# Mixed Equilibria in Simultaneous-Offers Bargaining*

by

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**Abstract**

This paper characterizes the inefficient mixed Nash equilibria of the complete information k-double auction. The purpose of the analysis is to find restrictions on the inefficiencies that may appear in equilibrium in a model that does not include sequential moves or incomplete contracts, as a result of coordination failure. Then, we explore extensions of the model. Can a model with complete information be rejected when trade inefficiencies are due to the presence of asymmetric information? Next, we extend the analysis to the case of repeated trading sessions, risk-aversion and disappointment-aversion.

(JEL: C7)

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## 1. INTRODUCTION

Bargaining mechanisms may not always yield efficient outcomes, even in the case of complete information. Many extensions of the alternating offers mechanism (Rubinstein 1982) document that its efficiency may not be robust to more general assumptions about the strategic interaction. This paper considers the same problem in the case of another well-known benchmark model: the complete information k-double auction. In several aspects, this mechanism can be seen as a polar

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opposite to alternating offers: it is one-shot, players propose simultaneously and effective transfers are a combination of the proposals. We characterize many inefficient equilibria that do not require any modification of the “standard” form of the game.

There exists a fertile recent literature that documents why rational players may fail to exhaust gains from the trade given complete information. The existence of a unique subgame-perfect equilibrium with immediate agreement in a sequential bargaining framework is based on a number of important assumptions. First, a realistic bargaining setting need not be stationary. These non-stationarities may be exogenous, for example when the payoffs from the game are stochastic (Furusawa and Wen 2003), or endogenous, for example in multilateral (Cai 2003) and multi-issue (Weinberger 2000a) bargaining. In these models, delays allow a participant to wait for more bargaining power as the game moves away from its initial state. A second set of models considers endogenous payoffs along the bargaining process, for example in the presence of endogenous disagreement payoffs (Houba 1997), history-dependent feasible proposals (Weinberger 2000b), strategic delays (Ma and Manove 1993) and endogenous deadlines (Mauleon and Vannetelbosch 2004). In these models, threats may be used to sustain inefficient equilibria or inefficient actions may raise the bargaining power. Third, inefficiencies may appear as players use delays to establish a reputation to be stubborn, when there is ex-ante commitment (Kambe 1999) or a positive probability of being irrational (Abreu and Gul 2000). Finally, frictions in the form of costly bargaining may also cause inefficient equilibria (Anderlini and Felli 2001). The participation cost generates a hold-up problem in that the proposer cannot commit to repay the cost to the other player.

A different set of models derives trade inefficiencies from the presence of restrictions on feasible contingent contracts, for example when gains from trade are private information. Although we believe that incomplete contracts are an essential aspect of many realistic settings, it is important to know whether the equilibria of the initial mechanism, in the absence of such restrictions, are efficient. There are two reasons for doing so. First, the existence of incomplete contracts is not observable and is often introduced as a rationale to account for certain empirical puzzles[1], one of them being trade inefficiencies; then, our purpose is to try to distinguish formally inefficiencies that may appear under complete information from inefficiencies that are incompatible with complete information. Second, inefficiencies with complete information are likely to carry over to models with incomplete contracts[2]. We believe that in more general models, the form of inefficiency that we document may coexist with other forms of inefficiencies. We consider this question in the context of the double auction model initially proposed by Chatterjee and Samuelson (1983) with

[1] See for example Chatterjee and Samuelson (1983): (p.836) “the complete information approach fails to mirror key features of actual negociations: (...) the occurence of ’unreasonable’ bargaining outcomes-breakdowns in negociations, strikes, and work stoppages-when mutually beneficial agreements are possible.”

[2] This feature is illustrated in Corollary 1 in the paper.

asymmetric information.

This paper investigates how lack of coordination, when two players must make simultaneous offers, may induce inefficient Nash equilibria. While it is well-known that multiplicity of equilibria may be costly in models with simultaneous offers, we attempt here to characterize what form such inefficiencies may take. Essentially, we will discuss implications of the mechanism when solving in mixed strategies. Specifically, choosing a particular pure strategy and assuming that players will follow this strategy relies on the assumption that players play the game repeatedly and will eventually coordinate or agree ex-ante on a focal point of the game. In a short-term interaction, however, a player may be initially uncertain about the allocation that will be acceptable to the other player. As suggested by Harsanyi (1973), each player will be matched to another player sampled from a population of players who have rationally chosen a pure strategy. We solve for the aggregate distribution of pure strategies that may be sustained in equilibrium. Problems of coordination are presented as an alternative to sequential bargaining and incomplete contracts to explain trade inefficiencies.

We show that there exist many inefficient mixed equilibria such that the proposal anticipated by each player is stochastic. More aggressive proposals, i.e. asking for a greater share of the surplus, raise the surplus that is extracted conditional on a successful negotiation. However, such proposals also reduce the probability of trade. The mixed strategies are sustained by a trade-off between bargaining power and trade breakdown. In comparison to many models with incomplete contracts, problems of coordination can be fully overcome by public intervention. But to achieve this, one needs first a formal theory that accounts precisely for trade inefficiencies due to coordination and allows to detect such problems.

Early work by Broman (1989) documents the finite mixed equilibria of the complete information double auction but there is, to our knowledge, no other author characterizing all the equilibria of the game. In the first part of the paper, we find a large set of mixed equilibria such that traders mix over a countably infinite or uncountable support[3]. Other papers that include simultaneous moves but in a sequential framework include Dekel (1990), Perry and Reny (1993) and Sàkovics (1993).

The first section investigates the Nash equilibria of the game. The strategy of each player can be described as a random proposal over the set of feasible outcomes. In equilibrium, both players must be indifferent to the same set of pure strategies. We characterize the mixed equilibria of the game when players mix over countably and uncountably infinite sets. In this process, we show that if strategies prescribe mixing over an interval, then the strategies admit a simple functional form

[3] In this process, we provide a counter-example to Broman's claim that mixed strategies over a countably infinite or uncountable support with only a finite number of discontinuities do not exist. We also derive explicitly mixed strategies over uncountable sets with singular parts which she leaves as an open question.

over this interval. Further, the distributions must be convex in the surplus obtained by each player: in order to guarantee indifference, inefficiencies arising from more aggressive proposals must be compensated by a sharper increase in the probability of trade.

In the second section, we perform several tests of the validity and robustness of the theory. First, since we argue that inefficiencies are not a specific feature of asymmetric information, we explore whether a model with lack of coordination with complete information may be rejected when trade breakdown occurs due to the presence of asymmetric information. We find that strategies with a support greater than four values will always distinguish between complete and asymmetric information. Second, we analyze whether the mixed equilibria remain in the presence of repeated trading, risk-aversion and violations of expected utility. We find that repeated trading can be restated strategy-wise as a reduction in the (subjective) gains from trade and risk-aversion can be restated as a change in the individual perception of the trading prices. The equilibria will be unchanged in the presence of risk-aversion but, for a given finite support, there may be no equilibrium in the presence of a high discount rate or multiple equilibria in the presence of disappointment aversion.

## 2. MIXED EQUILIBRIA

### 2.1. The Model

The model is the simplest version of the double auction mechanism as presented in Chatterjee and Samuelson (1983). A seller, who owns a single indivisible good, considers the sale of the asset to a buyer. There is a single period and any transaction surplus is socially lost after this period ends. Simultaneously, the buyer offers a bid $B$ and the seller submits an ask $S$. If $B$ is greater than $S$, trade occurs with probability[4] $k$ at price $B$ and $(1-k)$ at price $S$. The parameter $k \in (0,1)$ corresponds to the market power of the seller relative to the buyer. Traders are assumed to be risk-neutral.

The buyer has a valuation $V_b$ for the item, strictly greater than the valuation of the seller $V_s$. Both valuations are public information. We normalize utilities so that both agents obtain a reservation of zero in the absence of trade and the utility conditional on trade is equal to the valuation minus the monetary transfer for the buyer and the monetary transfer minus the valuation for the seller. There exists a continuum of pure-strategy efficient equilibria in which the two parties bid the same amount $B = S \in [V_s, V_b]$.

More generally, we denote $F_b$ (resp. $F_s$) the mixed strategy of the buyer (resp. seller), i.e. the distribution of $B$ (resp. $S$). We restrict our attention to strategies with support in $[V_s, V_b]$. It is

[4] Interpreting the final price as a random draw of each proposed price or as a weighted average of both prices is irrelevant in the risk-neutral case but the former is more tractable in the presence of risk-aversion.

useful first to mention all equilibria such that the buyer always bids $V_s$. These equilibria can be sustained by strategies $(B, S)$ of the form $P(S = V_s) \geq \max_X \mathbb{E}(V_b - kX - (1-k)S | S \leq X)$, for all $X \in [V_s, V_b]$. An analogous set of equilibria is obtained when the seller always bids one. We will put aside these (degenerate) equilibria in the rest of the analysis.

