The one object optimal auction and the desirability of exclusion

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Abstract

In this paper we investigate the exclusion of types in optimal auctions. We show that with multidimensional types exclusion is a general phenomenon.

1 Introduction

The mechanism design models of the monopoly pricing and optimal auction literature has relied mainly on one dimensional private information or uncertainty. What are the consequences for optimality if private information or uncertainty is multidimensional? In their pioneer paper, McAfee and McMillan (1988) analyse in a setting restricted to differentiable mechanisms the consequences of multidimensionality for incentive compatibility contraints. They introduce a generalized single crossing condition and apply their result to monopoly with unknown demands. One sees in their paper that multidimensional mechanism design is a difficult topic. In what concerns the present paper the next important step is Armstrong (1996). In his paper, Armstrong, study the multiproduct nonlinear pricing problem and obtain a general solution under a separability hypothesis. Another result in his paper is the desirability of exclusion. He shows that if the set of types is strictly convex with dimension greater than one then at the optimum a set of types with positive measure will not be served by the monopolist.

This result is general and holds for multiproduct and single product monopolist as well.

In this paper we investigate what changes have to be made in the one object optimal auction theory when the set of types is multidimensional. We will show that in contrast to the multiproduct nonlinear pricing theory the multidimensional one object optimal auction theory is not complicate. And we show that the desirability of exclusion is a very general property: the set of types doesn't have to be strictly convex. We finish this introduction with a few comments on multidimensionality and its mathematical aspects. In optimal screening models and in auctions models, the most usual assumption about the consumers types is that it is an infinite one dimensional set, say $T = [0, 1]$. This should–in principle– be enough. Multi-dimensional sets of types are measure theoretically equivalent (i.e. are isomorphic) to $[0, 1]$ with the Lebesgue's measure as long as the original space is atomless and separable. However in practice and for the study of deeper properties of the optimal solution it is more convenient to consider a multi-dimensional type set and maybe add some assumption on the economics characteristics. For example compare Myerson's (1981) solution of the regular case and the non-regular case. In brief, the isomorphism between atomless measure spaces may not be useful for two basic reasons:

1. Special assumptions are made on economic characteristics that depend on types. For example usually, utility is monotonic on types. Those assumptions are not preserved by isomorphisms;

2. To get intuition usually a more detailed information about the optimal solution is needed.

2 The model

There is one object to be sold at an auction with n bidders. Each bidder evaluate the object accordingly to several characteristics $t_i \in T_i$. If the object has characteristic t_i we say that the Bidder has type t_i. The set of possible types for Bidder i, namely T_i, is compact and connected set. The bidders have private independent types.

Bidder i utility is $U^i : T_i \to \mathbb{R}$. We suppose that U^i is continuous. Therefore $R^i = U^i(T_i) = [\alpha_i, \beta_i]$ is a closed interval. Our main assumption is that the distribution $F_i(u) = \Pr\left(\left\{t \in T_i; U^i(t) \leq u\right\}\right)$ has a density $f_i(u)$ continuous on $[\alpha_i, \beta_i]$ and positive on (α_i, β_i). To simplify[1] we consider the regular case:

$$u - \frac{1 - F_i(u)}{f_i(u)} \text{ is increasing in } u \in (\alpha_i, \beta_i).$$

The following theorem is proved in the appendix.

[1]The general case can be treated analogously as in Myerson's paper. This will become clear later.

Theorem 1 *The optimal auction is the one object optimal auction obtained in Myerson (1981) when Bidder i has valuation* $u^i \in [\alpha_i, \beta_i]$ *distributed with density* $f_i(u)$. *In particular the optimal auction deliver the object to the Bidder with the highest marginal value:*

$$U^i(t_i) - \frac{1 - F_i(U^i(t_i))}{f_i(U^i(t_i))}, i = 1, \ldots n.$$

Our main result is the following.

