Evolutionary Dynamics on Infinite Strategy Spaces*

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Abstract

The study of evolutionary dynamics was so far mainly restricted to finite strategy spaces. In this paper we show that this restriction is in most cases unnecessary. We give a mild condition under which the continuous time replicator dynamics are well defined for infinite strategy spaces. Furthermore, we provide conditions for the stability of rest points. Finally, we apply this general theory to a number of applications like the Nash demand game, the War of Attrition, Cournot and Bertrand oligopoly games, and mixed strategies.

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1 Introduction

The study of evolutionary game theory has focused on finite strategy spaces, especially when continuous time dynamics of an evolutionary game were involved (e.g. the replicator dynamics).[1] However, many economically relevant games have a continuous strategy space, e.g. bargaining games, oligopoly games, public good games or all games in which mixed strategies are considered. The reason for the restriction to finite strategy spaces is mainly that one has to study differential equations, and it is somewhat unfamiliar to do so with infinite dimensional spaces. However, most of the standard theory of ordinary differential equations carries over to the general case of Banach spaces.

It is the purpose of this paper to show that the restriction to finite strategy spaces is in most cases unnecessary. If one wants to study evolutionary dynamics on infinite strategy sets, one has to work with the space of probability measures over those strategies. It is important to note the difficulties that arise with such spaces. The main difference to the finite case is that on infinite dimensional spaces there are different "natural" norms that need not be equivalent. We propose to use the variational norm as it provides an intuitively sensible choice of distance in an evolutionary context and it is mathematically convenient. We show that endowed with the variational norm the space of probability measures becomes a Banach space. The usual Lipschitz continuity assumptions guarantee then a unique solution of the differential equation. We show that the replicator dynamics satisfy this condition if the underlying payoff function is bounded.

There are some further important differences to the finite case, though. In particular, the stability conditions are much more demanding. We provide an example showing that not even strict Nash equilibria are necessarily stable with respect to the replicator dynamics. This is in contrast to the finite case, where an ESS is a sufficient condition for asymptotic stability (Hofbauer et al. [12]). We show that a stronger condition, namely uninvadability, is sufficient for stability of a homogeneous population state.

We apply this general theory to a number of specific problems where a continuous strategy space is particularly natural. Namely, we consider the replicator dynamics in the context of the Nash demand game, the War

[1] Exceptions are Hopkins and Seymour [13] and Binmore and Seymour [4], who study the replicator dynamics but only on spaces of probability distributions with a continuous density, Bomze [5], [6], Seymour [18], and Friedman and Yellin [7]. The latter study gradient dynamics. The special case of mixed strategies has been studied by Akin [1], Hines [9], and Zeeman [22].

of Attrition, a Cournot duopoly, a Bertrand duopoly with homogeneous products, and in the context of mixed strategies.

In the Nash demand game the symmetric efficient equilibrium is stable and weakly attracting with respect to the replicator dynamics. In the War of Attrition the replicator dynamics converge to the unique mixed equilibrium from initial states which are within finite Kullback–Leiber distance from the equilibrium.

For the Cournot setting with a linear demand function we show that the replicator dynamics weakly converge to the Cournot–Nash equilibrium from all initial states with positive weight on the equilibrium strategy, generalizing a result obtained by Qin and Stuart [16] for the finite strategy case. While one can debate whether replicator dynamics are appropriate to study the interaction in duopoly, we think that those results are interesting nevertheless since many other games, like public good games, have a similar structure as the Cournot game.

With respect to the Bertrand case Hehenkamp [8] shows that if prices have to be chosen from a finite grid, the unique (globally) stable rest point of the dynamics is the price just above the Bertrand equilibrium. Note that this price is also a strict Nash equilibrium. In the continuous case this (second) equilibrium disappears leaving us with the unique Bertrand equilibrium, which is not even an ESS. While we can show that strategies bounded below marginal cost will vanish asymptotically, we have nothing to say about the shape of the limit distribution. We conjecture that the probability mass on intervals near the equilibrium increases.