## 2.2. Mixed Equilibria with a Unique Discontinuity

Suppose that $(F_b, F_s)$ are equilibrium distributions in the game. All proofs are delayed to the Appendix and provided only for one distribution when the result carries over to the other distribution by symmetry.

**LEMMA 2.1.** *Consider an open interval* $\Theta \subset (V_s, V_b)$, *bounded away from the boundaries.*

*(i) If* $F_b$ *(resp.* $F_s$*) is constant on* $\Theta$*, then* $F_s$ *(resp.* $F_b$*) is constant on* $\Theta$*.*

*(ii) If* $F_b$ *(resp.* $F_s$*) is strictly increasing on* $\Theta$*, then* $(F_b, F_s)$ *must have the following functional form on* $\Theta$*:*

$$F_b(x) = 1 - C_b/(x - V_s)^{1-k} \quad (1)$$

$$F_s(x) = C_s/(V_b - x)^k \quad (2)$$

*where* $C_b$ *and* $C_s$ *are two positive constants. Further, if there exists* $\epsilon > 0$ *such that* $F_b$ *(resp.* $F_s$*) is constant on* $(\inf \Theta - \epsilon, \inf \Theta)$ *(resp.* $(\sup \Theta, \sup \Theta + \epsilon)$*), then* $F_s$ *(resp.* $F_b$*) cannot be constant on* $(\inf \Theta - \epsilon, \inf \Theta]$ *(resp.* $[\sup \Theta, \sup \Theta + \epsilon)$*).*

Since $k \in (0, 1)$, each player will face a trade-off between an increasing probability of trade and a decrease of the terms of trade. If an interval is never chosen by one of the players, there will be no change in the probability of trade when deviating in the interior of this interval and therefore the terms of trade will prevail. Applying this argument for both players, it is clear that both strategies must be either simultaneously strictly increasing or constant. Note finally that if a particular value $x \in (V_s, V_b)$ is a best response to $F_s$ for the buyer, then it must necessarily be a best response to $F_b$ for the seller.

We also derive a functional form for the distributions in any interval where both distributions are strictly increasing. Equations (1) and (2) show that the strategy of each player is convex in the terms of trade. Both traders tend to follow "defensive" strategies in that the buyer often bids high and the seller often asks low. Since more aggressive proposals induce more inefficiencies, the loss of surplus must be compensated with a higher probability of trade.

It follows also from Lemma 2.1 that there is no absolutely continuous strategy with full support in $(V_s, V_b)$. However, an equilibrium may exhibit combinations of mass points and strictly increasing distributions following Equations (1) and (2). The next Proposition characterizes the discontinuities of the distributions.

**LEMMA 2.2.** *If $F_b$ (resp. $F_s$) is discontinuous at $x \in (V_s, V_b)$ then:*

*(i)* $\exists \epsilon > 0$ *such that $F_s$ (resp. $F_b$) is constant on $(x, x+\epsilon)$ (resp. $(x-\epsilon, x)$).*

*(ii) If there exists $\epsilon > 0$ such that $F_s$ (resp. $F_b$) is constant $(x-\epsilon, x)$ (resp. $(x, x+\epsilon)$), then $F_s$ (resp. $F_b$) is discontinuous at $x$.*

Any discontinuity due to the presence of an atom will imply that the effect from the probability of trade will always dominate the effect from the terms of trade. If the buyer plays an atom, the seller will prefer playing the atom to an ask arbitrarily close but above the atom; such strategy would yield terms of trade arbitrarily close to the atom but a non-marginal loss in the probability of trade.

**PROPOSITION 2.1.** *Suppose that:*

*(i) $F_b$ and $F_s$ are strictly increasing over an open interval $\Theta \subset (V_s, V_b)$, bounded away from the boundaries.*

*(ii) $F_b$ and $F_s$ have a finite number of discontinuities.*

*(iii) $F_b$ and $F_s$ admit a probability density function that can be decomposed into an absolutely continuous function non-zero over a countable union of intervals and a probability mass function.*

*Then: $(F_b, F_s)$ is an equilibrium if and only if there exists $(\underline{y}, \overline{y})$ with $V_s < \underline{y} \leq \inf \Theta < \sup \Theta \leq \overline{y} < V_b$ such that, for all $x \in [\underline{y}, \overline{y})$,*

$$\begin{aligned} F_b(x) &= 1 - ((\underline{y} - V_s)/(x - V_s))^{1-k} \\ F_s(x) &= ((V_b - \overline{y})/(V_b - x))^k \end{aligned}$$

*with $F_b(x) = F_s(x) = 0$ for $x < \underline{y}$ and $F_b(x) = F_s(x) = 1$ for $x \geq \overline{y}$.*

We characterize a family of distributions that has a unique discontinuity and an uncountable support. There exists a maximum acceptable threshold for the buyer that is played with non-zero probability. Any buyer playing this threshold is certain to trade but will pay a high price. A continuum of lower prices may also be chosen by the buyer, but never lower than a minimum threshold acceptable for the seller and asked with non-zero probability by the seller[5]. In the context of financial markets, this strategy may be interpreted as different types of orders. When the trader chooses the price that prescribes the least surplus, the probability of trade will be maximal but the terms of trade will be less favorable: such an order can be interpreted as a "market order" and we show that it is chosen with non-zero probability even in the presence of a continuum of other trading prices. An order above the threshold can be interpreted as a "limit order" as it may not be executed with strictly positive probability. The difference between each threshold can be interpreted as a particular form of bid-ask spreads.

Lemma 2.1 and Proposition 2.1 also provide two simple testable conditions that may be used to reject complete information, even when the true model may depart slightly from the framework presented here. First, the distributions $(F_b, F_s)$ must necessarily be convex in the surplus to guarantee indifference. Second, under the family of equilibria presented in Proposition 2.1, the distributions $(F_b, F_s)$ may never cross except when they are zero or one.

Along this family of equilibria, the profit of the buyer (resp. seller) is equal to $(V_b - \overline{y})^k/(V_b - \underline{y})^{1-k}$ (resp. $(\underline{y} - V_s)^{1-k}/(\overline{y} - V_s)^k$). We find that the profit of the buyer is decreasing in the threshold $\overline{y}$ corresponding to the maximum acceptable trading price. Here, this effect is not due to less favorable trading prices but rather to the fact that a high threshold implies that the buyer has little to lose from small trade inefficiencies, and thus the seller must compensate by inducing a greater level of inefficiencies. On the other hand, observe that the profit of the buyer is increasing in the threshold $\underline{y}$, i.e. choosing values that are more favorable to the seller (at a given maximum acceptable price) is beneficial. This result may seem counter-intuitive as one may think that a high value of $\underline{y}$ corresponds to low terms of trade for the buyer. In our setting, however, since inefficiencies are more severe for low bids, the aggregate level of inefficiencies that is necessary to sustain a particular mixed equilibrium is lower when the support includes low bids for the buyer[6].

### 2.3. Mixed Equilibria with more than one Discontinuity

We will now consider a countable support. The countable case yields a closed-form characterization of the equilibrium distributions and allows us to construct equilibria with uncountable support. Define $Y = (y_i)_{i=1}^N$, a countable subset of $(V_s, V_b)$, where $N$ is possibly infinite. Further,

[5] Proposition 2.1 is also a contradiction to Proposition 6 in Broman (1989) since it exhibits an equilibrium distribution for the buyer that can be decomposed into a probability mass function and an absolutely continuous function: p.161, "Proposition 6. Let p be a probability density function on [0,1] which can be decomposed into $p_1$, an absolutely continuous function, and $p_2$, a probability mass function defined on a set $\{a_i\}$ (i.e. a step function). Then, there is no equilibrium in which the buyer plays p as a mixed strategy."

[6] This particular example also shows why, without any modification to the mechanism, it may be difficult for two traders to negociate over the support of their strategies through pre-play communication. The buyer will always wish to decrease $\overline{y}$ or increase $\underline{y}$, whereas the seller will wish to do the reverse. A strict improvement in the utility of both traders can be achieved only by considering a simultaneous change in both bounds of the support.

$Y$ is supposed to have an infimum, $\underline{y}$, and a supremum, $\overline{y}$, bounded away from the boundaries[7]. We will consider mixed strategies with support $Y$.

Define as $(Y^n)_{n=1}^{\infty}$, an increasing sequence of subsets of $Y$ such that $Y^n$ has $\min(n, N)$ distinct terms ranked in increasing order, $Y^n = (y_j^n)_{j=1}^n$, and converging to $Y$.

