system.

In the following section, we derive a numerical representation for a preference relation without assuming transitivity or completeness. In Section 3, we prove the existence of a competitive equilibrium for a class of asset exchange economies and establish constrained Pareto optimality. Finally, the concluding section discusses some of the implications of these results and contains some remarks about extensions of the analysis.

2. Preliminaries

The subject matter of this section is the representation of preferences which may be incomplete or intransitive. An individual has preferences among alternatives. These preferences are described by a *binary relation* $\succ$ which stands for *strict preference.*

Let there be a finite number of consumers, indexed by $h \in \mathcal{H} = \{1, ..., H\}$. The consumption set of consumer h is given by $X^h \subset \Re^L$, where L denotes a finite number of commodities. Given a strict preference relation $\succ$ defined on $X \times X$, let $P(x) = \left\{y \in X^h : y \succ x\right\}$ and $P^{-1}(x) = \left\{y \in X^h : x \succ y\right\}$ be the *strict upper contour set* and *strict lower contour set*, respectively.

Assumption 1. *(a) Continuity The strict upper and lower contour sets are open subsets of* X^h $\forall x \in X^h$*; (b) Irreflexivity* $x \notin P(x)$, $\forall x \in X^h$*; (c) Convexity* $x \notin con(P)(x)$, $\forall x \in X^h$*, where* $con(A)$ *stands for the convex hull of* A.

Definition 1. If $\succ$ is a preference relation defined on X, then the *graph* of $\succ$ is given by

$$\Gamma(\succ) = \{(x,y) \in X \times X : y \succ x\}.$$

Moreover $\succ$ has an *open graph* if $\Gamma(\succ)$ is an open subset of $X \times X$.

3. The Economy

In this section, we analyze the properties of competitive equilibrium in the context of a finite asset exchange economy under uncertainty, where trade in assets is competitive. Economic activity occurs over two time periods, $t = 0, 1$. Uncertainty is described by states of the world, indexed by $s \in \mathcal{S} = \{1, ..., S\}$, a finite, non-empty set, and is resolved all in the second period. There is only one physical commodity so that the first period commodity space is $\mathbb{R}$ and the second period contingent commodity space is $\mathbb{R}^S$ making the total commodity space $\mathbb{R}^{S+1}$. However, we shall consider in the sequel an exchange economy where second period actions by consumers are restricted to trades in assets that offer linear combinations of contingent commodities. Therefore, we shall treat the assets to be the objects of choice rather than examining the contingent commodities explicitly.

There are a finite number of consumers, indexed by $h \in \mathcal{H}$. Each consumer h has a consumption set $X^h \subset \mathbb{R}^{S+1}$. Each consumer h is described by a preference relation $\succ^h$defined over state contingent consumption set X^h.

Assumption 2. *(a) For every $h \in \mathcal{H}$, the feasible set X^h is non-empty, closed, convex, and bounded below; (b) For every $h \in \mathcal{H}$, the initial endowment is in the interior of the consumption set, that is, $e^h \in intX^h$; (c) Continuity For every $h \in \mathcal{H}$, the preference relation $\succ^h$ defined on X^h has open graph; (d) Nonsatiation For each $x^h \in X^h$, $P^h(x) \neq \emptyset$; (e) Convexity For each $h \in \mathcal{H}$, $x^h \notin conP^h(x)$.*[3]

3.1. Induced Preferences.

The basic preferences over consumption will generate derived preferences over asset holdings. We shall refer to the latter as *induced preferences.*

Let there be J assets indexed by $j \in \mathcal{J} = \{1, ..., J\}$. Define the commodity space in the asset economy to be the space $\mathbb{R}^{J+1}$, where there are J assets and the first period commodity. In order to achieve consumption, consumer h holds assets $a^h \in \mathbb{R}^J$, which yield returns $\sum_{j \in \mathcal{J}} Z_j a^h_j$,

[3]We have assumed, without loss of generality, that $P^h(x)$ is convex and $x^h \notin P^h(x)$. Suppose not, then we can replace $\succ^h$: $X^h \to X^h$ by $\widehat{\succ}^h : X^h \to X^h$, where $\widehat{P}^h(x) = conP^h(x)$. The binary relation in question will still have open graph and by Assumption 2.1, $x^h \notin \widehat{P}^h(x)$ (see Border [2]).

