

# A New Weight Scheme for the Shapley Value\*

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## Abstract

It is well known since Owen (1968) that the weights in the weighted Shapley value cannot be interpreted as a measure of power (i.e. of the ability to bargain) of the players. This paper proposes a new weight scheme for the Shapley value. Weights in this framework have to be interpreted as a measure of bargaining power. Two different axiomatic characterizations of this new value are proposed: one including the weights in the axioms and one without.

**KEYWORDS** Shapley value, monotonicity, weights.

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## 1 INTRODUCTION

A cooperative game for some population of individuals describe the amount available to any subgroup of players. This field is sustained by two issues: the division problem and the formation of coalitions. This paper is focused to the first issue only. One of the most known solution concept in cooperative games is the Shapley value (Shapley 1953b). This value is based on four axioms: efficiency, dummy player, symmetry and additivity.<sup>1</sup> In his Ph.D. thesis, Shapley extended his value to the non symmetric case. His motivation is as follows:

“It is easy to imagine games or or game like situations in which the symmetry assumption is not appropriate, because of differences in the *external* characteristics of the players. For example [ . . . ], there might be differences in bargaining ability.”

This ability was introduced by Shapley by means of weights. To each player is assigned a weight, and the weighted Shapley value of some player in a unanimity game  $u_S$  is the relative weight of that player in that coalition. Using the mathematical properties of the Shapley value, this weighted value was defined for any game. Nevertheless, Owen (1968) brightly showed that weights cannot be interpreted as a measure of power, but rather as a measure of slowness to reach the grand coalition.

This paper proposes another way to introduce weights in the Shapley value. Weights in this framework have to be interpreted as a measure of bargaining power. Our value is strongly based on the decomposition of games over the basis of unanimity games. Indeed, as shown by Shapley, any game can be decomposed as a linear combination of unanimity games. We argue that the coefficients of this decomposition provide some information about the relationships between the coalitions (and also the players) in the game.

Consider any game  $v$ . Rewrite this game in terms of a linear combination of unanimity games. To each coalition  $S$ , subset of the grand coalition, corresponds then a unanimity game  $u_S$  and a coefficient  $\alpha_S$ . This coefficient has been called by Harsanyi (1963) the ‘dividend of the game’ for the coalition  $S$ .  $\alpha_S$  is the part in  $v(N)$  (the worth of the grand coalition) due to the formation of the subcoalition  $S$ . According to the Shapley value, players in  $S$  have to share this amount  $\alpha_S$ . But this latter can be either positive or negative. If positive, then each player will receive a positive amount of money. Conversely, if  $\alpha_S$  is negative, then each player will have to “pay,” i.e. each player will receive a negative amount of wealth. It is our contention that the sharing rule cannot be the same for all cases ( $\alpha_S$  positive or negative). Indeed, when the coalition  $S$  wil have to share a positive amount of

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<sup>1</sup>Shapley originally proposed only three axioms: efficiency and dummy player were replaced by the carrier axiom

money, every player will try to maximize its profit. But when  $\alpha_S < 0$ , the players will try to minimize their losses, i.e.: to pay as less as possible. The modification we propose for the weight scheme is that for “negative” games, the order of strengths (i.e. of weights) must be inverted.

The paper is organized as follow. In section 2, we introduce the definitions and notations used. We then provide in section 3 a complete characterization of our modified weighted Shapley value. Several axioms are introduced, and two existence theorems are stated: one including the weights in the axioms, and one without. Section 4 studies the monotonicity properties of our new value and section 5 is devoted to the proofs of the theorems.