Let us define the functions $F_b^n$ and $F_s^n$ as follows. For all $i \in [1, n-1]$, for $x \in [y_i^n, y_{i+1}^n)$,

$$
\begin{aligned}
F_b^n(x) &= 1 - \prod_{j=1}^{i} \frac{y_j^n - V_s}{k y_j^n + (1-k) y_{j+1}^n - V_s} && (3) \\
F_s^n(x) &= \prod_{j=i}^{n-1} \frac{V_b - y_{j+1}^n}{V_b - k y_j^n - (1-k) y_{j+1}^n} && (4)
\end{aligned}
$$

with for all $x \geq y_n^n$, $F_b^n(x) = F_s^n(x) = 1$ and for all $x < y_1^n$, $F_b^n(x) = F_s^n(x) = 0$.

**PROPOSITION 2.2.** *For a countable set $Y$, suppose that $(F_b^n, F_s^n)_{n=1}^{\infty}$ are constructed following Equations* (3) *and* (4)*.*

*(i) $(F_b^n, F_s^n)$ converges pointwise to a limit $(F_b, F_s)$ that does not depend on $(Y^n)_{n=1}^{\infty}$.*

*(ii) If $F_b$ and $F_s$ are probability distributions, there exists a unique mixed equilibrium with support $Y$ such that both players are indifferent to all pure strategies in $Y$. It is given by $(F_b, F_s)$.*

We obtain an equilibrium strategy that induces indifference over a countable number of values as the limit of equilibria with finite support; $(F_b^n, F_s^n)$ do not necessarily converge to a distribution but, if they do, this distribution is a solution. We also show that in this case, there will be a unique solution such that traders follow completely mixed strategies over $Y$. It should also be noted that the limit of $(F_b^n, F_s^n)$ does not depend on the choice of an approximating sequence $(Y^n)$, so that Proposition 2.2 also yields a numerical method to check if a particular countable set may support an equilibrium. Note that $F_s(\underline{y})$, and thus the profit of the buyer, is increasing with the bargaining power $(1-k)$. This effect should be explained by more favorable terms of trade in the ex-post trading terms. An increase of $(1-k)$ reduces the payoff from more aggressive bids and thus weakens incentives to induce inefficiencies.

There are many equilibria that can be written as a combination of a point mass and an absolutely continuous function with more than a unique discontinuity. Proposition 2.3 derives a family of such equilibria where equilibrium distributions have a support that is a countable subset and an interval.

[7] It is easy to verify that if a player chooses with strictly positive probability a value arbitrarily close to a surplus of zero, then we must obtain an equilibrium such that this player obtains zero profit.

**PROPOSITION 2.3.** *Consider an interval $[\underline{\theta}, \overline{\theta}]$, a strictly increasing (resp. strictly decreasing) sequence $(\underline{y}_i)_{i=1}^{\infty}$ (resp. $(\overline{y}_i)_{i=1}^{\infty}$) converging to $\underline{\theta}$ (resp. $\overline{\theta}$). There exists a unique equilibrium with completely mixed strategies over $(\underline{y}_i)_{i=1}^{\infty} \cup [\underline{\theta}, \overline{\theta}] \cup (\overline{y}_i)_{i=1}^{\infty}$ and it is given by:*

*For $i \in [1, \infty)$, for $x \in [\underline{y}_i, \underline{y}_{i+1})$,*

$$F_b(x) = 1 - \prod_{i=1}^{k} \frac{\underline{y}_i - V_s}{k\underline{y}_i + (1-k)\underline{y}_{i+1} - V_s} \tag{5}$$

$$F_s(x) = \left(\frac{V_b - \overline{\theta}}{V_b - \underline{\theta}}\right)^{1-k} \prod_{i=1}^{\infty} \frac{V_b - \overline{y}_i}{V_b - k\overline{y}_{i+1} - (1-k)\overline{y}_i} \prod_{i=k}^{\infty} \frac{V_b - \underline{y}_{i+1}}{V_b - k\underline{y}_i - (1-k)\underline{y}_{i+1}} \tag{6}$$

*For $i \in [2, \infty)$, for $x \in [\overline{y}_i, \overline{y}_{i-1})$,*

$$F_b(x) = 1 - \left(\frac{\underline{\theta} - V_s}{\overline{\theta} - V_s}\right)^k \prod_{i=1}^{\infty} \frac{\underline{y}_i - V_s}{k\underline{y}_i + (1-k)\underline{y}_{i+1} - V_s} \prod_{i=k-1}^{\infty} \frac{\overline{y}_{i+1} - V_s}{k\overline{y}_{i+1} + (1-k)\overline{y}_i - V_s} \tag{7}$$

$$F_s(x) = \prod_{i=1}^{k-1} \frac{V_b - \overline{y}_i}{V_b - k\overline{y}_{i+1} - (1-k)\overline{y}_i} \tag{8}$$

$$\tag{9}$$

*For $x \in [\underline{\theta}, \overline{\theta}]$,*

$$F_b(x) = 1 - \left(\frac{\underline{\theta} - V_s}{x - V_s}\right)^{1-k} \prod_{i=1}^{\infty} \frac{\underline{y}_i - V_s}{k\underline{y}_i + (1-k)\underline{y}_{i+1} - V_s} \tag{10}$$

$$F_s(x) = \left(\frac{V_b - \overline{\theta}}{V_b - x}\right)^k \prod_{i=1}^{\infty} \frac{V_b - \overline{y}_i}{V_b - k\overline{y}_{i+1} - (1-k)\overline{y}_i} \tag{11}$$

*with for $x < \underline{y}_1$, $F_b(x) = F_s(x) = 0$ and for $x \geq \overline{y}_1$, $F_b(x) = F_s(x) = 1$.*

A similar argument as Proposition 2.3 can be used to construct equilibria that admit absolutely continuous parts on a finite number of disjoint intervals. In this case, the equilibrium admits a simple expression on the interval, but the equilibrium conditions at the boundary of each interval require a converging sequence of isolated values in the support. A more surprising fact is that any such sequence will sustain an equilibrium.

In the following section, we illustrate how to understand problems of coordination in more general settings. In doing so, we restrict the attention to mixed equilibria over finite sets given that most countably and uncountably infinite sets can be obtained as the limit of the finite case.

## 3. EXTENSIONS

### 3.1. Identification of Complete Information

We investigate now how much information can be recovered from the model using only observations from bids and asks. In particular, we consider two related questions. Can valuations be identified in the class of complete information games? Can a model with asymmetric information be distinguished from a model with complete information? In other words, we characterize the identification of the valuations in the class of complete and asymmetric information games[8].

Suppose that many observations of the same bargaining setup may be recorded. Let us denote $Y = (y_i)_{i=1}^n$ a finite grid and we assume that all players have complete information and play each of its elements with strictly positive probability. The probability that each element of $Y$ is played by the buyer (resp. seller) is denoted $(p_i)_{i=1}^n$ (resp. $(q_i)_{i=1}^n$). Now, we define a candidate asymmetric information game. Let $(V_b^j, b_j)_{j=1}^J$ (resp. $(V_s^j, s_j)_{j=1}^J$) be a countable set, where $(V_b^j)_{j=1}^J$ (resp. $(V_s^j)_{j=1}^J$) corresponds to randomly chosen initial valuations and $(b_j)_{j=1}^J$ (resp. $(s_j)_{j=1}^J$) the associated probabilities for the buyer (resp. seller).

The set $Y$ as well as the probability of occurrence of each of its elements will be observed for a sufficiently large number of observations. If for some set $Y$, there exists $(V_b, V_s)$ such that the solutions to Proposition 2.2 are also solution to an asymmetric information game where private valuations are drawn from $(V_b^j, b_j)_{j=1}^J$ (resp. $(V_s^j, s_j)_{j=1}^J$) for the buyer (resp. seller), then a double auction with complete information will be observationally equivalent to a double auction with asymmetric information.

**PROPOSITION 3.1.** *Suppose $Y$ is a set with n distinct elements,*

*(i)* $(V_b, V_s)$ *is identified if and only if* $n \geq 2$.

*(ii) There exists no* $(Y, (p_i, q_i)_{i=1}^n)$ *that is a mixed strategy equilibrium given complete information* $(V_b, V_s)$ *and a pure strategy equilibrium given asymmetric information* $((V_b^j, V_s^j, b_j, s_j)_{j=1}^J)$ *if and only if* $n \geq 4$.

In Proposition 3.1, we show that the identification of a game of complete information requires a sufficiently large support[9]. In the extreme case, when the support is a singleton, since playing this particular value is solution to any valuation, the model does not identify $(V_b, V_s)$. An observationally equivalent asymmetric information game can always be constructed for $n \leq 3$. The equilibrium strategies in the complete information game are constructed so that players will be indifferent to all values in the support when their valuation is $(V_b, V_s)$. For any different valuation, by linearity, their optimum will lie at one of the boundaries of $Y$. Any value of $Y$ that is neither its maximum or minimum must be induced by the valuation of the complete information game.