where $Z_j \in \mathbb{R}^S$ since

$$\begin{bmatrix} Z_{11} & ... & Z_{1J} \\ . & ... & . \\ Z_{S1} & . & Z_{SJ} \end{bmatrix} \begin{bmatrix} a_1^h \\ . \\ a_J^h \end{bmatrix} = \begin{bmatrix} \sum_{j\in \mathbf{J}} Z_{1j} a_j^h \\ . \\ \sum_{j\in \mathbf{J}} Z_{Sj} a_j^h \end{bmatrix}.$$

In order to derive consumer preferences over assets, we shall define a function $\Lambda : \mathbb{R}^{J+1} \to \mathbb{R}^{S+1}$ by $Z'\beta = \alpha$, where $\beta \in \mathbb{R}^{J+1}$ and $\alpha \in \mathbb{R}^{S+1}$ and Z' is the $(S+1) \times (J+1)$ semi-positive matrix, that is,

$$\begin{bmatrix} 1 & 0 \\ 0 & Z \end{bmatrix}.$$

The function Λ is linear and onto the range Q, which is a vector subspace of dimension $(J+1)$. Define $V^h = Q \cap X^h$, where $V^h \neq \emptyset$ since $\{0\} \subset Q \cap X^h$. Define consumer h's feasible portfolio space A^h by $\Lambda^{-1} : V^h \to \mathbb{R}^{J+1}$ such that $A^h \equiv \Lambda^{-1}\left(V^h\right)$. Hence, induced preferences $\succ_a^h$ over assets can be derived from commodity preferences by way of the linear mapping Λ^{-1}. In other words, assets are desired solely for their returns, therefore, preferences over assets are derived preferences. We shall now give some properties of induced preferences.

Lemma 1. *If X^h is non-empty, convex, closed and bounded below and associated preferences $\succ^h$ satisfy the properties (c)-(e) of Assumption 2 and the condition that for any $x^h \in V^h$, $\exists y^h \in V^h$ such that $y^h \succ^h x^h$, then: (a) A^h is non-empty, closed, convex, and bounded below*[4]*; (b) Continuity For every $a \in A^h$, the sets $P(a)$ and $P^{-1}(a)$ are open; (c) Convexity For every $a^h \in A^h$, $a^h \notin conP^h(a)$; (d) Nonsatiation For every $a^h \in A^h$, $P^h(a) \neq \emptyset$.*

Proof. The proof of Lemma 1 can be found in Milne [10] (Lemma 1) which was given for the weak preference relation. In our case, this involves a trivial modification for the strict preference relation. Therefore, we will omit the proof. ■

3.2. **Equilibria in Bounded Economies.** In this section, we wish to prove existence of an economy where arbitrary bounds are imposed on trades. Then, in Section 3.3, we prove existence in general by letting these bounds tend to infinity.

Let $\mathcal{E} = \left(A^h, \overline{a}^h, P_a^h\right)_{h\in\mathcal{H}}$ be an asset exchange economy in which each consumer h has a portfolio space A^h, an initial endowment of assets $\overline{a}^h \in \mathbb{R}^J$, and a preference relation $P_a^h \subset$

[4]The assumption that each A^h is closed and bounded below can be replaced by the assumption that each A^h is compact, which is standard, see Debreu [3].

$A^h \times A^h$. Consumer h can trade this portfolio to obtain a new portfolio of assets. Let

$$B^h(q) = \left\{a^h \in A^h : q \cdot a^h \leq q \cdot \overline{a}^h\right\}$$

be the consumer h's budget set for a given price system $q \in \mathbb{R} \setminus \{0\}$. Define

$$\Omega = \left\{a \in \prod_{h \in \mathcal{H}} A^h : \sum_{h \in \mathcal{H}} a^h = \sum_{h \in \mathcal{H}} \overline{a}^h\right\}.$$

Definition 2. An equilibrium for an asset economy $\mathcal{E} = \left(A^h, \overline{a}^h, P_a^h\right)_{h \in \mathcal{H}}$ is a collection (a^*, q^*) of asset holdings $a^* \in \mathcal{A}$ and prices $q^* \in \mathbb{R}^J \setminus \{0\}$ such that

a. For every $h \in \mathcal{H}$, $q^* a^{*h} = q^* \overline{a}^h$;

b. For each $h \in \mathcal{H}$, $P_a^h\left(a^{*h}\right) \cap B^h(q^*) = \emptyset$.