Theorem 2 (Exclusion of types) *Suppose the set of types is* $T = \Pi_{i=1}^n [a_i, b_i]$ *and that the distribution of types has a positive continuous density* $f : T \to \mathbb{R}_{++}$. *Suppose also that* $U^i : T \to \mathbb{R}$ *is continuous differentiable and that* $\frac{\partial U^i}{\partial t_i} > 0, i = 1, \ldots, n$. *Finally suppose that* U^i *has a density* f_i *which is bounded and positive in the interior of* $U^i(T)$. *Then in the optimal auction there is exclusion of types.*

Proof. It suffices to show that for some bidder i, $\alpha_i - \frac{1}{f_i(\alpha_i)} < 0$. Then every type near α_i will be excluded. We will show that $f_i(\alpha_i) = 0$ and this will finish the proof. Define $a = (a_1, \ldots, a_n)$ and $u_a = U^i(a)$. Let M be a bound for f_i. If $u > u_a$,

$$F_{U^i}(u) = \int f(t)\, \chi_{\{v \in A; U^i(v) \le u\}}(t)\, dt \le M\lambda\left(\left\{v \in T; U^i(v) \le u\right\}\right).$$

Now define $\delta = \min\left\{\frac{\partial U^i}{\partial t_i}(t); i = 1, \ldots, n, t \in A\right\} > 0$. For any $t \in T$,

$$U^i(t) - U^i(a) = \int_0^1 \frac{d}{ds}\left(U^i(a + s(t-a))\right) ds = \int_0^1 \left(U^i\right)'(a + s(t-a)) \cdot (t-a)\, ds \ge \delta \sum_{i=1}^n (t_i - a_i) \ge \delta \max_i (t_i - a_i).$$

Thus if $U(t) \le u$ then $\max_i (t_i - a_i) \le \frac{u - u_a}{\delta}$. Therefore

$$\lambda\left(\left\{v \in T; U^i(v) \le u\right\}\right) \le \lambda\left(\left\{z \in \mathbb{R}^n; |z_i - a_i| \le \frac{u - u_a}{\delta} \text{ for all } i\right\}\right) = \left(\frac{u - u_a}{\delta}\right)^n.$$

Since $f_i(\alpha_i) = \lim_{u \to u_a} \frac{F_{U^i}(u) - F_{U^i}(u_a)}{u - u_a}$ and $n > 1$ we conclude that $f_i(\alpha_i) = 0$ ending the proof. ■

Example 1 *As an example we calculate the optimal auction in a simple cases with the uniform distribution. Consider* $U^i(x_i) = x_{i1} + x_{i2}$. *We suppose that* x_i *is uniformly distributed in* $[0, 1]^2$. *The distribution of* U^i *is*

$$F_U(u) = \begin{cases} \frac{u^2}{2} & \text{if} \quad u \le 1 \\ 1 - \frac{(2-u)^2}{2} & \text{if} \quad 1 \le u \le 2 \end{cases} \quad \text{has density } f_U(u) = \begin{cases} u & \text{if} \quad u \le 1 \\ 2 - u & \text{if} \quad 1 \le u \le 2. \end{cases}$$

Note that the density function is continuous and it is zero for $u = 0, 2$. *The optimal auction will deliver the object to the bidder i with highest*

$$\frac{3u^i}{2} - \max\left\{\frac{1}{u^i}, 1\right\}, u^i = U^i(x_i).$$

Appendix

In this appendix we prove theorem 1. For convenience the auctioneer will be bidder 0. We define $\mathcal{P}$ as the set of probability distributions on $\{0, 1, \ldots, n\}$. Thus $\mathcal{P} = \left\{ (q_j)_{j=0}^n \geq 0, \sum_{j=0}^n q_j = 1 \right\}$. The interpretation is that q_j is the probability that bidder j receives the object. Thus q_0 is the probability that the auctioneer keeps the object. We consider on $T = \Pi_{i=1}^n T_i$ the product probability measure.

The auction proceed as follows:

1. The auctioneer announces the measurable functions[2] $q : T \to \mathcal{P}$ and $P = \left(P^i\right)_{i=1}^n$ where $P^i : T \to \mathbb{R}$;

2. The bidder i confidentially announces to the auctioneer types $t_i \in T_i$; The auctioneer forms the vector $t = (t_i)_{i=1}^n$

3. The objects are delivered accordingly to $q(t) \in \mathcal{P}$ and bidder i pays $P^i(t)$.

For a given direct mechanism (q, P) we define

$$Q_i(t_i) = E_{-i}[q_i(t)] \text{ and } P^i(t_i) = E_{-i}\left[P^i(t)\right]. \tag{1}$$

The mechanisms must satisfy incentive compatibility and individual rationality constraints.