[8] An alternative procedure is the rejection of the asymmetric information model, as featured in Hollifield, Miller, and Sandas (2004).

[9] We do not attempt to solve for the identification of the distributions when the valuations may vary in the cross-section.

If both players may follow a mixed strategy in the presence of asymmetric information, the complete information setting is nested as a special case. In fact, an analogous statement as Proposition 3.1 can be made in this case. For any $j$ and for any $i$, we denote $b_j^i$ (resp. $s_j^i$), the probability that a buyer (resp. a seller) whose valuation is $V_b^j$ bids (resp. asks) $y_i^n$. In the next Corollary, let us denote $\epsilon_b^+$ (resp. $\epsilon_b^-$) the probability that the valuation is above (resp. below) the complete information valuation of the buyer and $\epsilon_s^+$ (resp. $\epsilon_s^-$), the probability that the valuation is above (resp. below) the complete information valuation of the seller.

**COROLLARY 3.1.** *Suppose $n \geq 4$. For any complete information mixed equilibrium $(Y, (p_i, q_i)_{i=1}^n)$, any observationally equivalent non-degenerate asymmetric information game must satisfy:*

*(i)* $\epsilon_b^+ + \epsilon_b^- \leq 1 - \max_{i \in [2,n-1]} p_i$ *(resp.* $\epsilon_s^+ + \epsilon_s^- \leq 1 - \max_{i \in [2,n-1]} q_i$*).*

*(ii)* $\epsilon_b^+ \leq p_n \leq 1 - \epsilon_b^-$ *(resp.* $\epsilon_s^- \leq q_1 \leq 1 - \epsilon_s^+$*)*

*(iii)* $\epsilon_b^- \leq p_1 \leq 1 - \epsilon_b^+$ *(resp.* $\epsilon_s^+ \leq q_n \leq 1 - \epsilon_b^-$*)*

Values that are not extrema of $Y$ can be chosen only when a valuation exactly equal to the complete information game has been drawn. It follows that introducing mixed strategies does not substantially modify the argument. Valuations different from the valuation that generated the complete information game will always induce an extremum pure strategy. However, the mixed equilibrium can now be replicated using a mixed strategy when $V_b$ is drawn.

## 3.2. Repeated Trading with Discounting

In the standard model, if the object is not traded, its value is completely lost; in other words, we do not allow for repeated trading sessions. We suppose now that both traders are patient so that the value of the item for the buyer (resp. seller) is discounted by $\beta_b$ (resp. $\beta_s$) after each trading session. Both discount rates[10] are in $[0, 1)$. Next, we assume that whenever trade fails, each trader is matched to a new trader so that the expected profit at each new occurrence of the game does not change. Let us denote $Y = (y_i)_{i=1}^n$ a finite subset of $(V_s, V_b)$, with values ranked in increasing order and $p_i$ (resp. $q_i$), the probability that $y_i$ is chosen by the buyer (resp. seller).

**PROPOSITION 3.2.** *There is always a unique equilibrium with completely mixed strategies over $Y = \{y_1, y_2\}$.*

We consider now the case of a support that includes more than two values.

[10] Note that for a discount rate equal to unity, since the trader becomes infinitely patient, the equilibrium strategy must prescribe a support that has a single point and may not be completely mixed.

**LEMMA 3.1.** $(Y, (p_i, q_i)_{i=1}^n)$ *is an equilibrium if and only if:*

$$\forall i \in [1, n-1], \quad \frac{p_i}{1-F_b(y_{i-1})} = \frac{(1-k)(y_{i+1}-y_i)}{ky_i + (1-k)y_{i+1} - V_s^{\beta_s}} \tag{12}$$

$$\forall i \in [2, n], \quad \frac{q_i}{F_s(y_i)} = \frac{k(y_i - y_{i-1})}{V_b^{\beta_b} - ky_{i-1} - (1-k)y_i} \tag{13}$$

$$\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$$

*where:*

$$V_b^{\beta_b} = \frac{1-q_1\beta_b}{1-\beta_b(1-q_1)}V_b + \frac{q_1\beta_b}{1-\beta_b(1-q_1)}y_1$$

$$V_s^{\beta_s} = \frac{1-p_n\beta_s}{1-\beta_s(1-p_n)}V_s + \frac{p_n\beta_s}{1-\beta_s(1-p_n)}y_n$$

The equilibrium conditions can no longer be represented as a linear system of equations and the existence of an equilibrium may now depend on the support of the strategies. The numerator of Equation (12) remains analogous to the case of one-shot trading. The direct effect of the bargaining power $(1-k)$ and the grid step $(y_{i+1} - y_i)$ can be separated from the effect of repeated trading. The denominator reflects the compensation for the inefficiencies due to absence of trade and are changed by repeated trading. More precisely, the equivalent valuation of the buyer $V_b$ in the repeated game is now modified to $V_b^{\beta_b} \leq V_b$. A new valuation $V_b^{\beta_b}$ arises as a linear combination of $y_1$ and $V_b$, and is always smaller than $V_b$. The presence of repeated trading reduces (resp. increases) the valuation of the buyer (resp. seller) in an equivalent one-shot game.

**PROPOSITION 3.3.** *Suppose that* $(Y, V_b, V_s, \beta_b, \beta_s, k)$ *is such that:*

$$(1-\beta_b)V_b + \beta_b y_1 \geq \max ky_{i-1} + (1-k)y_i$$

$$(1-\beta_s)V_s + \beta_s y_n \leq \min ky_i + (1-k)y_{i+1}$$

$$1 - \sum_{j=2}^n \frac{k(y_j - y_{j-1})}{(1-\beta_b)V_b + \beta_b y_1 - ky_{j-1} - (1-k)y_j} \geq 0$$

$$1 - \sum_{j=1}^{n-1} \frac{k(y_{j+1} - y_j)}{ky_j + (1-k)y_{j+1} - (1-\beta_s)V_s - \beta_s y_n} \geq 0$$

*there is always an equilibrium with completely mixed strategies over* $Y$. *Else there exists* $\epsilon(k, Y) > 0$ *such that for* $0 \leq \min(\beta_b, \beta_s) < \max(\beta_b, \beta_s) < \epsilon(k, Y, V_b, V_s)$, *such that there is an equilibrium with completely mixed strategies over* $Y$.

In Proposition 3.3, we propose two conditions that guarantee the existence of an equilibrium with completely mixed strategies over a given support $Y$. The first condition is global and requires the lower bound on the equivalent one-shot valuation to be sufficiently high (resp. low) for the buyer (resp. seller). In general, the first two conditions are easier to verify whenever the support of $Y$ is away from the extrema $V_b$ and $V_s$. For excessively unfavorable trading prices, a trader may prefer to delay trade rather than accept this price[11]. In the following two equations, this problem is more severe when trading prices are further apart so that the trader can wait for better trading prices. The second condition is local and ensures that for a discount rate that is sufficiently low there will always be an equilibrium.

### 3.3. Risk Aversion

Mixed strategies also introduce risk in the bargaining process and therefore, risk-aversion may play an important role in the analysis. Until now, we supposed that both agents value their utility from transfers linearly, we will now suppose that the seller (resp. the buyer) has a strictly increasing, concave and twice-differenciable utility function $U_b$ (resp. $U_s$). We normalize $U_b(0) = 0$ (resp. $U_s(0) = 0$) and $U_b'(0) = 1$ (resp. $U_s'(0) = 1$). Following these normalizations, the absolute risk-aversion is captured by the convexity of the utility function.

**PROPOSITION 3.4.** *With risk-averse traders, for any $Y$, a finite set with $n$ distinct values, there is a unique equilibrium $(F_b, F_s)$ with support $Y$ and it is obtained by Equation (3) after relabelling $Y$ to $Y_b = (y_i^b)_{i=1}^n$ and Equation (4) after $Y$ is relabelled to $Y_s = (y_i^s)_{i=1}^n$, where:*

$$y_i^b = V_b - U_b(V_b - y_i) \tag{14}$$

$$y_i^s = U_s(y_i - V_s) + V_s \tag{15}$$

In the presence of risk-aversion, the perception of the trading values in the grid in $Y$ are subjective. Mixed equilibria with finite support, as obtained in Proposition 2.2, carry over to risk-aversion after a transformation of the objective grid to a risk-neutral grid where each agent will behave as if risk-neutral. It is important to note that in general both subjective grids will be different so that each agent will appear to be following a mixed strategy for a particular grid.