Theorem 1. *Suppose that $\mathcal{E} = \left(A^h, \overline{a}^h, P_a^h\right)_{h \in \mathcal{H}}$ satisfies the conditions of Lemma 3 for every $h \in \mathcal{H}$, then a competitive equilibrium exists.*

As A^h's are bounded below, let $\underline{b} \in \mathbb{R}^{J+1}$ be such that

$$\underline{b} < \sum_{h \in \mathcal{G}} A^h \quad \forall \mathcal{G} \subseteq \mathcal{H}.$$

Let also $\overline{b} \in \mathbb{R}^{J+1}$ be such that

$$\sum_{h \in \mathcal{H}} \overline{a}^h < \overline{b}.$$

Define

$$\widehat{A}^h = \left\{a^h \in A^h : a^h \leq \overline{b} - \underline{b}\right\}$$

$\forall h \in \mathcal{H}$. Let $\widehat{A} = \prod_{h \in \mathcal{H}} \widehat{A}^h$.

Lemma 2. *If $a = \left(a^1, ..., a^H\right) \in \widehat{A}$ and $\sum_{h \in \mathcal{H}} a^h = \sum_{h \in \mathcal{H}} \overline{a}^h$, then $a^h \ll \overline{b} - \underline{b} \;\; \forall h \in \mathcal{H}$.*

Proof. Suppose $a_i^{\widehat{h}} = \overline{b}_i - \underline{b}_i$. Then

$$\sum_{h \in \mathcal{H}} \overline{a}_i^h = a_i^{\widehat{h}} + \sum_{h \neq \widehat{h}} a_i^h = \overline{b}_i - \underline{b}_i + \sum_{h \neq \widehat{h}} a_i^h > \overline{b}_i,$$

a contradiction. ∎

Corollary 1. *Let $a = \left(a^1, ..., a^H\right) \in \widehat{A}$ and $\sum_{h \in \mathcal{H}} a^h = \sum_{h \in \mathcal{H}} \overline{a}^h$ then (a) $a \in int\widehat{A}$; (b) $a^h \in int\widehat{A}^h \; \forall h \in \mathcal{H}$.*

Here, we shall only sketch the proof since it is standard. For each $h \in \mathcal{H}$ and each $a^h \in \widehat{A}^h$, let

$$\widehat{P}_a^h\left(a^h\right) = \left\{\widehat{a}^h = \lambda a^h + (1-\lambda)\,\widetilde{a}^h \quad \text{for } 0 \le \lambda < 1 \quad \text{and } \widetilde{a}^h \in P_a^h\left(a^h\right)\right\}.$$

The economy $\widehat{\mathcal{E}} = \left(\widehat{A}^h, \overline{a}^h, \widehat{P}_a^h\right)_{h\in\mathcal{H}}$ so constructed satisfies the following conditions:

i. $\widehat{A}^h$ is non-empty, convex, and compact;

ii. $\overline{a}^h \in int\widehat{A}^h$;

iii. $\widehat{P}_a^h$ is open in $\widehat{A}^h \times \widehat{A}^h$;

iv. For each $a^h \in \widehat{A}^h$, $a^h \notin con\widehat{P}_a^h\left(a^h\right)$;

v. For each $a = \left(a^1, ..., a^H\right) \in \widehat{A}$ such that $\sum_{h\in\mathcal{H}} a^h = \sum_{h\in\mathcal{H}} \overline{a}^h$, $a^h \in bd\widehat{P}_a^h\left(a^h\right)$, where "*bd*" stands for the "*boundary*".

Therefore, by Shafer [16], $\widehat{\mathcal{E}}$ has an equilibrium (a^*, q^*), that is,

a. $a^{*h} \in \widehat{B}^h(q^*) = \left\{a^h \in \widehat{A}^h : q^* \cdot a^{*h} \le q^* \cdot \overline{a}^h\right\}$;

b. $\widehat{P}_a^h\left(a^{*h}\right) \cap \widehat{B}^h(q^*) = \emptyset$;

c. $\sum_{h\in\mathcal{H}} a^h = \sum_{h\in\mathcal{H}} \overline{a}^h$.