However, if we restrict the strategy space by excluding an open interval (c, b) of prices above marginal cost c, we obtain the result that the price b is strongly attracting, which is the analog to the discrete case.

Finally, we show that mixed strategies also fit in our framework. Here we can use results already obtained by Hines [9] and Zeeman [22]. The advantage of using replicator dynamics on the space of mixed strategies is that – in contrast to dynamics on pure strategy spaces – evolutionary stable strategies are characterized by the asymptotically stable rest points of the replicator dynamics.

The paper is organized as follows. In the next section we describe the framework for analyzing dynamics on infinite dimensional spaces. In Section 3 we introduce the replicator dynamics and show that they are well defined on infinite strategy spaces. Section 4 is devoted to the derivation of stability conditions. Section 5 contains the applications, Nash demand game, War of Attrition, Cournot duopoly, Bertrand duopoly, and mixed strategies. Finally, Section 6 concludes. Some proofs are relegated to an appendix.

2 Formulation of the game

Let $S \subset \mathbb{R}^n$. Elements of S are called strategies. Typical examples are the discrete case, $S = \{x_1, \ldots, x_n\}$, the set of all possible quantities in a Cournot game, $S = \mathbb{R}_+$, or some subset of $\mathbb{R}^n$ with the usual metric.

Let $\mathcal{S}$ be the Borel σ–field on S. In the discrete case, this is the power set of S (and not mentioned because it plays no role); in the continuous case, this is the smallest σ-field containing all open sets.

For notational simplicity we restrict ourselves to the case of symmetric two–player games, though the setup can be extended to the asymmetric case. Let $f : S \times S \to \mathbb{R}$ be the payoff function. For the beginning, the only condition we impose on f is that it be measurable. $f(x, y)$ is the payoff for player 1 when she plays x and player 2 plays y.

A *population* is described by a probability measure P on the measure space $(S, \mathcal{S})$. The simplex of all populations is denoted by Δ. The (expected) payoff of population P against population Q is

$$E(P, Q) = \int_S \int_S f(x, y) Q(dy) P(dx) .$$

The aim is to study the evolution of populations over time. For this purpose, we need a Banach space to which all populations belong. We say that ν is a *signed measure* on $(S, \mathcal{S})$ if there are two finite measures μ^1 and μ^2 such that for all sets $A \in \mathcal{S}$, $\nu(A) = \mu^1(A) - \mu^2(A)$. The space of all signed measures is denoted by $\mathcal{M}^e(S, \mathcal{S})$. It is clearly a vector–space. What kind of norm is appropriate to describe the "distance" between two populations? We propose the supremum or variational norm.

Definition 1 *The variational norm on $\mathcal{M}^e(S, \mathcal{S})$ is given by*

$$\|\mu\| = \sup_f \left| \int f \, d\mu \right| ,$$

where the sup *is taken over all measurable functions $f : S \to \mathbb{R}$ bounded by $1, \sup_{s \in S} |f(s)| \leq 1$.*

Endowed with the variational norm, $\mathcal{M}^e$ is a Banach space (cf. Alt [2]), that is, every Cauchy sequence in $\mathcal{M}^e$ converges to an element in $\mathcal{M}^e$.

Since we want to study the dynamics of populations as time unfolds, we have to deal with curves $m : \mathbb{R}_+ \to \mathcal{M}^e$. Such a function is called

continuously differentiable if there exists a continuous function $m' : \mathbb{R}_+ \to \mathcal{M}^e$ with

$$\lim_{h \to \mathbf{0}} \frac{m(t+h) - m(t)}{h} = m'(t)\,,$$

where the limit has to be taken with respect to $\|\cdot\|$.

Now, it is easy to introduce ordinary differential equations, too. Let $F : \mathcal{M}^e \to \mathcal{M}^e$ be continuous. A continuously differentiable function m with

$$m'(t) = F(m(t)), \quad m(0) = \mu \tag{1}$$

is called a solution to the ordinary differential equation (1).