## 2 DEFINITIONS

A game in characteristic function form is defined by a couple  $(N, v)$  where  $N$  is the set of players,  $N = \{1, \dots, n\}$ , and  $v$  is a map from the power set of  $N$  to the real line:  $v(S)$  designs the worth of the coalition  $S \in 2^N$ , with  $v(\emptyset) \equiv 0$ . The space of all games with player set  $N$  is denoted by  $\Gamma_N$ . Throughout the paper, capital latin letters will denote coalitions and their corresponding lower case will denote their cardinality (except  $v$  which denotes the characteristic function):  $S$  is a coalition of size  $s$ . For convenience, we shall write  $i$  instead  $\{i\}$ . Set inclusion is supposed to be strict:  $T \subset S$  means that  $T$  is a subset of  $S$  and that  $T \neq S$ . The sum of two games  $v$  and  $v'$  is defined by  $(v + v')(S) = v(S) + v'(S)$  and the multiplication by a scalar  $\lambda$  is defined by  $(\lambda v)(S) = \lambda v(S)$ . A unanimity game  $u_S$  is a game such that  $u_S(T) = 1$  if  $T \supseteq S$  and  $u_S(T) = 0$  otherwise. It is well known that the family of games  $(u_S)_{S \subseteq N}$  forms a basis for  $\Gamma_N$ . Thus, any game can be decomposed as a linear combination of unanimity games:  $v = \sum_{S \subseteq N} \alpha_S \cdot u_S$ , where  $\alpha_S = \sum_{T \subseteq S} (-1)^{s-t} v(T)$ .  $\alpha_S$  is called the dividend of the game (see Harsanyi). A solution is a map  $\phi : \Gamma_N \mapsto \mathbb{R}^n$ , and its weighted counterpart is a map  $\phi^\omega : \Gamma_N \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ .  $\phi_i(v)$  is the payoff of player  $i$  in the game  $v$  with the solution  $\phi$ . We shall now introduce some definitions.

**DEFINITION 1** *A game is positive (negative) iff  $\alpha_S \geq 0$  ( $\leq 0$ ) for all  $S \subseteq N$ . A game is sign oriented if it is either positive or negative.*

**DEFINITION 2** *Two games  $v$  and  $v'$  are comparable iff the coefficients of their decomposition in unanimity games have all the same sign, i.e.  $\alpha_S^v \cdot \alpha_S^{v'} \geq 0, \forall S$ .*

The Shapley value (Shapley 1953b) is the the solution  $\varphi$  defined by

$$\varphi_i(v) = \sum_{S \ni i} \alpha_S \cdot \varphi_i(u_S(S)) = \sum_{S \ni i} \alpha_S \cdot \frac{1}{s}, \quad \forall i \in N \quad (1)$$

The weighted Shapley value is defined as follow (Shapley 1953a). For each unanimity games  $u_S$  the weighted Shapley value  $\varphi^\omega$  is:

$$\varphi_i^\omega(u_S) = \begin{cases} \frac{\omega_i}{\omega_S} & \text{if } i \in S \\ 0 & \text{else} \end{cases} \quad (2)$$

where  $\omega_S = \sum_{j \in S} \omega_j$ . If  $\omega_i = \omega_j, \forall i, i \in N$ , then  $\varphi^\omega = \varphi$ .

DEFINITION 3  $\varphi^\omega$  is monotonic with respect to the weights if and only if for any game  $v$ , for all player  $i \in N$ , one of this two statements is true:

1.  $\frac{\partial \varphi_i^\omega(v)}{\partial \omega_i} \geq 0, \quad \forall \omega_i \in \mathbb{R}_+,$
2.  $\frac{\partial \varphi_i^\omega(v)}{\partial \omega_i} \leq 0, \quad \forall \omega_i \in \mathbb{R}_+,$

### 3 MODIFIED WEIGHTED SHAPLEY VALUE

Let  $\omega$  be a weight vector:  $\omega = (\omega_i)_{i \in N}$ , with  $\omega_i > 0, \forall i \in N$ . These weights describe external characteristics of the players. We define  $\omega^+$  and  $\omega^-$  as:

$$\omega^+ = (\omega_1, \dots, \omega_n) \quad (3)$$

$$\omega^- = \left( \frac{1}{\omega_1}, \dots, \frac{1}{\omega_n} \right) \quad (3')$$

The modified weighted Shapley value is defined as the solution  $\psi$  such that:

$$\begin{aligned} \psi_i^\omega(v) &= \sum_{S \ni i} |\alpha_S| \cdot \psi^\omega(\text{sgn}(\alpha_S) \cdot u_S) = \sum_{S \ni i} \alpha_S \cdot \omega_{i/S}^v, \\ \text{whith } \omega_{i/S}^v &= \begin{cases} \frac{\omega_i^+}{\sum_{j \in S} \omega_j^+} & \text{if } \alpha_S \geq 0, \\ \frac{\omega_i^-}{\sum_{j \in S} \omega_j^-} & \text{if } \alpha_S < 0, \end{cases} \end{aligned} \quad (4)$$

where  $\text{sgn}(\alpha_S)$  is the sign of  $\alpha_S$ . Any game can be decomposed in a linear combination of unanimity games. In some way, this decomposition reflects the attractions and the antagonisms in the game.  $\alpha_S$  positive (negative) means that the formation of the coalition  $S$  is beneficial (penalizing) for the grand coalition. In other words, if  $\alpha_S > 0$  ( $\alpha_S < 0$ ), then coalition  $S$  contributes positively (negatively) to  $v(N)$ . Players have then to share the benefits of the game as well as the losses. For beneficial situations, each player will try to maximize her share, whereas for the other situations, the converse will hold. We now introduce several axioms. Some of these axioms are labeled “semi.” This means that the property they refer to is restricted to games with some properties of regularity.