PROPOSITION 1. *(a^*, q^*) is also an equilibrium for $\mathcal{E} = \left(A^h, \overline{a}^h, P_a^h\right)_{h\in\mathcal{H}}$.*

Proof. Clearly $a^{*h} \in \widehat{B}^h(q^*) \subseteq B^h(q^*)$ and $\sum_{h\in\mathcal{H}} a^h = \sum_{h\in\mathcal{H}} \overline{a}^h$. Thus, it is sufficient to show that $P_a^h\left(a^{*h}\right) \cap B^h(q^*) = \emptyset\ \forall h \in \mathcal{H}$.

Suppose $P_a^i\left(a^{*i}\right) \cap B^i(q^*) \ne \emptyset$ for some i. Let $\widetilde{a}^i \in P_a^i\left(a^{*i}\right) \cap B^i(q^*)$. Then for each $0 < \lambda < 1$, define $a_\lambda^i = \lambda \widetilde{a}^i + (1-\lambda)\,a^{*i}$. Since $q^* \cdot a^{*i} \le q^* \cdot \overline{a}^i$ and $q^* \cdot \widetilde{a}^{*i} \le q^* \cdot \overline{a}^i$, we have $q^* \cdot a_\lambda^{*i} \le q^* \cdot \overline{a}^i$. By Corollary 1 and $\sum_{h\in\mathcal{H}} a^h = \sum_{h\in\mathcal{H}} \overline{a}^h$, we also know that $a^{*i} \in int\widehat{A}^h$. Therefore, there exists sufficiently small $\widehat{\lambda}$ such that $a_{\widehat{\lambda}}^i \in \widehat{A}^h$, which implies that $a_{\widehat{\lambda}}^i \in \widehat{B}^i(q^*)$ and $a_{\widehat{\lambda}}^i \in \widehat{P}_a^i\left(a^{*i}\right)$, but this contradicts $\widehat{P}_a^i\left(a^{*i}\right) \cap \widehat{B}^i(q^*) = \emptyset$. This completes the proof. ■

3.3. **Equilibria in Unbounded Economies.** The fact that we treat assets as claims to contingent consumption in the second period has an important effect on the problem of the existence of competitive equilibria. In this section, we shall allow for the possibility that consumers can go arbitrarily short in asset trading. Since consumers are allowed to sell short assets, we will work with portfolio space without a prior lower bound. Thus, we shall provide a basic result that shows the existence of equilibrium allocations in an economy with unbounded asset trade sets.

We shall take $X^h \subset \mathbb{R}^{S+1}$ to be the consumption possibility set of each consumer defined by

$$X^h = \left\{ x^h \in \mathbb{R}_+^{S+1} : x^h \geq 0 \right\}.$$

One unit of the jth asset is a promise of a return $Z_j \in \mathbb{R}_+^S$, contingent upon the realization of a state of the world. Let $(Z_j)_{j \in \mathcal{J}}$ be the asset structure in the economy. As before, a portfolio of assets is defined as a vector $a^h \in \mathbb{R}^J$, where a_j^h defines the number of the jth asset held by consumer h . We shall assume that a_j^h may be positive or negative. A consumer holding a portfolio $a^h \in \mathbb{R}^J$ will have a control over the net commodity bundle given by $\sum_{h \in \mathcal{H}} Z_j a_j^h$. For each h, define $sp\,(Z_j)_{j \in \mathcal{J}}$ to be the span of $(Z_j)_{j \in \mathcal{J}}$.

In the presence of asset markets with an incomplete structure, the consumption set of each consumer h can be specified as follows:

$$X_A^h = X^h \cap \left\{ x^h \in \mathbb{R}_+^{S+1} : x^h \in sp\,(Z_j)_{j \in \mathcal{J}} \right\},$$

that is, the allocations attainable by way of the exchange of assets. Asset markets so constructed may be incomplete in the sense that the available assets do not span X^h. Define the asset set of each consumer as follows:

$$A^h = \left\{ a^h \in \mathbb{R}^{J+1} : \sum_{j \in \mathcal{J}} Z_j a_j^h \in X_A^h \right\}.$$

Notice that A^h is assumed to have no lower bound. Let $A = \prod_{h \in \mathcal{H}} A^h$.