A crucial fact is that with Lipschitz–conditions one has always a unique solution to such initial value problems.

Theorem 1 *Suppose that F is bounded and satisfies a global Lipschitz condition:*

$$\exists K > 0 \text{ s.t. } \forall \mu, \nu \in \mathcal{M}^e,\ \|F(\mu) - F(\nu)\| \leq K \|\mu - \nu\|\ .$$

Then the ordinary differential equation (1) has a unique solution on $[0, \infty)$.

For a proof see e.g. Zeidler [23] Corollary 3.9. The important property needed in the proof is Lipschitz–continuity. It allows to define a contraction

$$m \mapsto \left(t \mapsto \mu + \int_{\mathbf{0}}^{t} F(m(s))ds \right)$$

on the space of all continuous functions that map $\mathbb{R}_+$ into $\mathcal{M}^e$ and the typical fixed–point argument gives existence and uniqueness.

It is useful to have some tools at hand that make calculations of the variational norm easier and lead at the same time to a better understanding of the induced topology.

For the rest of this section, let $P, Q \in \Delta$ denote probability measures. For probability measures we have that (cf. Shiryaev [19, p. 360])

$$\|P - Q\| = 2 \sup_{A \in \mathcal{S}} |P(A) - Q(A)|\,.$$

Thus, the maximum distance between two probability measures is 2 and is reached when the measures are orthogonal,

$$P \perp Q \Rightarrow \|P - Q\| = 2\,.$$

To see this, take a set A with $P(A) = 1$ and $Q(A^c) = 1$ and let $f = 1_A - 1_{A^c}$, where 1_A denotes the indicator function. Then

$$\int f d(P - Q) = P(A) + Q(A^c) = 2\,.$$

Therefore, $\|\cdot\|$ is a very strong measure of distance. For example, even if $x_n \to x$ in $\mathbb{R}$ and $x_n \neq x$, the corresponding Dirac–measures δ_{x_n} do not converge to δ_x in the sense of the variational norm because of $\|\delta_{x_n} - \delta_x\| = 2$. Since we are interested in populations, there may be an argument in favor of such a strong norm of the following type. The Dirac measure δ_x describes a homogeneous population. From an evolutionary point of view, two distinct homogeneous societies are quite far from one another in evolutionary terms because everyone in a population has to mutate in order to convert one population into the other, which indeed would be a very rare event.

It may seem quite difficult to calculate the variational norm. However, this is not so if we have densities. Let $\mu = aP - bQ$, for some nonnegative $a, b \geq 0$, be a signed measure. Assume that there is a third probability measure R that dominates[2] P, Q, hence also μ. Then the Radon-Nikodym densities $\phi = \frac{dP}{dR}\,, \psi = \frac{dQ}{dR}$ and $\xi = \frac{d\mu}{dR} = a\phi - b\psi$ exist. We have

Theorem 2 *The variational norm of μ is given by*

$$\|\mu\| = \int_S |\xi|\, dR\,.$$

In particular, the distance between probability measures is given by

$$\|P - Q\| = \int_S |\phi - \psi|\, dR\,.$$

Proof. See appendix. ■

The preceding theorem implies that the convergence of $P^n \to P$ in the variational norm is equivalent to the convergence of the densities $\frac{dP^n}{dR} \to \frac{dP}{dR}$ in the $\mathcal{L}^1$–sense with respect to the dominating measure R if the sequence (P^n) and the measure P are dominated by R.

We state this as a corollary for the case of the Lebesgue measure.

Corollary 1 *Let P^n and P have densities $v^n(x)$ and $v(x)$ with respect to the Lebesgue measure. Then*

$$P^n \to P \Leftrightarrow \int_{-\infty}^{\infty} |v^n(x) - v(x)|\, dx \to 0\,.$$

[2] Every set A with $R(A) = 0$ has $P(A) = 0$.

Finally, let us have a look at the classical discrete case.