$$Q_i(t_i) U^i(t_i) - P^i(t_i) \geq Q_i(t_i') U^i(t_i) - P^i\left(t_i^{'}\right), \forall t_i', t_i \in T_i, \tag{IC}$$

$$Q_i(t_i) U^i(t_i) - P^i(t_i) \geq 0. \tag{IR}$$

Let us define the auxiliar function, $V^i : T_i \to \mathbb{R}$,

$$V^i(t_i) = Q_i(t_i) U^i(t_i) - P^i(t_i).$$

The compatibility of incentives constraints can be rewritten as

$$V^i(t_i) - V^i(t_i') \geq Q_i(t_i')\left(U^i(t_i) - U^i(t_i')\right), \forall t_i, t_i' \in T_i. \tag{IC'}$$

The following lemma is important:

Lemma 1 *There exists a convex function $\phi_i : R^i \to \mathbb{R}$ such that $V^i = \phi_i \circ U^i$. Moreover $Q_i(t_i) \in \partial\phi_i\left(U^i(t_i)\right)$.*

Proof: Note first that if a and c are elements of $\left(U^i\right)^{-1}(r), r \in R^i$ then (IC') implies $V^i(a) - V^i(c) \geq Q_i(c) \cdot 0 = 0$. Analogously $V^i(c) - V^i(a) \geq 0$. Thus $V^i(a) = V^i(c)$. Therefore

$$\phi_i : R^i \to \mathbb{R}, \phi_i(r) := V^i(x_r), \text{where } x_r \in \left(U^i\right)^{-1}(r), r \in R^i$$

[2] Henceforth called direct mechanisms.

is well defined. Since

$$\phi_i(\lambda) = V^i(x_\lambda) = \sup_{x' \in T_i} Q_i(x')\lambda - P^i(x')$$

it follows that ϕ_i is a convex function. And if $\mu = U^i(t_i)$

$$\phi_i(\lambda) - \phi_i(\mu) \geq Q_i(t_i)(\lambda - \mu), \lambda, \mu \in R^i$$

we have that $Q_i(t_i) \in \partial\phi_i\left(U^i(t_i)\right)$. QED

For every $u = \left(u^1, \ldots, u^n\right) \in R = \prod_{i=1}^n [a_i, b_i]$ define

$$\vec{q}_i(u) = E\left[q_i(x)|\, u = \left(U^j(x_j)\right)_{j=1}^n\right] \text{ and}$$
$$\vec{P}^i(u) = E\left[P^i(x)\Big|\, u = \left(U^j(x_j)\right)_{j=1}^n\right].$$

Note first that $\sum_{i=0}^n \vec{q}_i(u) = 1$. Let us calculate $\vec{Q}_i\left(u^i\right) = E\left[\vec{q}_i(u)|\, u^i\right]$. We calculate first

$$E\left[\vec{q}_i(u)|\, u^i\right] = E\left[E\left[q_i(x)|\, u\right]|\, u^i\right] = E\left[q_i(x)|\, u^i\right] =$$
$$E\left[E\left[q_i(x)|\, x_i\right]|\, u^i = U^i(x_i)\right] = E\left[Q_i(x_i)|\, u^i = U^i(x_i)\right] \in \partial\phi_i\left(u^i\right).$$

The last relation is true since $Q_i(x_i) \in \partial\phi_i\left(u^i\right) = \left[\phi^-\left(u^i\right), \phi^+\left(u^i\right)\right]$ and that

$$\phi^-\left(u^i\right) = E\left[\phi^-\left(u^i\right)|u^i\right] \leq E\left[Q_i(x_i)|u^i = U^i(x_i)\right] \leq E\left[\phi^+\left(u^i\right)|u^i\right] = \phi^+\left(u^i\right).$$

The incentive compatibility constraints can be rewritten as

$$\vec{Q}_i\left(u^i\right)u^i - E\left[\vec{P}^i(u)\Big| u^i\right] \geq \vec{Q}_i(u')u^i - E\left[\vec{P}^i(u)\Big| u'\right], \tag{IC'}$$

The individual rationality constraint can be rewritten

$$\vec{Q}_i\left(u^i\right)u^i - E\left[\vec{P}_i(u)|u^i\right] \geq 0.$$

The auctioneer expected revenue is

$$E\left[\sum_{i=1}^n P^i(x)\right] = E\left[E\left[\sum_{i=1}^n P^i(x)\Bigg| u\right]\right] = E\left[\sum_{i=1}^n \vec{P}^i(u)\right].$$

Thus the optimal auction problem is reduced to the one-dimensional optimal auction solved by Myerson. There is one twist however. Myerson's assumptions imply that the density function has a positive minimum. In our case simple examples will not satisfy this assumption. One can however easily see that Myerson proof work as well if the density is positive and continuous in the interior of R^i.

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