Assumption 3. *For each $h \in \mathcal{H}$, $\overline{a}^h$ is in the interior of A^h, that is $\overline{a}^h \in intA^h$.*

Definition 3. Given a subset $X \subset \mathbb{R}^J$, we say that $y \in \mathbb{R}^J$ is a *direction of recession* for X if $x + \lambda y \in X$ for all $\lambda \geq 0$ and $x \in X$. We shall denote by O^+X the set of all recession directions of X. If X is a closed convex set, then O^+X is a closed convex cone containing the origin. Equivalently, $O^+X = \left\{ y \in \mathbb{R}^J : X + y \subset X \right\}$. Therefore, the *recession cone* O^+A^h corresponding to the asset set A^h is a closed convex cone containing the origin.

Since each unit of asset $j \in \mathcal{J}$, is a contract that promises to pay a fixed non-negative vector $Z_j \in \mathbb{R}^S$, defining the matrix $Z = [Z_1, ..., Z_J]$ and assuming that consumer h has no other source of wealth in the second period, one can obtain the following result.

LEMMA 3. *Assume that* $rank\,(Z) = J+1$ *and* $X^h = \mathbb{R}_+^{S+1}$. *Then the derived asset set satisfies the following condition:*

$$O^+ A \cap O^+(-A) = \{0\}.$$

Proof. Note that $X^h = \mathbb{R}_+^{S+1}$ implies $X = \mathbb{R}_+^{S+1}$. Given the fact that Z is a semi-positive matrix, by definition, we have

$$A = \left\{a \in \mathbb{R}^{J+1} : Za \geq 0\right\}$$

and hence

$$-A = \left\{a \in \mathbb{R}^{J+1} : Za \leq 0\right\}.$$

If $a' \in A \cap (-A)$, then $Za' = 0$. However, $rank\,(Z) = J+1$ implies $a' = 0$. This complete the proof. ■

Let $\mathcal{E} = \left(A^h, \succ^h, \overline{a}^h\right)_{h\in\mathcal{H}}$ denote the *unbounded asset exchange economy* with each consumer h having an asset set $A^h \subset \mathbb{R}^J$ and an endowment of assets $\overline{a}^h \in A^h$. Consumer h's preferences over A^h are specified by a strict preference relation P^h. For each $a \in A^h$, consumer h's *preferred set* is given by

$$P^h(a^h, q) = \left\{a' \in A^h : a' \succ^h a^h\right\}.$$

Let $\mathcal{B} = \left\{q \in \mathbb{R}^J : \|q\| \leq 1\right\}$ be the set of relative prices. Throughout we shall assume that $P^h(a^h)$ exhibits the following properties:

ASSUMPTION 4. *For each* $h \in \mathcal{H}$, *the set* $P^h(a^h)$ *is non-empty and convex,* $a^h \notin P^h\left(a^h\right)$ *for all* $a^h \in A^h$, *and* $a^h \in clP^h\left(a^h\right)$ *for all* $a^h \in A^h$, *where "cl" stands for the "closure".*[5]

ASSUMPTION 5. *The graph of* $P^h(a^h)$ *is open in* $A^h \times A^h$.

Let

$$\Omega = \left\{a \in \prod_{h\in\mathcal{H}} A^h : \sum_{h\in\mathcal{H}} a^h = \sum_{h\in\mathcal{H}} \overline{a}^h \right\}$$

be the attainable state of the economy.

[5]Assumption 4 says that $a^{h'} \succ^h a^h$ implies $\lambda a^{h'} + (1-\lambda)\,a^h \succ^h a^h$ for all $\lambda \in (0,1]$ and $a^h \in bdP^h\left(a^h\right)$, where "*bd*" stands for the "*boundary*".