Corollary 2 *Let* $S = \{1, \ldots, n\}$. *Set* $p_i^n = P^n(\{i\})$ *and* $p_i = P(\{i\})$. *Then*

$$P^n \to P \Leftrightarrow p_i^n \to p_i \,, \forall i$$

Proof. See appendix. ■

As we see, in the discrete case, the topology induced by the variational norm is equivalent to the pointwise convergence of probabilities. The classical results will therefore be contained in our results.

3 The Replicator Dynamics

The dynamics most widely studied in the literature on evolutionary game theory are the replicator dynamics. They formalize the idea that in a dynamic process of evolution a strategy x should increase in frequency if it is a successful strategy in the sense that individuals playing this strategy obtain a higher than average payoff. Formally, the success (or lack of success) of a strategy x if the population is Q is given by the difference

$$\sigma(x, Q) := \int_S f(x, y) Q(dy) - \int_S \int_S f(x, y) Q(dy) Q(dx) = E(\delta_x, Q) - E(Q, Q) \,.$$

The idea of replicator dynamics is that the relative increment of the frequency of a set of strategies is given exactly by the success of that set. Formally,

Definition 2 *The ordinary differential equation*

$$Q'(t) = \int \sigma(x, Q(t))\, Q(t)(dx), \qquad Q(0) = P$$

or in more extensive form,

$$Q'(t)(A) = \int_A \sigma(x, Q(t))\, Q(t)(dx), \qquad Q(0) = P$$

for all $A \in \mathcal{S}$, *is called* replicator dynamics.

Note that by taking $A = \{x\}$ we get the usual formulation of the replicator dynamics for the finite strategy case.

Our first message in this paper is that the replicator dynamics are well defined in this general setting if the payoff function f is bounded. The remainder of this section is devoted to proving this statement.[3]

Denote by $F(Q) = \int \sigma(x,Q)Q(dx)$ the right–hand side of the replicator dynamics. The strategy for proving the statement is the following. Since it is easier to work with the vector space $\mathcal{M}^e$ rather than with Δ, in the following lemma we state sufficient conditions on σ for the existence of a Lipschitz continuous function $\tilde{F}$ defined on $\mathcal{M}^e$. $\tilde{F}$ is constructed so that it coincides with F on Δ. In Lemma 2 we will then verify that the conditions on σ are satisfied if f is bounded. Finally, in Theorem 3 we show that Δ is invariant with respect to the differential equation $Q^{'}(t) = \tilde{F}(Q(t))$.

Lemma 1 *Suppose the following Lipschitz and boundedness conditions hold for* σ

$$\|Q\|, \|R\| \leq 2 \Rightarrow \sup_x |\sigma(x,Q) - \sigma(x,R)| \leq L \|Q - R\| \tag{2}$$

$$\sup_{Q:\|Q\|\leq 2} |\sigma(x,Q)| \leq \sigma_\infty , \tag{3}$$

where L *and* σ_∞ *are some constants with* L, $\sigma_\infty < \infty$.*Then there exists a bounded, Lipschitz continuous function* $\tilde{F} : \mathcal{M}^e \to \mathcal{M}^e$, *which coincides with* F *on* Δ,

$$\tilde{F}(P) = F(P), \ \forall P \in \Delta.$$

Proof. See appendix. ■

Lemma 2 *If the payoff function* f *is bounded, then conditions (2) and (3) of Lemma 1 are satisfied.*

Proof. See appendix. ■

By the preceding lemmata we know that the ordinary differential equation

$$Q'(t) = \tilde{F}(Q(t)) \tag{4}$$

has a unique solution. For the replicator dynamics to make sense we need furthermore that the set Δ of all populations is invariant under these dynamics.

[3]The following analysis owes much to the work of Bomze [5], [6]. See also the recent paper by Seymour [18], who develops a similar framework to ours for the asymmetric case.