**COROLLARY 3.2.** *For a given $Y$, the probability of trade is increasing with risk-aversion.*

[11] Numerically, we could find examples such that there is no equilibrium for a given finite support $Y$, however, existence seemed to hold in most cases. However, we did not find any example with multiple equilibria. The system of polynomials always yields $n$ solutions but only one of them prescribed $q_n$ positive in our simulations; we could not prove this in general.

Mixed equilibria exhibit additional sources of inefficiencies as agents bear risk about the occurrence of trade and the final trading price. These inefficiencies induce a lower value for favorable but unlikely trading prices. In equilibrium, the lower comparative advantage of better prices must be compensated with a greater probability of trade. In this respect, although risk-aversion introduces inefficiencies due to imperfect insurance, it also dissipates inefficiencies due to trade breakdown.

To finish this section on risk-aversion, note that one can also use the change of variables in Proposition 3.4 to find equilibria with uncountable supports. Similarly, Equations (1) and (2) can be adapted to the case of risk-aversion. Consider the differential equation for $F_s$:

$$0 = \Pi_b'(x) = F_s'(x)U_b(V_b - x) - k\int_x^{V_b} U_b'(V_b - x)dF_s(x)$$

$F_s(x)$ will be more convex than in the case of risk-neutrality. Risk-aversion, therefore, is compatible with greater convexity in the foregone surplus.

## 3.4. Disappointment Aversion

We will now consider the bargaining setting when agents do no not have expected utility preferences. Although it is widely understood that agents may not always evaluate lotteries according to expected utility, generalized preferences have rarely been applied to bargaining problems since, often, the outcome prediction arising from the model is deterministic. In our model, however, proper consideration of the random nature of the strategy chosen by the other trader is crucial and thus behavioral biases will affect the trading strategy.

Let us consider the case of disappointment aversion introduced by Gul (1991). Disappointment aversion can be obtained as a result of violations of the independence axiom[12]. For final outcomes lower than the expected ex-ante outcome of the lottery, the buyer will experience a penalty increasing in the magnitude of the deviation. For simplicity, we assume that only the buyer has such preferences and the seller is an expected utility maximizer.

Formally, consider a finite lottery $L = ((L_i, p_i)_{i=1}^n)$, where $L_i$ is a payoff and $p_i$ its probability in the lottery $L$. Define $\tilde{U}_b$ the utility obtained for a lottery $L$. We assume that:

$$\tilde{U}_b(L) = \sum_{i=1}^{n} p_i(U_b(L_i) - \rho(U_b^{-1}(\tilde{U}_b(L)) - L_i)) \tag{16}$$

where $U_b$ is a utility function that satisfies our earlier assumptions and $\rho$ is a continuous positive increasing penalty function such that $\rho(0) = 0$.

[12] See Schweinzer (1999) for an axiomatic presentation of different preferences that violate the independence axiom.

We show now that there is always an equilibrium for a given finite support of the strategies[13].

**PROPOSITION 3.5.** *There is always an equilibrium with completely mixed strategies over $Y$.*

Let us denote $U_b^i$ as $U_b(V_b - y_i^n)$ and denote $\Pi_b$, the equilibrium utility obtained by the buyer $\tilde{U}_b(L)$ where $L$ is the lottery over trading prices when the seller plays the equilibrium mixed strategy $F_s$. Observe that for any support $Y$, the buyer will always be disappointed at $y_n$. Suppose our claim does not hold, then $U_b^n > \Pi_b$, but then the equilibrium Equation when bidding $y_n^n$ would yield that: $U_b^n > \Pi_b = kU_b^n + (1-k)\sum_{j=1}^n q_j^n U_b^j$. This last inequality would yield a contradiction.

With disappointment aversion, there may be a multiple equilibria for a given support. The occurrence of multiple solutions can be explained simply. A high profit induces high disappointment. This effect implies that low bids hurt more. As a result, the seller must follow a strategy that induces an even higher level of profit for the buyer. Here, disappointment plays a role that is analogous to risk-aversion. When the gains in profit due to risk-aversion dominate the loss from disappointment, there may be multiple equilibria: an equilibrium such that the buyer expects disappointment and a high surplus and an equilibrium such that the buyer expects low disappointment and a lower surplus.

## 4. CONCLUSION

It is widely believed that trade inefficiencies are evidence of asymmetric information. In this paper, we analyze the Nash equilibria of the complete information double auction and show that problems of coordination may also appear whenever traders do not find a particular coordination rule. As in the existing litterature on complete information bargaining, we try to exhibit the inefficiency of this particular mechanism and argue for the role of more sophisticated coordination devices or intervention in a simple decentralized market. Further, we believe that reputations in long-term bargaining settings are often used as coordination rules; in this respect, inefficiency losses arising from coordination failure may also be interpreted as a transaction cost before a reputation has been established.

Many questions are left for further work. What remains identified in the complete information game whenever valuations are allowed to vary but remain omitted in the data? Is repeated trading socially beneficial? What is the speed of convergence to an efficient equilibrium when two traders interact repeatedly and commit to an increasingly favorable sequence of offers? What is the equilibrium with a discrete support and a large number of buyers and sellers and how does it respond

[13]This property is due to the assumption of disappointment aversion; in the presence of other preferences that violate the independence, it may not hold. For example, with weighted utilities, we find that there may be no equilibrium for a particular finite support with more than two values.

to a change in the share of buyers and sellers? What form of learning may or may not induce convergence to a mixed equilibrium in a population of traders?

## APPENDIX

PROOF OF LEMMA 2.1: (i) Suppose that $F_b$ is constant on $\Theta$ but there exists $x \in \Theta$ such that the utility of the seller is maximized at $x$. Suppose first that $F_b$ is one on $\Theta$, but then by asking $x$, the seller obtains zero profit; this has been excluded previously. Suppose now that $F_b$ is different from one on $\Theta$. Then, by asking any $\overline{x} > x$ such that $\overline{x}$ is in $\Theta$, the seller would achieve strictly more profit. To show (ii), we integrate by parts the profit of the seller on $\Theta$. Let us denote $\Pi_s(x)$, the profit of the seller when asking $x$.

$$\begin{aligned}\Pi_s(x) &= \int_x^{V_b} \{kB + (1-k)x - V_s\} dF_b(B) \\ &= [(kB + (1-k)V_b - V_s)F_b(B)]_x^{V_b} - k\int_x^{V_b} F_b(B)dB\end{aligned}$$

Since the derivative of $\Pi_s$ exists and $\int_x^{V_b} F_b(B)dB$ is differenciable, $F_b$ must also be differenciable. Taking the first-order condition on $\Pi_s$ yields that: $0 = \Pi_s'(x) = (1-k)(1-F_b(x)) - F_b'(x)(x - V_s)$. Equation (1) follows readily and Equation (2) is analogous. Finally, suppose by contradiction that there exists $\epsilon > 0$ such that $F_s$ is constant in $[x-\epsilon, x]$ and strictly increasing on $[x, x+\epsilon]$. But at $x - \epsilon/2$, the profit of the buyer is greater than at $x$.

*Q.E.D.*

PROOF OF LEMMA 2.2: (i) By contradiction, suppose there exists a sequence $(y_n)_{n=1}^{\infty}$ converging to $y$ and such that $(F_s(y_n))_{n=1}^{\infty}$ is strictly decreasing to $F_s(y)$ and such that $y_n$ maximizes $\Pi_s(y)$.

$$\Pi_s(y_n) \to \lim \int_{y_n}^{V_b} \{kB + (1-k)y - V_s\} dF_b(B)$$

$$< \int_y^{V_b} \{kB + (1-k)y - V_s\} dF_b(B) = \Pi_s(y)$$

(ii) is a direct application of Lemma 2.1.

*Q.E.D.*

PROOF OF PROPOSITION 2.1: Let us prove first that the distributions presented in the Proposition are an equilibrium; we will prove that $F_s$ constructed by (i) and (ii) makes the buyer indifferent between all values in $[\underline{y}, \overline{y}]$. Notice first that since $F_s$ has the functional form given in Lemma (2.1), the buyer must be indifferent between all values in $[\underline{y}, \overline{y})$. Clearly, it is never optimal for the buyer to play outside of $[\underline{y}, \overline{y}]$. Let us now prove the "only if" part. It is convenient to extend $\Theta$ to $\overline{\Theta}$ defined as the largest open interval that includes $\Theta$ such that $F_s$ is strictly increasing on $\overline{\Theta}$ (it is clear this construction is valid here). We argued earlier that $F_s$ must have at least one discontinuity. Suppose that $F_s$ has more than one discontinuity.