Definition 4. An equilibrium for an economy $\mathcal{E} = \left(A^h, \succ^h, \overline{a}^h\right)_{h\in\mathcal{H}}$ is an $(H+1)$-tuple of vectors $< a^{*1}, ..., a^{*H}, q^* >$ such that (a) $a^* \in \Omega$; (b) $q^* \in \mathcal{B}\backslash\{0\}$; (c) $\sum_{h\in\mathcal{H}} a^{*h} = \sum_{h\in\mathcal{H}} \overline{a}^h$; (d) $P^h\left(a^{*h}\right) \cap B^h\left(q^*\right) = \emptyset$.

Definition 5 (Debreu [3]). H cones $X^1, ..., X^H$ (with *vertex* 0) are said to be *positively semi independent* if $x^h \in X^h$ $\forall h \in \mathcal{H}$, and $\sum_{h=1}^{H} x^h = 0$ implies $x^h = 0$ $\forall h$. Obviously, two cones X^h, X^i with vertex 0 are positively semi independent if and only if $X^h \cap X^i = \{0\}$ for $i, h \in \mathcal{H}$.

Proposition 2. *Given an economy $\mathcal{E}$, the set Ω of attainable states is bounded if and only if its recession cone $O^+\Omega$ consists of the zero vector alone.*

Proof. Clearly, the set of attainable states Ω of the asset exchange economy is *closed* and *convex.* The set A^h may be unbounded. To show that Ω is *bounded*, it is sufficient to prove that the recession cone $O^+\Omega = \{0\}$.

By Definition 3, one can define the recession cone of a closed convex subset X of $\mathbb{R}^J$ by

$$O^+X = \left\{y \in \mathbb{R}^J : X + y \subset X\right\}.$$

We shall first show that O^+X implies

$$O^+\Omega = \left\{a \in \mathbb{R}^{H(J+1)} : a^h \in \mathbb{R}^{J+1} \quad \forall h, \quad \sum_{h\in\mathcal{H}} a^h = 0\right\}.$$

Define $O^+\Omega = \left\{a \in \mathbb{R}^{H(J+1)} : \Omega + a \subset \Omega\right\}$. Let $a \in O^+\Omega$ and $b \in \Omega$, where $b^h \in A^h$. Since $O^+\Omega \subset \Omega$, then $a + b \in \Omega$. Hence, summing over h, one has $\sum_{h\in\mathcal{H}}\left(a^h + b^h\right) = \sum_{h\in\mathcal{H}} \overline{a}^h$, where $\sum_{h\in\mathcal{H}} b^h \in \Omega$. Now define $\sum_{h\in\mathcal{H}} b^h = \sum_{h\in\mathcal{H}} \overline{a}^h$. This implies $\sum_{h\in\mathcal{H}}\left(a^h + \overline{a}^h\right) = \sum_{h\in\mathcal{H}} \overline{a}^h$. Hence, one has $\sum_{h\in\mathcal{H}} a^h = 0$ as desired.

Next we shall show that $O^+A^1, ..., O^+A^H$ are positively semi-independent. By Lemma 6, one has $X^h = \mathbb{R}_+^{S+1}$ which implies $X = \mathbb{R}_+^{S+1}$, Z is the same for all $h \in \mathcal{H}$, and $O^+A \cap O^+(-A) = \{0\}$. Define

$$A^h = \left\{a^h \in \mathbb{R}^{J+1} : Za^h \geq 0\right\}$$

and

$$-A^h = \left\{a^h \in \mathbb{R}^{J+1} : Za^h \leq 0\right\},$$

thus $A^h \cap \left(-A^h\right) = \left\{a^h \in \mathbb{R}^{J+1} : Za^h = 0\right\}$. But because $rank\,(Z) = J + 1$, one has $A^h \cap \left(-A^h\right) = \{0\}$. Since $O^+A^h \subset A^h$ and $-O^+A^h \subset -A^h$, then $O^+A^h \cap \left(-O^+A^h\right) \subset A^h \cap \left(-A^h\right)$. Also $0 \in O^+A^h \cap \left(-O^+A^h\right) \subset A^h \cap \left(-A^h\right) = \{0\}$ implies $O^+A^h \cap \left(-O^+A^h\right) = \{0\}$.