Theorem 3 *Let $(Q(t))$ be the unique solution to (4) with initial condition $P \in \Delta$. Then $(Q(t)) \subset \Delta$, that is, the set of all populations Δ is invariant with respect to (4).*

Moreover, $Q(t)$ and P are uniformly equivalent in the sense that there exists constants $\gamma_t, \Gamma_t > 0$ such that

$$\gamma_t Q(t)(A) \leq P(A) \leq \Gamma_t Q(t)(A)$$

for all sets $A \in \mathcal{S}$. The density of $Q(t)$ with respect to P satisfies

$$\frac{dQ(t)}{dP}(x) = \exp\left(\int_0^t \sigma(x, Q(s))ds\right) . \tag{5}$$

Proof. See appendix. ■

That is, if we start with a population $P \in \Delta$, the solution of $(Q(t))$ stays in the set of populations Δ. On Δ the functions $\tilde{F}$ and F coincide. Combining Lemmata 1 and 2 with Theorem 3 we get the following

Theorem 4 *If the payoff function f is bounded, then the replicator dynamics are well defined.*

4 Stability concepts

We start with the classical concept of evolutionary stability introduced by Maynard Smith [15].

Definition 3 *A population P is called an* evolutionary stable state *(ESS) if for every "mutation" Q, there is an invasion barrier $\varepsilon(Q) > 0$ such that for all $\eta \leq \varepsilon$,*

$$E(P, (1-\eta)P + \eta Q) > E(Q, (1-\eta)P + \eta Q) . \tag{6}$$

Two stronger notions are better suited for the dynamical aspects of evolutionary game theory.

Definition 4 *A population is called* uninvadable *if there is a uniform invasion barrier, that is, an $\varepsilon > 0$ such that (6) holds for all Q.*

Uninvadability requires a uniform invasion barrier for all possible mutations Q. In general, one could require even more. Up to now, we have considered the case in which a small fraction η of the populations changes *arbitrarily*. More generally, one could look at the case where the whole population is allowed to mutate, but only in a manner that the *distance* between the original and the mutated population remains small.

Definition 5 (Bomze) *A population P is called* strongly uninvadable *if there is a barrier $\varepsilon > 0$ such that for all populations R in the ε–neighborhood of P, $R \neq P$,*

$$\|R - P\| \leq \varepsilon,$$

we have

$$E(P, R) > E(R, R)\,.$$

We put together some simple facts about the various stability concepts in

Lemma 3 *1. Every strongly uninvadable population is uninvadable, and every uninvadable population is evolutionary stable.*

2. If a Dirac measure $P = \delta_x$ is uninvadable, then it is also strongly uninvadable.

3. If the strategy space S is discrete, then the three concepts coincide.

Proof. The first statement is obvious. For the second, note that if $\|\delta_x - Q\| \leq \varepsilon$, then

$$\varepsilon \geq |\delta_x(\{x\}) - Q(\{x\})| = |1 - Q(\{x\})| \tag{7}$$

and $Q(\{x\}) \geq 1 - \varepsilon$ follows. Hence, Q can be written as $Q = (1 - \eta)\delta_x + \eta R$ for some $\eta \leq \varepsilon$ and some population R. In this case, the conditions for strong uninvadability and uninvadability coincide.

That every ESS is uninvadable in the discrete case is well known (cf. [21]). Since in the discrete case all populations are convex combinations of Dirac measures, a similar argument as for the second statement gives that every uninvadable population is also strongly uninvadable. ■

Finally, we recall the following dynamic stability notions.