We provide a sketch of the argument; a more detailed proof is available in a supplemental material section. (a) Suppose that the limit of $F_s$ when x converges from below to $\sup \overline{\Theta}$ is different from one. We claim first that $F_s$ cannot be discontinuous at $\sup \overline{\Theta}$ or $F_b$ would be constant on some neighborhood $(\sup \overline{\Theta} - \epsilon, \sup \overline{\Theta})$ (Lemma 2.2). Now, if $F_s$ is continuous at $\sup \overline{\Theta}$ and constant on neighborhood $(\sup \overline{\Theta}, \sup \overline{\Theta} + \epsilon)$, $F_b$ must be constant as well on this neighborhood. Consider now the next price that maximizes (since the number of discontinuities is finite, this price is attained) the profit of the buyer. But now, the buyer can bid marginally less and achieve a greater profit. If on the other hand, $F_s$ is never constant on any neighborhood $(\sup \overline{\Theta}, \sup \overline{\Theta} + \epsilon)$, let us choose first this neighborhood sufficiently small so that it will not include any discontinuities. In this neighborhood, $F_s$ must alternate between intervals where it is strictly increasing and Qintervals where it is constant (Assumption (iii)). But then, it is clearly optimal for the seller to ask marginally more than one of the upper boundaries of those intervals. (b) Suppose that the limit of $F_s$ when $x$ converges from below to $\sup \overline{\Theta}$ is not zero. The contradiction is then analogous to (a) by symmetry.

Cases (a) and (b) show that there must a single discontinuity at the upper (resp. lower) boundary of the interval $\overline{\Theta}$ for the buyer (resp. seller).

*Q.E.D.*

PROOF OF PROPOSITION 2.2: (i) First, we prove the pointwise convergence of $F_b^n$. Choose $y \in Y$. Denote $j^n(y) = \max(j / y_j^n \leq y)$. We can write $F_b^n(y)$ as:

$$F_b^n(y) = 1 - \prod_{j=1}^{j^n(y)} \frac{y_j^n - V_s}{k y_j^n + (1-k) y_{j+1}^n - V_s}$$

Define $\tilde{y}^n$ such that: $Y^{n+1} = Y^n \cup \{\tilde{y}^n\}$. There are four possible cases; we will show that for a given $y$, $F_b^n(y)$ is increasing and since it is bounded, must converge.

**Case 1:** $y_{j^n(y)+1}^n < \tilde{y}^n$. Then: $F_b^{n+1}(y) = F_b^n(y)$.

**Case 2:** $y_{j^n(y)}^n < \tilde{y}^n < y_{j^n(y)+1}^n$. Then:

$$\frac{1 - F_b^{n+1}(y)}{1 - F_b^n(y)} = \frac{k y_{j^n(y)}^n + (1-k) y_{j^n(y)+1}^n - V_s}{k y_{j^n(y)}^n + (1-k)\tilde{y}^n - V_s} > 1$$

**Case 3:** There exists $i < j_n(y) - 1$ such that: $y_i^n < \tilde{y} < y_{i+1}^n$. Then:

$$\frac{1 - F_b^{n+1}(y)}{1 - F_b^n(y)} = (1 + \frac{k(1-k)(\tilde{y}^n - y_i^n)(y_{i+1}^n - \tilde{y}^n)}{(k y_i^n + (1-k)\tilde{y}^n - V_s)(k\tilde{y}^n + (1-k) y_{i+1}^n - V_s)}) > 1$$

**Case 4:** $\tilde{y} < y_1^n$. Then:

$$\frac{1 - F_b^{n+1}(y)}{1 - F_b^n(y)} = (1 + \frac{(1-k)(\tilde{y}^n - y_1^n)}{k\tilde{y}^n + (1-k) y_1^n - V_s}) > 1$$

(ii) The rest of the proof proceeds in three parts: (a) we argue that the Proposition holds when $Y$ is finite, (b) the limit distribution $F_s$ makes the buyer indifferent between all values of $Y$, (c) the solution is unique.

(a) To show this, we show that $F_b^n$ is the (unique) equilibrium distribution with finite support $Y^n$. This step is essentially a reformulation of Broman (1989) (Proposition 1, p.140). Let $\Pi_b^*$ denote the optimal profit of the buyer. Equilibrium conditions for the buyer correspond to the following system of $(n+1)$ linear equations in $(n+1)$ unknowns:

$$\begin{pmatrix} V_b - y_1^n & 0 & \dots & & 0 & -1 \\ & & & & & \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & \\ m_{i,j} & & \dots & & V_b - y_n^n & -1 \\ 1 & & \dots & & 1 & 0 \end{pmatrix} \begin{pmatrix} q_1^n \\ \\ \vdots \\ \\ q_n^n \\ \Pi_b^* \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \vdots \\ \\ 0 \\ 1 \end{pmatrix}$$

where $m_{i,j} = V_b - ky_i^n - (1-k)y_j^n$. Denote $M_{n+1}$ this $(n+1) \times (n+1)$ matrix.

There is a unique solution that is given by, for all $1 < k \leq n-1$,

$$q_i^n = \frac{(1-k)(y_i^n - y_{i-1}^n) \prod_{j=i}^{n-1} (V_b - y_{j+1}^n)}{\prod_{j=i-1}^{n-1} (V_b - ky_j^n - (1-k)y_{j+1}^n)}$$

and $q_1^n = 1 - \sum_{i=1}^n q_i^n$. Summing these probabilities, the optimal strategy $F_s$ follows.

(b) Define $\mathcal{M}$, a continuous operator defined from the set of all probability distributions over $[V_s, V_b]$ to $\mathbb{R}^{\mathbb{N}}$ as follows:

$$\mathcal{M}(F) = \left( \int_{V_s}^{y_i} (V_b - ky_i - (1-k)x) dF(x) \right)_{i=1}^{\infty}$$

Suppose now that the sequence of distributions $(F_s^n)$ is constructed as in Equations (3) and (4) and converges to a distribution $F_s$. Thus: $\lim \mathcal{M}(F_s^n) = \mathcal{M}(F_s)$.

$$\mathcal{M}(F_s^n) = (\mathbb{1}_{y_i \in Y^n} \Pi_b^n + (1 - \mathbb{1}_{y_i \notin Y^n}) \Pi_b^n(y_i))_{i=1}^{\infty}$$

where: $\Pi_b^n = (V_b - y_1^n) \prod_{i=1}^n \frac{V_b - y_{i+1}^n}{V_b - ky_i^n - (1-k)y_{i+1}^n}$ and $\Pi_b^n(y_i)$ is some value in $[0, \Pi_b^n)$.

But for any $y_i \in Y$, there exists a $n$ such that for any $n' \geq n$, $y_i$ is in $Y^{n'}$; therefore, in the limit, $\mathcal{M}(F_s)$ has all its components equal to $\Pi_b$, where $\Pi_b$ is the limit of $\Pi_b^n$.

(c) The euclidian space is endowed with the $L^1-$norm. To show uniqueness, we argue first that for any matrix $M_{n+1}$ constituted from choosing a finite number of values $(\tilde{y}_i^n)_{i=1}^n$ of $Y$, denoting $M^{n+1}$ its inverse, then for each row $i$, $\sum_j |M_{i,j}^{n+1}|$ can be bounded from above by a finite value that does not depend[14] on $(\tilde{y}_i^n)_{i=1}^n$ or $n$. Let us denote this bound as $R$. Then, the norm of all eigenvalues of $M^{n+1}$ must be bounded by $R$ (this is a trivial implication of Gershgorin Circle Theorem).

[14]This claim is checked by writing in closed form the expression of $M^{n+1}$; the complete expression is provided in the supplemental section.

Denote $Z_0^n = \{0, \ldots, 0, 1\}'$, a $(n+1) \times 1$ column vector with only zeros but for the last component, and $X_0^n$, a $(n+1) \times 1$ column vector where the first $n$ components correspond to the completely mixed solution over $(\tilde{y}_i^n)_{i=1}^n$ and the $(n+1)^{th}$ component is the profit achieved in equilibrium. Then, $X_0^n = M^{n+1} Z_0^n$ (for notational simplicity, we do not explicitly index $X_0^n$ and $M^{n+1}$ by the current support when there is no possible confusion). Consider another set of vectors $(X, Z)$ such that $X = M^{n+1} Z$. Writing the singular value decomposition of $M^{n+1}$: $M^{n+1} = UDV'$, where $U$ and $V'$ are unitary matrices and $D$ is zero but for the singular values of $M^{n+1}$ on the diagonal. Then: $X_0^n - X = UDV'(Z_0 - Z)$. Necessarily then, for any $\eta > 0$, $\{M^{n+1} y / y \in \mathcal{B}(Z_0, \eta)\} \subset \mathcal{B}(X_0^n, R\, \eta)$, where $Z_0 = (0, \ldots, 0, 1)'$. In other words, if a distribution nearly satisfies the equilibrium Equations, then it must be arbitrarily close to the solution of $M^{n+1}$.