Since by symmetry $A^h = A^{h'}$ $\forall h, h' \in \mathcal{H}$, one has $O^+A^h \cap \left(-O^+A^{h'}\right) = \{0\}$. This implies $O^+A^1, ..., O^+A^H$ are positively semi-independent.

Finally, let $a \in O^+\Omega$ which implies $\sum_{h\in\mathcal{H}} a^h = 0$. Since $\Omega \subset \prod_{h\in\mathcal{H}} A^h$, then $a \in O^+\Omega \subset O^+ \prod_{h\in\mathcal{H}} A^h \subset \prod O^+A^h$. This implies $a^h \in O^+A^h$ $\forall h \in \mathcal{H}$ and $\sum_{h\in\mathcal{H}} a^h = 0$, which in turn implies $a^h = 0$ because $O^+A^1, ..., O^+A^H$ are positively semi-independent. Therefore $O^+\Omega = \{0\}$ and Ω is bounded. ■

Consider now a *compact economy* $\widehat{\mathcal{E}}_n = \left(\widehat{A}^h_n, \overline{a}^h, P^h\right)_{h\in\mathcal{H}}$ such that

(1) For all h, $\widehat{A}^h_n \subset \widehat{A}^h_{n+1}$, for all n;
(2) $\lim_{n\to\infty} \widehat{A}^h_n = A^h$ for all h;
(3) $\overline{a}^h \in intA^h$ for all h;
(4) $\Omega \subset \prod_{h\in\mathcal{H}} \widehat{A}^h_1$.

This implies that, for all n, there exists (a^*_n, q^*_n) which is an equilibrium of $\widehat{\mathcal{E}}_n$ by Shafer and Sonnenschein [15]. This in turn implies $(a^*_n, q^*_n) \in \Omega \times \mathcal{B}$ for all n. Hence, equilibrium sequence $\{(a^*_n, q^*_n)\} \subset \Omega \times \mathcal{B}$. But since $\Omega \times \mathcal{B}$ is compact, $\{(a^*_n, q^*_n)\}$ has a converging subsequence. Let (a^*, q^*) be the limit of this subsequence, that is, $\lim_{n\to\infty} (a^*_n, q^*_n) = (a^*, q^*)$.

PROPOSITION 3. *(a^*, q^*) is an equilibrium for $\mathcal{E}$.*

Proof. First we will show that, for all h, $q^* \cdot a^{*h} = q^* \cdot \overline{a}^h$. Suppose $q^* \cdot a^{*h} \neq q^* \cdot \overline{a}^h$. Since $(a^*, q^*) = \lim_{n\to\infty} (a^*_n, q^*_n)$ for n sufficiently large one must have $q^*_n \cdot a^{*h}_n \neq q^*_n \cdot \overline{a}^h$, a contradiction.

Next we will show that $P^h\left(a^{*h}\right) \cap B^h(q^*) = \emptyset$. Since $(a^*_n, q^*_n) \to (a^*, q^*)$, then $a^* \in \Omega$ for $\{a^*_n\}_{n=1}^{\infty} \subset \Omega$. Thus, $\sum_{h\in\mathcal{H}} a^{*h} = \sum_{h\in\mathcal{H}} \overline{a}^h$ which implies $q^* \cdot a^{*h} = q^* \cdot \overline{a}^h$. Let $a^h \in A^h$ such that $q^* \cdot a^h < q^* \cdot \overline{a}^h$. This implies, for sufficiently large n, $q^*_n \cdot a^h \leq q^* \cdot \overline{a}^h$, $a^h \in A^h_n$, and hence $a^h \in B^h(q^*_n)$. Consequently, one has $a^h \notin P^h\left(a^{*h}_n\right)$. Therefore, since $a^h \in A^h$ and $q^* \cdot a^h < q^* \cdot \overline{a}^h$, then $a^h \notin P^h\left(a^{*h}\right)$. Now take any point $b^h \in B^h(q^*)$. This implies that b^h can be approximated by a sequence $\left\{b^h_n\right\}_{n=1}^{\infty} \subset A^h$ such that $q^* \cdot b^h_n < q^* \cdot \overline{a}^h$ for all n and $\lim_{n\to\infty} b^h_n = b^h$. For sufficiently large n, $b^h_n \in A^h$ and $q^* \cdot b^h_n < q^* \cdot \overline{a}^h$ imply $b^h_n \notin P^h\left(a^{*h}\right)$ which in turn implies $b^h \notin P^h\left(a^{*h}\right)$. Therefore, one has $P^h\left(a^{*h}\right) \cap B^h(q^*) = \emptyset$. This establishes the existence of equilibrium (a^*_n, q^*_n) for $\mathcal{E}$ as desired. ■