AXIOM 1 (EFFICIENCY)  $\sum_{i \in N} \phi_i(v) = v(N)$ .

AXIOM 2 (DUMMY PLAYER) *If  $i$  is a dummy player (i.e.,  $v(S \cup i) = v(S)$ ,  $\forall S \subseteq N$ ) then  $\phi_i(v) = 0$ .*

AXIOM 3 (SEMI-ADDITIVITY) *if  $v$  and  $w$  are comparable, then  $\phi(v + w) = \phi(v) + \phi(w)$ .*

AXIOM 4 (SEMI- $\omega$ -SYMMETRY) *If  $i$  and  $j$  are symmetric players (i.e.,  $v(S \cup i) = v(S \cup j)$ ,  $\forall S \subseteq N \setminus \{i, j\}$ ) and  $v$  is sign oriented, then  $\omega_j \phi_i(v) = \omega_i \phi_j(v)$ .*

AXIOM 5 (POWER INVERSION)  $\phi_i(u_S)/\phi_j(u_S) = \phi_j(-u_S)/\phi_i(-u_S)$ ,  $\forall i, j \in S$ .

Axiom 1 and 2 are usual. The main argument for axiom 3 and axiom 4 is that a comparison between players or games can be done only when the games exhibit some kind of regularity conditions. Two different games may not have the same decomposition, and then will not treat the coalitions (and then the players) equally likely. Axiom 3 is a weaker version of the additivity axiom. This latter states that two situations (i.e. two games) can be analyzed separatly. Axiom 3 narrows this to the case when the games exhibit the same characteristics about the antagonisms between the players. For instance, in some game  $v$  forming coalition  $S$  is penalizing ( $\alpha_S^v \leq 0$ ), and for another game  $v'$  the formation of same coalition is beneficial ( $\alpha_S^{v'} > 0$ ). Obviously, the game  $w = v + v'$  will hide some of the characteristics of the coalition  $S$ . Likewise, semi- $\omega$ -symmetry is the usual  $\omega$ -symmetry, but applied only when the game is sign oriented. The last axiom is new. For a unanimity game, the relative strength of one player with respect to another player will be inverted if the sign of the game changes. This captures our main idea. Players with a high ability to bargain will obtain the highest share when the game is positive. Conversely, if the game is negative, these players will get the lowest payoof (in absolute value). Thus, the order of shares for a negative game as to be the inverse of the order of shares for positive game. In a unanimity game, all the non dummy players are symmetric, and their shares are precisely their relative weights. Thus, weights in negative games shall be the inverse of weights in positive games.

THEOREM 1 *A solution  $\phi$  with a weighth vector  $\omega$  satisfies the efficiency, dummy player, semi-additivity, semi- $\omega$ -symmetry and power inversion axioms if and only if  $\phi$  is the modified weighted shapley  $\psi^\omega$ .*

AXIOM 6 (SEMI-GENERALIZED SYMMETRY) *If  $S$  is a coalition where all players in  $S$  are symmetric in  $v$  and  $w$ , and such that  $v$  and  $w$  are both positive (negative), then for all  $i \in S$ :*

$$\frac{\phi_i(v)}{\sum_{j \in S} \phi_j(v)} = \frac{\phi_i(w)}{\sum_{j \in S} \phi_j(w)}.$$

Semi-generalized symmetry simply states that for any game in which two players are symmetric, then their relative payoffs between each other is unchanged.