Definition 6 *Let $Q^\star$ be a rest point of the replicator dynamics,*

$$\sigma(\cdot, Q^\star) = 0 \quad Q^\star - a.e.$$

Then

- *$Q^\star$ is called* Lyapunov stable *if for all $\varepsilon > 0$ there exists an $\eta > 0$ such that $\|Q(0) - Q^\star\| < \eta \Rightarrow \|Q(t) - Q^\star\| < \varepsilon$ for all $t > 0$.*
- *$Q^\star$ is called* strongly attracting *if there exists $\varepsilon > 0$ such that $\|Q(0) - Q^\star\| < \varepsilon \Rightarrow \|Q(t) - Q^\star\| \to 0$.*

- $Q^\star$ *is called* weakly attracting *if there exists* $\varepsilon > 0$ *such that* $\|Q(0) - Q^\star\| < \varepsilon \Rightarrow Q(t) \to Q^*$ *in distribution.*

As we shall see, convergence in variation is in most cases too much to hope for. Therefore, the concept of weak attraction is more appropriate for our purpose. Nevertheless, we will later encounter an example where a rest point is strongly attracting (Section 5.4).

We have the following useful result, which connects the static and the dynamic stability concepts.

Theorem 5 *If* $Q^\star = \delta_x$ *is an uninvadable, homogeneous population, then* $Q^\star$ *is Lyapunov stable.*

Proof. Since δ_x is uninvadable (and therefore strongly uninvadable by Lemma 1), there exists an $\varepsilon > 0$ such that for all $R \in \Delta$ with $\|R - \delta_x\| \leq \varepsilon$

$$E(\delta_x, R) > E(R, R).$$

For $Q(0) = \delta_x$ there is nothing to show. Assume $Q(0) \neq \delta_x$. The function $H(t) := Q(t)(\{x\})$ is continuously differentiable. By Lemma 3 we have

$$H(t) = H(0) \exp\left(\int_0^t \sigma(x, Q(s))ds\right),$$

hence

$$\frac{H'(t)}{H(t)} = \sigma(x, Q(t)). \tag{8}$$

We claim that H is strictly increasing. By assumption, $H'(0) = \sigma(x, Q(0)) > 0$. Suppose that H' eventually became zero and set

$$t_0 := \inf\left\{t \geq 0; H'(t) = 0\right\}.$$

Then $H'(t_0) = 0$ since the set $\{t \geq 0; H'(t) = 0\}$ is closed. For all $s < t_0$ we have $H'(s) > 0$ which implies $H(s) > H(0)$, hence $\|Q(s) - \delta_x\| < \varepsilon$. Continuity of the trajectory yields $\|Q(t_0) - \delta_x\| \leq \varepsilon$, hence $H'(t_0) = \sigma(x, Q(t_0)) > 0$ by uninvadability, a contradiction. Therefore, H is strictly increasing which implies that $\|Q(t) - \delta_x\| < \varepsilon$. ■

The proof of the preceding Theorem shows that the replicator dynamics increase the weight on the pure strategy x if δ_x is uninvadable. Therefore, the weight $Q(t)(\{x\})$ converges and the growth rate of the strategy must vanish. We state this fact as a corollary.

Corollary 3 *If $Q^\star = \delta_x$ is an uninvadable, homogeneous population, then the fitness differential vanishes:*

$$\sigma(x, Q(t)) \to 0.$$

Proof. By (8), the convergence of $Q(t)(\{x\}) = H(t)$ implies $\int_0^\infty \sigma(x, Q(s))ds < \infty$. The result follows if we show the equicontinuity of the map $t \mapsto \sigma(x, Q(t))$. Since the conditions of Lemma 1 are satisfied, we have by (2),

$$|\sigma(x, Q(t)) - \sigma(x, Q(s))| \leq L \|Q(t) - Q(s)\| .$$

The boundedness of σ on Δ, (3), and replicator dynamics yield for every set A

$$\begin{aligned} |Q(t)(A) - Q(s)(A)| &= \left| \int_s^t Q'(u)(A)du \right| \\ = \left| \int_s^t \int_A \sigma(\xi, Q(u))Q(u)(d\xi)du \right| &\leq \sigma_\infty |t - s| . \end{aligned}$$

Hence, using (2),

$$|\sigma(x, Q(t)) - \sigma(x, Q(s))| \leq 2L\sigma_\infty \|t - s\| ,$$

which implies Lipschitz and hence equicontinuity of the map $t \mapsto \sigma(x, Q(t))$. ■