3.4. **Optimality of Competitive Allocations.** In the following, we shall give a definition of Pareto optimal allocations which is a special case of what is referred to as *Constrained Core*, since it only excludes Pareto improvement brought by exchanging the existing assets. There is no reason to expect an equilibrium allocation to be Pareto optimal. In fact, it is Pareto optimal, in general, only if the market structure is essentially complete.

Definition 6. A Constrained Core with respect to the preference relation $\succ$ is an allocation $\left\langle a^{*1}, ..., a^{*H} \right\rangle$ of one portfolio for each individual such that there does not exist an allocation $\left\langle a^{1}, ..., a^{H} \right\rangle$ and a non-empty subset $\mathcal{I} \subset \mathcal{H}$ for which $a^h \in P^h\left(a^{*h}\right)$ $\forall h \in \mathcal{I}$ such that $\sum_{h\in\mathcal{I}} a^h = \sum_{h\in\mathcal{I}} \overline{a}^h$.[6]

Theorem 2. *In an asset exchange economy* $\left(A^h, \succ^h, \overline{a}^h\right)_{h\in\mathcal{H}}$ *every competitive equilibrium* $< a^*, q^* >$ *is in the Constrained Core.*

Proof. Let $< a^*, q^* >$ be the equilibrium allocation and price system. Suppose that there exists a non-empty subset $\mathcal{I} \subset \mathcal{H}$ and $a^h \in A^h$ for $h \in \mathcal{I}$ such that $\sum_{h\in\mathcal{I}} a^{*h} = \sum_{h\in\mathcal{I}} \overline{a}^h$ and $a^h \in P^h\left(a^{*h}\right)$. Since $P^h\left(a^*\right) \cap B^h(q^*) = \emptyset$, we have $q^*a^h > q^*\overline{a}^h$ for all $\forall h \in \mathcal{I}$. Summing over $h \in \mathcal{I}$, one can get $q^* \sum_{h\in\mathcal{I}} a^h > q^* \sum_{h\in\mathcal{I}} \overline{a}^h$. But this contradicts the fact that

$$q^* \sum_{h\in\mathcal{I}} \overline{a}^h = q^* \sum_{h\in\mathcal{I}} a^h = q^* \sum_{h\in\mathcal{I}} a^{*h}$$

obtained from summing over the budget constraint. ∎

4. Conclusion

In this paper, we have given simple and direct equilibrium existence results for an asset exchange economy when unlimited short selling was allowed. Throughout it has been assumed that consumer preferences were given by an irreflexive binary relation with open graph, that preferences were possibly incomplete or intransitive, and that the portfolio space was non-compact and finite dimensional. Our study therefore generalize various results in the existing literature of economic theory.

Some comments are in order. First of all, in the proof of existence for the unbounded economy, it was assumed that there is an independent set of asset returns. This assumption ensures the result of Lemma 3, and rule out the possibility of a consumer taking an unbounded position in dependent assets. That is, with dependent assets, it is reasonable for the consumer to issue a set of dependent assets that give the same returns as another asset held long, without violating contractual feasibility, see Milne [10]. In general, a dependent asset equilibrium can easily

[6]Note that an element of the constrained core is constrained Pareto optimal.

be derived from an independent asset equilibrium by taking appropriate linear combination of quantities and prices of independent assets, see Milne [11] and Yalçin and Kelsey [19].

Second, since the asset market is possibly incomplete and has a competitive equilibrium, it follows that the asset economy achieves a Pareto Optimal allocations of resources which coincides with the notion of a Constrained Optimum due to Diamond [4].

Finally, the obvious limitation of the model is that the analysis has been restricted to a one-physical commodity case. Inclusion of many commodities would introduce the possibility of commodity price uncertainty in the second period. Despite this restriction, we believe that the model provides some useful implications for the pure theory of financial markets.

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