We prove first that the limit distribution $F_s$ does not depend on the choice of the sets $(Y^n)$. Suppose by contradiction that there exists a valid sequence of sets $(\tilde{Y}_n)$ such that the corresponding sequence $\tilde{F}_s^n$ converges to $\tilde{F}_s$ and there exists $x_0$ such that $F_s(x_0) \neq \tilde{F}_s(x_0)$. Let us denote $\Pi_b^F(x)$ (or $\Pi_b^F$ when there is no ambiguity), the profit to the buyer bidding $x$ when the seller follows $F$. For any $\epsilon > 0$, there exists $n_0$ such that for any $n'$ and $n''$ greater than $n_0$:

1. $|\Pi_b^{F_s^{n'}}(x) - \Pi_b^{F_s}| < \epsilon$, for any $x \in Y$; 2. $|F_s^{n'}(x_0) - \tilde{F}_s^{n''}(x_0)| > |F_b(x_0) - \tilde{F}_b(x_0)|/2$

Let us consider $n_1$ defined as an integer greater than $n_0$ and sufficiently large so that $Y^{n_0} \subset \tilde{Y}^{n_1}$. Consider the distribution $\tilde{F}_s^{n_1}$ and select the vector $\tilde{Q} = (\tilde{q}_1, \ldots, \tilde{q}_{n_0}, 0)'$, where $\tilde{q}_i$ is the probability (in $\tilde{F}_s^{n_1}$) that the seller plays the $i^{th}$ value of $Y^{n_0}$. Denote $\tilde{M}^{n+1}$ the matrix associated to the support $\tilde{Y}^{n_1}$ and $\tilde{X}_0 = M^{n+1} Z_0^n$. Then, (from 1.) $M_{n+1} \tilde{Q} \in \mathcal{B}(\tilde{X}_0, \epsilon)\}$ therefore $X \in \mathcal{B}(X_0^n, R\, \epsilon)$. Choosing $\epsilon = |F_s(x_0) - \tilde{F}_s(x_0)|/(4R)$, $X \in \mathcal{B}(X_0^n, |F_s(x_0) - \tilde{F}_s(x_0)|/4)$. Necessarily, $|F_s^{n_1}(x_0) - \tilde{F}_s^{n_0}(x_0)| < |F_s(x_0) - \tilde{F}_s(x_0)|/2$ which is a contradiction to 2. .

Then, we show that any other distribution that makes the buyer indifferent (even if it cannot be obtained as a limit of $(F_s^n)$) must be equal to $F_s$. Consider now an equilibrium distribution $G$ that yields $\Pi_b^G$ to the buyer. And denote $F^\epsilon$ a step function with steps over $(y_i^\epsilon)_{i=1}^{N(\epsilon)}$ chosen in $Y$ and that converges to $G$ when $\epsilon$ goes to zero. Then, for any $\eta > 0$, there exists $\epsilon > 0$ such that for all $0 < \epsilon' < \epsilon$, for all $y_i \in Y$, $|\Pi_b^{F^\epsilon}(y_i) - \Pi_b^G| < \eta$ and $|F_b^\epsilon(y_i) - G(y_i)| < \eta$. Considering the matrix $\overline{M}^{N(\epsilon)+1}$ associated to the steps of $F^\epsilon$ and denoting $Y^\epsilon \equiv (\Pi_b^{F^\epsilon}(y_1^\epsilon) - \Pi_b^G, \ldots, \Pi_b^{F^\epsilon}(y_{N(\epsilon)}^\epsilon) - \Pi_b^G, 1)'$, we know that $Y^\epsilon \in \mathcal{B}(Z_0^{N(\epsilon)}, \eta)$, thus $(q_1^\epsilon, \ldots, q_{N(\epsilon)}^\epsilon, \Pi_b^G)' \in \mathcal{B}(\overline{M}^{N(\epsilon)+1} Z_0^{N(\epsilon)}, R\eta)$, where $q_i^\epsilon$ is the probability that $F^\epsilon$ assigns to its ith term. Therefore the limit of the $F^\epsilon$ must be $F_s$ and thus $G$ and $F_s$ must coincide.

*Q.E.D.*

PROOF OF PROPOSITION 2.3: First, let us define $(\tilde{y}_i)_{i=1}^\infty$, a sequence dense in $[\underline{\theta}, \overline{\theta}]$. To construct an equilibrium, consider a sequence of sets $((\underline{y}_i)_{i=1}^n, (\tilde{y}_i)_{i=1}^n, (\overline{y}_i)_{i=1}^n)_{n=1}^\infty$, such that $\underline{y}_i)_{i=1}^n$ is increasing and $(\overline{y}_i)_{i=1}^n$ is decreasing.

First, Equation (8) is immediate and characterizes the distribution $F_s$ over $(\overline{\theta}, \overline{y}_1]$. Consider now the interval $[\underline{\theta}, \overline{\theta}]$. For any $y$ in this interval, we define as before $j^n(y) = \max(j / y_j^n \leq y)$. The finite difference derivative of $F_s^n$ at $I(x, n)$ will be:

$$\frac{F_s^n(y^n_{j^n(y)+1}) - F_s^n(y^n_{j^n(y)})}{y^n_{j^n(y)+1} - y^n_{j^n(y)}} = \frac{kF_s^n(y^n_{j^n(y)})}{V_b - y^n_{j^n(y)+1}}$$

Taking the limit over this equation yields a differential equations of the same functional form as in Lemma 2.1: $F_s(x) = C_s/(V_b - x)^k$. Equations (11) and (6) follow readily (up to a constant). We now need to verify that the limit function is a distribution.

To ensure continuity at $\overline{\theta}$, the constant must be chosen:

$$C_s = (V_b - \overline{\theta})^k \prod_{i=1}^{\infty} \frac{V_b - \overline{y}_i}{V_b - k\overline{y}_i - (1-k)\overline{y}_{i+1}}$$

Reinjecting $C_s$ into $F_s(x)$, we obtain Equation (11). Solving for $F_s(\underline{\theta})$ and reinjecting, Equation (6) follows. To conclude, it is necessary to verify that $F_s(\overline{\theta})$ is not zero (or else the limit function will not be a distribution). To do so, we write:

$$\ln(F_s(\overline{\theta})) = \sum_{i=1}^{\infty} \ln\left(\frac{V_b - \overline{y}_i}{V_b - k\overline{y}_{i+1} - (1-k)\overline{y}_i}\right)$$

A Taylor expansion yields that: $\ln\left(\frac{V_b - \overline{y}_i}{V_b - k\overline{y}_i - (1-k)\overline{y}_{i+1}}\right) = \frac{1-k}{V_b - \overline{\theta}}(\overline{y}_i - \overline{y}_{i+1}) + o(\overline{y}_i - \overline{y}_{i+1})$.

But the right-hand side term corresponds to a convergent series therefore the initial series must also be convergent. This argument shows that any sequence $(\overline{y}_i)_{i=1}^{\infty}$ can be used to sustain an equilibrium.

*Q.E.D.*

PROOF OF PROPOSITION 3.1: (i) If $n = 1$, bidding $y^1$ is always a best response for the buyer for all $V_b$. Conversely, suppose that $n \geq 2$, then from Proposition 2.2:

$$F_s(y^n_{n-1})/F_s(y^n_n) = (V_b - y^n_n)/(V_b - ky^n_{n-1} - (1-k)y^n_n)$$

This expression yields that: $V_b = y^n_n + kF_s(y^n_{n-1})(y^n_n - y^n_{n-1})/(F_s(y^n_n) - F_s(y^n_{n-1}))$.

(ii) The case when $n = 1$ is obvious. Suppose that a distribution is given by Proposition 2.2 and is such that $n = 2$. Denote $V_b$ the valuation given by the previous Equation. Observe that with complete information $\Pi_b(x)$ is linear in $V_b$ with slope $F_s(x)$. Therefore, with asymmetric information, if $V_b^j > V_b$ (resp. $V_b^j < V_b$), the optimum for the buyer will be $y^n_n$ (resp. $y^n_1$). Therefore, given $F_s$, the support of a pure strategy with asymmetric information may have at most three point. Conversely, notice that if $n = 3$, we can choose the following asymmetric information game: $V_b^1 = V_b/2$ with probability $b_1 = F_b(y^3_1)$;$V_b^2 = V_b$ with probability $b_2 = F_b(y^3_2) - F_b(y^3_1)$;$V_b^3 = 3V_b/2$ with probability $b_3 = 1 - F_b(y^3_2)$.

*Q.E.D.*

PROOF OF COROLLARY 3.1: As in Proposition 3.1, denote $V_b$ the unique solution to the complete information game. Denote $Y^{-2}$, the set $Y$ when excluding its maximum and minimum. Then, values in $Y^{-2}$

must necessarily be played when a valuation of $V_b$ is drawn in the asymmetric information game. Let us first consider an asymmetric information game with only three possible valuations. The asymmetric information game must be constructed as follows.