**THEOREM 2** *A solution  $\phi$  satisfies the efficiency, dummy player, semi-additivity, power inversion and semi-generalized symmetry axioms if and only if there exists a weight vector  $\omega$  such that  $\phi$  is the modified weighted Shapley value  $\psi^\omega$ .*

#### 4 MONOTONICITY

Consider the weighted Shaley value of any player of some game  $(N, v)$ . We say that the value is monotonic if an increase of the weight of some player induces an increase in her value, given that the oponents' weights unchanged. Consider the following 3 players game:  $v(i) = 0, \forall i, v(12) = 12, v(13) = v(23) = 6$  and  $v(123) = 12$ . The decomposition in unanimity games is:

$$v(S) = 12 \cdot u_{(12)}(S) + 6 \cdot u_{(13)}(S) + 6 \cdot u_{(23)}(S) - 12 \cdot u_{(123)}(S)$$

The weighted Shapley value with weight vector  $\omega = (1, 1, 1)$  is  $\varphi^\omega(v) = (5, 5, 2)$ . With  $\omega' = (1, 1, 3)$ , one obtains:  $\varphi^{\omega'}(v) = (5.1, 5.1, 1.8)$ . Clearly, the weighted Shapley value may not be monotonic, although the game is. This result has been pointed out by Owen (1968).

On the contrary, our new weight scheme makes the weighted Shapley value beeing always monotonic, even for non-monotonic games. Yet, it is worth to point out that negative values can be obtained. Consider the same game as above, with  $\omega'' = (1, 2, 3)$ . The modified weighted Shapley value is:  $\psi^{\omega''}(v) \approx (-1.05, 7.13, 5.92)$ .

#### 5 PROOFS OF THE THEOREMS

**LEMMA 1**

*Let  $i$  be a dummy player in the game  $v$ . Then  $\alpha_S = 0, \forall S \ni i$ .<sup>2</sup>*

**Proof** Recall that if  $i$  is dummy, then  $v(S \cup i) = v(S) \forall S \subseteq N \setminus i$ . We

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<sup>2</sup>This lemma has already been stated by Kalai and Samet. However, their proof is different. They use an induction principle. They show that for a dummy player  $i$ , if  $\alpha_S = 0$  for all  $s \leq k$ , then  $\alpha_S = 0$  for  $s = k + 1$ , for all  $S \ni i$ .

have:

$$\begin{aligned}
 \alpha_S &= v(S) - \sum_{T \subset S} (-1)^{s-t} v(T) \\
 &= v(S \setminus i) - \sum_{\substack{T \subset S \\ T|t=s-1}} v(T) + \sum_{\substack{T \subset S \\ T|t=s-2}} v(T) \dots - \sum_{j \in S} v(j) \\
 &= v(S \setminus i) - \sum_{\substack{T \subset S \\ T|t=s-1 \\ i \notin T}} v(T) - \sum_{\substack{T \subset S \\ T|t=s-1 \\ i \in T}} v(T) + \sum_{\substack{T \subset S \\ T|t=s-2 \\ i \notin T}} v(T) + \sum_{\substack{T \subset S \\ T|t=s-2 \\ i \in T}} v(T) - \dots \\
 &= v(S \setminus i) - v(S \setminus i) - \sum_{\substack{T \subset S \\ T|t=s-2 \\ i \notin T}} v(T) + \sum_{\substack{T \subset S \\ T|t=s-2 \\ i \notin T}} v(T) + \sum_{\substack{T \subset S \\ T|t=s-3 \\ i \notin T}} v(T) \\
 &\quad - \sum_{\substack{T \subset S \\ T|t=s-3 \\ i \notin T}} v(T) + \dots = 0
 \end{aligned}$$

□

LEMMA 2

Let  $\phi$  be a solution that satisfies the dummy player, efficiency and either semi-generalized symmetry or semi- $\omega$ -symmetry axioms. Then  $\phi(\alpha_S u_S) = |\alpha_S| \phi(\text{sgn}(\alpha_S) u_S)$

**Proof** First note that if  $\phi$  with a weight vector  $\omega$  satisfies the semi- $\omega$ -symmetry axiom, then  $\phi$  also satisfies the semi-generalized symmetry axiom. Thus, it suffices to prove the case for the semi-generalized symmetry axiom. Suppose that  $\alpha_S$  is positive. Take any player  $i$  and  $j$  in  $S$  (The proof for  $s = 1$  is obvious). By semi-generalized symmetry, we have:

$$\frac{\phi_i(u_S)}{\phi_i(\alpha_S u_S)} = \frac{\phi_j(u_S)}{\phi_j(\alpha_S u_S)}, \text{ which yields } \phi_i(u_S) = \phi_i(\alpha_S u_S) \frac{\phi_j(u_S)}{\phi_j(\alpha_S u_S)}. \text{ Sum}$$

over  $i$ . By efficiency, we get  $1 = \alpha_S \frac{\phi_j(u_S)}{\phi_j(\alpha_S u_S)} \Rightarrow \phi_j(\alpha_S u_S) = \alpha_S \phi_j(u_S)$ .