If the payoff function f is continuous – which it is frequently *not* –, then we have the following

Corollary 4 *If the payoff function f is continuous, every uninvadable, homogeneous population $Q^* = \delta_x$ is weakly attracting.*

Proof. For continuous f, the expected payoff difference $E(\delta_x, Q) - E(Q, Q)$ is continuous in Q with respect to the weak topology. The set Δ of all populations is compact in the weak topology. Let P be a weak accumulation point of the trajectory $(Q(t))$. By the preceding corollary $0 = \lim \sigma(x, Q(t)) = \lim [E(\delta_x, Q(t)) - E(Q(t), Q(t))] = E(\delta_x, P) - E(P, P)$. By Theorem 5, δ_x is stable, hence P is close to δ_x. By uninvadability, $P = \delta_x$. ■

We should contrast the stability results obtained in this section with those for the finites case. As is well known in the finite case an ESS is sufficient for asymptotic stability (Hofbauer et al. [12]). A fortiori, all strict equilibria are asymptotically stable. One may wonder whether this

result survives in the infinite case. Somewhat surprisingly it does not as the following example shows.

Consider a game with strategy set $S = \mathbb{R}$ and payoff function

$$f(x,y) = \begin{cases} 1 - |x-1| & \text{if } y = 1 \\ 0 & \text{if } x = 1, y \neq 1 \\ 1 & \text{otherwise.} \end{cases}$$

It can easily be checked that $(1,1)$ is a strict Nash equilibrium. However, it is not uninvadable and it is *not* Lyapunov stable. For any $\varepsilon > 0$, there exists some x such that $\sigma(x, (1-\varepsilon)\delta_1 + \varepsilon\delta_x) > 0$. Therefore, δ_1 is not uninvadable. Now pick some $Q(0) = (1 - \eta(0))\delta_1 + \eta(0)\delta_x$ in the ε–neighborhood of δ_1 and choose $x = 1 + \eta(0)/2$. Simple calculations show that

$$\sigma(x, Q(t)) = \tfrac{1}{2}\eta(t)\left(1 - \eta(t)^2\right) > \tfrac{1}{2}\eta(0)\left(1 - \eta(0)^2\right) > 0.$$

Thus, $Q(t)(\{x\})$ increases exponentially to 1.[4]

5 Applications

For many games it is more natural to think of strategies as continuous, in particular, if strategies involve the timing of actions or the choice of prices or locations. Even if smallest measurement units for quantities or prices exist, it is often more convenient to model them as continuous. In this section we present a number of examples for how the replicator dynamics can be applied to infinite strategy spaces.

5.1 Nash demand game

One simple example is the Nash demand game. Two players have to decide how to divide a resource of size 1. Both players simultaneously submit demands, x and y (i.e. $S = \mathbb{R}_+$). If the demands are feasible, both get what they demanded. If not, both receive nothing. Thus the payoff function is

$$f(x,y) = \begin{cases} x & \text{if } x + y \leq 1 \\ 0 & \text{if } x + y > 1 \end{cases}.$$

The unique efficient symmetric equilibrium of this game is $(\frac{1}{2}, \frac{1}{2})$. Since this equilibrium is strict, it is an ESS. The next proposition shows that it is uninvadable and weakly attracting.

[4] Note that the payoff function in the example is discontinuous. However, it can be smoothed out without changing the result.

Proposition 1 *In the Nash demand game the homogenous population $\delta_{1/2}$ is uninvadable, Lyapunov stable, and weakly attracting.*

Proof. To establish uninvadability, we need to show that $E(P,P) < E(\delta_{1/2},P)$, for all $P \neq \delta_{1/2}$ with $\|\delta_{1/2} - P\| \leq \varepsilon$. All P in the ε–neighborhood of $\delta_{1/2}$ can be written as

$$P = \alpha Q^- + \beta Q^+ + (1-\alpha-\beta)\delta_{1/2}, \tag{9}$$

where Q^- and Q^+ are some probability measures with $Q^-([\frac{1}{2},1]) = 0$ and $Q^+([0,\frac{1}{2}]) = 0$, and $\alpha+\beta \leq \varepsilon$. For probability measures $R, R^{'}$ concentrated on $\left[0,\frac{1}{2}\right]$, one has $E(R,R^{'}) = m(R) \leq 1/2$, where $m(R) = \int xR(dx)$ denotes the mean value of population R. If $\beta = 0$, one has that $E(P,P) = m(P) < \frac{1}{2} = E(\delta_{1/2},P)$.