(i) $V_b^1 < V_b$, $V_b^2 = V_b$ and $V_b^3 > V_b$

(ii) $b_1 = \epsilon_b^-$, $b_3^n = b_1^1 = 1$, $b_3 = \epsilon_b^+$

(iii) $b_2 = 1 - \epsilon_b^+ - \epsilon_b^-$, $b_2^i = p_i/(1 - \epsilon_b^+ - \epsilon_b^-) \qquad \forall i \in [2, n-1]$

(iv) $b_2^n = (p_n - \epsilon_b^+)/(1 - \epsilon_b^+ - \epsilon_b^-)$, $b_2^1 = (p_1 - \epsilon_b^-)/(1 - \epsilon_b^+ - \epsilon_b^-)$

The restrictions follow from the fact that probabilities must lie in $[0, 1]$. The same argument is extended easily when more than three different valuations can be drawn.

*Q.E.D.*

PROOF OF PROPOSITION 3.2: (i) Consider first the case when $Y = \{y_1, y_2\}$. Solving Equations (12) and (13), we obtain two possible solutions:

$$\underline{q}_1 = (\overline{Q} - \sqrt{\Delta})/(2(y_2 - y_1)(1-k)\beta_b)$$

$$\overline{q}_1 = (\overline{Q} + \sqrt{\Delta})/(2(y_2 - y_1)(1-k)\beta_b))$$

where:

$$\begin{aligned}
\overline{Q} &= (k + \beta - k\beta)(y_2 - y_1) + (1-\beta)(V_b - y_2) \\
\Delta &= 4(y_2 - y_1)(1-k)k\beta_b + \underline{Q} \\
\underline{Q} &= V_b(1-\beta_b) + y_2(-1 + k + k\beta_b) + y_1(-k + \beta_b - k\beta_b)
\end{aligned}$$

First, we show existence. Notice first that $\underline{q}_1$ cannot be zero when $\beta_b \in (0, 1)$ and from Proposition 2.2, it is non-zero when $\beta_b = 0$. By continuity, $\underline{q}_1 > 0$ for $0 \leq \beta_b < 1$. Next,

$$1 - \underline{q}_1 = (-\underline{Q} + \sqrt{\Delta})/(2(y_2 - y_1)(1-k)\beta_b)$$

This term is also positive since $\Delta$ dominates $\underline{Q}$. Second, we show uniqueness.

$$1 - \overline{q}_1 = (-\underline{Q} - \sqrt{\Delta})/(2(y_2 - y_1)(1-k)\beta_b))$$

From similar arguments, this term is negative.

*Q.E.D.*

PROOF OF LEMMA 3.1: $\forall i \in [1, n]$,

$$\Pi_b = \sum_{j=1}^{i} q_j (V_b - k y_i - (1-k) y_j) + (1 - F_s(y_i)) \beta_b \Pi_b$$

First-differencing and rearranging, $\forall i \in [2, n]$,

$$q_i = \frac{k F_s(y_i)(y_i - y_{i-1})}{V_b - (1-k) y_i - k y_{i-1} - \beta_b \Pi_b}$$

$$\Pi_b = q_1 (V_b - y_1)/(1 - (1 - q_1)\beta_b)$$

Substituting $\Pi_b$ yields the result.

*Q.E.D.*

PROOF OF PROPOSITION 3.3: Suppose that Y has n terms. Let us consider the transformation $\mathcal{T}$ from $\Delta^{n-1}$ (the (n-1)-dimensional simplex) to $\mathbb{R}^n$ defined as follows, for a given $Q = (q_1, \ldots, q_n)$, $\mathcal{T}(Q)_1 = 1 - \sum_{i=2}^{n} q_i$ and $\forall i \in [2, n]$,

$$\mathcal{T}(Q)_i = \frac{k \sum_{j=1}^{i} q_j (y_i - y_{i-1})}{V_b^{\beta_b} - (k y_{i-1} + (1-k) y_i)}$$

Note first that $V_b^{\beta_b} \geq (1 - \beta_b) V_b + \beta_b y_1$. Then, under the Assumptions of the Proposition, for all $i \in [2, n]$, $\mathcal{T}(Q)_i \geq 0$ and $\mathcal{T}(Q)_1 \geq 0$. Therefore: $\mathcal{T}(Q) \in \Delta^{n-1}$. Further, $\mathcal{T}$ is also continuous therefore by Brouwer's fixed point theorem, $\mathcal{T}$ has a fixed point and by Lemma 12, this fixed point is a solution.

Let us define $\tilde{\mathcal{T}}$ as, for all $Q$, $\tilde{\mathcal{T}}(Q) = Q - \mathcal{T}(Q)$. To show the second part of the claim, we apply the implicit function theorem on $\tilde{\mathcal{T}}$ at a neighborhood of $(\beta_b, \beta_s) = (0, 0)$. To verify the existence of a solution on some neighborhood $(\beta_b, \beta_s) \in [0, \epsilon(k, Y))^2$, we need to show that the matrix $D_Q \tilde{\mathcal{T}}$ is invertible at $(q_1^*, \ldots, q_n^*)$ (given by Equation (4)) and $(\beta_b, \beta_s) = (0, 0)$. This matrix is denoted $\tilde{M}$ with components $(\tilde{M}_{i,j})_{i,j}$.

$$\begin{array}{rcl} \tilde{M}_{1,j} = & 1 & \text{For } 1 \leq j \leq n \\ \tilde{M}_{i,j} = & \frac{-k(y_i - y_{i-1})}{V_b - k y_{i-1} - (1-k) y_i} & \text{For } 1 \leq j < i \leq n \\ \tilde{M}_{i,i} = & \frac{V_b - y_i}{V_b - k y_{i-1} - (1-k) y_i} & \text{For } 2 \leq i \leq n \\ \tilde{M}_{i,j} = & 0 & \text{For } 2 \leq i < j \leq n \end{array}$$

Then: $\text{Det}(\tilde{M}) = \prod_{i=1}^{n-1} \frac{V_b - k y_i - (1-k) y_{i+1}}{V_b - k y_{i+1} - (1-k) y_i} > 0$.

*Q.E.D.*

PROOF OF PROPOSITION 3.4: Writing the equilibrium conditions for the buyer:

$$\begin{pmatrix} U_b(V_b-y_1) & 0 & \dots & & 0 & -1 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & \\ m_{i,j} & & \dots & & U_b(V_b-y_n) & -1 \\ 1 & & \dots & & 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ \\ q_n \\ \Pi_b^* \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \\ 0 \\ 1 \end{pmatrix}$$

where $m_{i,j} = kU_b(V_b - y_i) + (1-k)U_b(V_b - y_j)$. Therefore, defining $y_i^b = V_b - U_b(V_b - y_i)$, the result is an application of Proposition 2.2.

*Q.E.D.*

PROOF OF COROLLARY 3.2: Suppose that $\forall x \in [V_s, V_b]$, $-\underline{U}_b''(x) > -\overline{U}_b''(x)$. Under the normalization $\underline{U}_b'(0) = \overline{U}_b'(0) = 1$, the utility function $\underline{U}$ corresponds to greater absolute risk-aversion than $\overline{U}$. Necessarily:

$$\frac{\overline{U}_b(V_b - y_{i+1})}{k\overline{U}_b(V_b - y_i) + (1-k)\overline{U}_b(V_b - y_{i+1})} < \frac{\underline{U}_b(V_b - y_{i+1})}{k\underline{U}_b(V_b - y_i) + (1-k)\underline{U}_b(V_b - y_{i+1})}$$

Then necessarily, $\overline{F}_s(y_i) < \underline{F}_s(y_i)$ for all $k < n$, where $\overline{F}_s$ (resp. $\underline{F}_s$) corresponds to the mixed strategy associated to $\overline{U}_b$ (resp. $\overline{F}_s$).

*Q.E.D.*

PROOF OF PROPOSITION 3.5: To show existence, consider the following transformation $\mathcal{D}$: $\mathcal{D}(\Pi) = \Pi^*$, where $\Pi^*$ is the profit that is achieved by the buyer with an expected utility defined such that:

$$\overline{U}_b(x) \equiv U_b(x) - \rho(U_b^{-1}(\Pi) - x)$$

We will define this transformation on $[U_b(0) - \rho(U_b(V_b - y_1^n)), U_b(V_b - y_1)]$. It must also necessarily have values on this interval. A fixed point of $\mathcal{D}$ is an equilibrium for the disappointment aversion case. Now, from Proposition 3.4, $\mathcal{D}$ must be continuous. Applying Brouwer's fixed point theorem, the result follows.

*Q.E.D.*

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