If  $\alpha_S < 0$ , then define  $\beta_S := -\alpha_S$ . This yields:  $\phi(\alpha_S u_S) = \phi(\beta_S - u_S) = \beta_S \phi(-u_S)$  because  $\beta_S > 0$ , which is tantamount to  $|\alpha_S| \phi(\text{sgn}(\alpha_S) u_S)$ . □

**Proof of theorem 1** We first show that  $\psi$  satisfies axioms 1–5. Efficiency follows from the definition of  $\psi$ :

$$\begin{aligned}
 \sum_{i \in N} \psi_i(v) &= \sum_{i \in N} \sum_{S \ni i} \alpha_S \frac{\omega_i}{\sum_{j \in S} \omega_j} = \sum_{S \subseteq N} \alpha_S \left( \sum_{i \in S} \frac{\omega_i}{\sum_{j \in S} \omega_j} \right) \\
 &= \sum_{S \subseteq N} \alpha_S u_S(S) = v(N).
 \end{aligned}$$

By lemma 1, if  $i$  is a dummy player, then  $\alpha_S = 0, \forall S \ni i$ . Then  $\psi_i(v) = 0$ . Take  $v$  and  $w$  such that  $v$  and  $w$  are comparable, then  $\alpha_S^{v+w}$  has the same

sign as  $\alpha_S^v$  and  $\alpha_S^w$ . Let  $N_+ = \{S \subseteq N \mid \alpha_S \geq 0\}$  and  $N_- = \{S \subseteq N \mid \alpha_S < 0\}$ .

$$\begin{aligned}
 \psi_i(v+w) &= \sum_{S \ni i} \alpha_S^{v+w} \cdot \omega_{i/S} \\
 &= \sum_{S \in N_+} \alpha_S^{v+w} \cdot \frac{\omega_i}{\sum_{j \in S} \omega_j} + \sum_{S \in N_-} \alpha_S^{v+w} \cdot \frac{\omega_i}{\sum_{j \in S} \omega_j} \\
 &= \sum_{S \in N_+} (\alpha_S^v + \alpha_S^w) \cdot \frac{\omega_i^+}{\sum_{j \in S} \omega_j^+} + \sum_{S \in N_-} (\alpha_S^v + \alpha_S^w) \cdot \frac{\omega_i^-}{\sum_{j \in S} \omega_j^-} \\
 &= \sum_{S \ni i} \alpha_S^v \omega_{i/S} + \sum_{S \ni i} \alpha_S^w \omega_{i/S} \\
 &= \psi_i(v) + \psi_i(w).
 \end{aligned}$$

To prove that  $\psi$  satisfies the semi- $\omega$ -symmetry, consider a game  $v$  such that  $v$  is positive (take  $-v$  if  $v$  is negative). Let  $i$  and  $j$  be symmetric players. Then

$$\forall S \subseteq N \setminus \{i, j\}, \alpha_{S \cup i} = \alpha_{S \cup j} \quad (5)$$

Thus,  $\sum_{S \ni i} \alpha_S \frac{\omega_i}{\sum_{j \in S} \omega_j} = \frac{1}{\omega_i} \psi_i(v)$ . Applying equation (5), this yields  $\frac{1}{\omega_i} \psi_i(v) = \frac{1}{\omega_j} \psi_j(v)$ . To show that  $\psi$  satisfies axiom 5, it suffices to show that  $\frac{\omega_i^+ / \sum_{k \in S} \omega_k^+}{\omega_j^+ / \sum_{k \in S} \omega_k^+} = \frac{\omega_j^- / \sum_{k \in S} \omega_k^-}{\omega_i^- / \sum_{k \in S} \omega_k^-}$ . But this is equivalent to  $\omega_i^+ / \omega_j^+ = \omega_j^- / \omega_i^-$ , which follows from equations 3 and 3'.

It remains to show that a solution satisfying axioms 1–5 is the modified weighted Shapley value  $\psi^\omega$ . Let  $\phi$  be a solution that satisfies these 5 axioms. Any game  $v$  can be decomposed in a linear combination of unanimity games:  $v = \sum_{S \subseteq N} \alpha_S u_S$ . Define  $v'$  and  $v''$  by:

$$\begin{aligned}
 v' &= \sum_{S \in N_+} \alpha_S u_S + \sum_{S \in N_-} 0 \cdot u_S \\
 v'' &= \sum_{S \in N_+} 0 \cdot u_S + \sum_{S \in N_-} \alpha_S u_S
 \end{aligned}$$

It is obvious that  $v'$  and  $v''$  are comparable. Moreover,  $v'$  ( $v''$ ) can be seen as a sum of  $n_+$  ( $n_-$ ) games:  $v' = \sum_{K \in N_+} v'_k$ , where  $v'_k = \sum_{R \subseteq N} \alpha_R u_R$  with  $\alpha_R = \alpha_S$  if  $R = S$  and 0 else. All the  $v'_k$  are comparable (they are all positive). Then  $\phi(v') = \sum_{S \in N_+} \phi(\alpha_S u_S)$ . The same property holds for  $v''$ . Because  $\phi$  satisfies the dummy player axiom, it suffices to sum over the sets containing  $i$ , as  $\phi_i(u_S) = 0$  if  $i \notin S$ . By lemma 2,  $\phi_i(v) = \sum_{S \ni i} \phi_i(\alpha_S^v u_S) = \sum_{S \ni i} |\alpha_S^v| \phi_i(\text{sgn}(\alpha_S) u_S)$ . It remains to show that  $\psi^\omega$  and  $\phi$  coincide on each unanimity game. It is obvious that for all  $S$ ,  $u_S$  is sign oriented. Suppose first that  $\alpha_S \geq 0$ . Then by  $\omega$ -symmetry, we have:

$$\omega_j \phi_i(u_S) = \omega_i \phi_j(u_S)$$

and

$$\omega_j \psi_i^\omega(u_S) = \omega_i \psi_j^\omega(u_S).$$

Divide the first equality by the second:

$$\frac{\phi_i(u_S)}{\psi_i^\omega(u_S)} = \frac{\phi_j(u_S)}{\psi_j^\omega(u_S)} = \lambda.$$

Notice that  $\lambda$  is constant,  $\forall i, j \in S$ . This implies that  $\phi_i(u_S) = \lambda \psi_i^\omega(u_S)$ . Summing over  $S$ , and because  $\psi$  satisfies the efficiency and dummy player axioms, we obtain:

$$\sum_{i \in S} \phi_i(u_S) = \lambda \cdot \sum_{i \in S} \psi_i^\omega(u_S) = 1.$$

Thus  $\lambda = 1$ , which implies that  $\phi_i(u_S) = \psi_i^\omega(u_S)$ .

Suppose now that  $\alpha_S < 0$ . Then, as both  $\psi^\omega$  and  $\phi$  satisfies the power inversion axiom we get:

$$\frac{\phi_i(u_S) \phi_i(-u_S)}{\psi_i^\omega(u_S) \psi_i^\omega(-u_S)} = \frac{\phi_j(u_S) \phi_j(-u_S)}{\psi_j^\omega(u_S) \psi_j^\omega(-u_S)} = \lambda.$$

Again, notice that  $\lambda$  is constant,  $\forall i, j \in S$ . Moreover, we know that  $\phi_i(u_S) = \psi_i^\omega(u_S)$ . Then, applying the same reasoning as before, we get  $\phi_i(-u_S) = \psi_i^\omega(-u_S)$ .

We now show that axiom 5 implies that there are two weight schemes, one for  $\alpha_S \geq 0$ , and one for  $\alpha_S < 0$ , such that  $\omega_i^+ = 1/\omega_i^-$ ,  $\forall i \in N$ . If  $\alpha_S < 0$ , then  $\alpha_S \phi(u_S) = \alpha_S \phi(-u_S)$ . By axiom 5 we have:

$$\frac{\phi_i(u_S)}{\phi_j(u_S)} = \frac{\phi_j(-u_S)}{\phi_i(-u_S)}$$

which implies, using efficiency:

$$\begin{aligned} \sum_{j \in N} \phi_j(-u_S) &= \phi_i(u_S) \phi_i(-u_S) \left( \sum_{j \in S} \frac{1}{\phi_j(u_S)} \right) = 1 \\ \Rightarrow \phi_i(-u_S) &= \frac{1/\phi_j(u_S)}{\sum_{i \in S} \phi_i(u_S)} \\ \Rightarrow \frac{\omega_i^-}{\sum_{j \in N} \omega_j^-} &= \frac{1/\omega_i^+}{\sum_{j \in N} 1/\omega_j^-} \end{aligned}$$

This holds for all  $i \in S$ , for all  $S$ . Then by identity we have  $\omega_i^- = 1/\omega_i^+$ . As  $S$  was arbitrary, this completes the proof.  $\square$

**Proof of theorem 2** By theorem 1, we only have to show that  $\psi$  satisfies the semi-generalized symmetry axiom. First notice that for all players  $j$  symmetric to player  $i$  we have the following property:

$$\psi_j(v) = \omega_j \sum_{S \ni i} \alpha_S \frac{1}{\omega_S} \quad (6)$$

Let  $T$  be the set of all players  $j$  symmetric to  $i$  (including  $i$  itself). It is obvious that property 6 holds for any subgame  $w$  of  $v$  relative to any subset of  $T$ , i.e.  $\psi_j(w) = \omega_j \sum_{S \ni i} \alpha_S \cdot (1/\omega_S)$ . But this implies that

$$\begin{aligned} \sum_{j \in T} \psi_j(v) &= \psi_i(v) \frac{1}{\omega_j} \sum_{j \in T} \omega_j \\ \Rightarrow \frac{\psi_i(v)}{\sum_{j \in T} \psi_j(v)} &= \frac{\omega_j}{\sum_{j \in T} \omega_j} \end{aligned}$$

But this calculus can be done for any other game  $w$  such that all the players in  $T$  are symmetric. This yields:  $\frac{\phi_i(v)}{\sum_{j \in T'} \phi_j(v)} = \frac{\omega_j}{\sum_{j \in T'} \omega_j} = \frac{\phi_i(w)}{\sum_{j \in T'} \phi_j(w)}$ .

It remains to prove that a solution  $\phi$  that satisfies the efficiency, dummy player, semi-additivity, power inversion, and semi-generalized symmetry axioms is the modified weighted Shapley value  $\psi^\omega$  for some weight vector  $\omega$ . Most of this part of the proof is similar to the proof of theorem 1, except for the proof that  $\psi^\omega$  and  $\phi$  coincide on positive unanimity games.<sup>3</sup> By the semi-generalized symmetry axiom applied to  $\psi^\omega$  and  $\phi$ , we get:

$$\phi_j(u_S) \phi_i(u_{(ij)}) = \lambda \cdot \psi_j^\omega(u_S) \psi_i^\omega(u_{(ij)})$$

$\forall i, j \in S$ . Again,  $\lambda$  is constant. Summing over  $i$ , using the efficiency and dummy player axioms, we get:

$$\phi_j(u_S) = \lambda \cdot \psi_j^\omega(u_S)$$

We now sum over  $j$ , which leads:

$$v(N) = \sum_{j \in N} \phi_j(v) = \lambda \cdot \sum_{j \in N} \psi_j^\omega(v) = \lambda \cdot v(N).$$

Which implies that  $\lambda = 1$ .

It remains now to show that there exists a weight vector  $\omega = (\omega_1, \dots, \omega_n)$  such that for all unanimity game  $u_S$ ,  $\phi_i(u_S) = \omega_i / (\sum_{j \in S} \omega_j)$ . Define  $\lambda_i := \phi_i(u_N)$ . Notice that any unanimity game  $u_N$  and  $u_S$  are both positive and that all players in coalition  $S$  are symmetric. Thus, by axiom 6, we have  $\phi_i(u_S) = \lambda_i / (\sum_{j \in S} \lambda_j)$ . Moreover, by efficiency,<sup>4</sup>  $\sum_{i \in N} \lambda_i = 1$ . This implies that there exist a vector  $\omega$  such that  $\lambda_i = \frac{\omega_i}{\sum_{j \in N} \omega_j}$ . We claim now that all the  $\lambda_i$  are positive. Suppose that  $\exists S$ , such that  $\lambda_i < 0, \forall i \in S$ . By efficiency,  $\sum_{i \in S} (\lambda_i / \lambda_S) = 1$ . This implies that there exists at least one  $j \in S$  such that  $\lambda_j > 0$ , which is a contradiction. This completes the proof.  $\square$

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<sup>3</sup>In the previous proof, we used the semi- $\omega$ -symmetry axiom

<sup>4</sup>Note that efficiency is only required for  $\phi(u_N)$  and not for the other unanimity games. Efficiency on these games is implied by the semi-generalized symmetry axiom.

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