Next consider $\beta > 0$. In general, $E(\delta_{1/2},P) = \frac{1}{2}(1-\beta)$. Note that $E(Q^+,P) = \alpha E(Q^+,Q^-) \leq \alpha m(Q^+) \leq \alpha$. The bilinearity of E yields

$$\begin{aligned}
&\sigma\left(\tfrac{1}{2},P\right) = \tfrac{1}{2}(1-\beta) - E(P,P) \\
&= \tfrac{1}{2}(1-\beta) - \alpha E(Q^-,P) - (1-\alpha-\beta)E(\delta_{1/2},P) - \beta E(Q^+,P) \\
&\geq \tfrac{1}{2}(1-\beta) - \alpha m\left(Q^-\right) - \tfrac{1}{2}(1-\alpha-\beta)(1-\beta) - \alpha\beta m\left(Q^+\right) \qquad (10) \\
&\geq \tfrac{1}{2}(1-\beta) - \tfrac{1}{2}\alpha - \tfrac{1}{2}(1-\alpha-\beta)(1-\beta) - \alpha\beta \\
&= \tfrac{1}{2}\beta\left(1-\beta-3\alpha\right). \qquad (11)
\end{aligned}$$

Thus, $\sigma(1/2,P) > 0$ if $\beta > 0$ and $3\alpha+\beta < 1$, which is satisfied for ε small. Hence $\delta_{1/2}$ is uninvadable. It follows from Theorem 5 that $\delta_{1/2}$ is Lyapunov stable.

By Corollary 3 the fitness difference $\sigma\left(\frac{1}{2},Q(t)\right)$ vanishes. Defining $\beta(t),\alpha(t)$ for $Q(t)$ as in equation (9), we obtain by (11) that $\beta(t) \to 0$. (10) implies then that $\alpha(t) \to 0$ or $m(Q(t)^-) \to \frac{1}{2}$. In both cases, $Q(t)^-$ converges in $\mathcal{L}^1$, hence weakly, to $\delta_{1/2}$. Therefore, the $\delta_{1/2}$ is also weakly attracting. ■

5.2 The War of Attrition

An example for a game in which timing is the relevant choice is the well known War of Attrition, which has important applications in economics and biology. Consider two players fighting for a prize worth V to both players.

A strategy is to choose a length of time $x \in [0, M]$ for which one is prepared to stay in the race. Fighting is costly. The payoffs are given as follows

$$f(x,y) = \begin{cases} V - y & \text{if } x > y \\ \frac{V}{2} - x & \text{if } x = y \\ -x & \text{if } x < y \end{cases}$$

that is, a player gets the prize if he stay longer in the race than his rival but has to share if they stay equally long. We assume that $M > V/2$. Otherwise waiting until the end is always profitable.

It is obvious that no pure strategy Nash equilibrium exists. But as shown by Bishop and Cannings [3] there is a unique, completely mixed Nash equilibrium, which has the following equilibrium distribution $P^\star$. Let $t^\star = M - V/2$.

$$P^\star([0,x]) = \begin{cases} 1 - e^{-x/V} & \text{if } x \le t^\star \\ 1 - e^{-t^*/V} & \text{if } t^\star < x < M \\ 1 & \text{if } x = M \end{cases}$$

Bishop and Cannings [3] show that $P^\star$ is an ESS. In fact, they show [3, p. 118] that the fitness differential between the equilibrium distribution $P^\star$ and any mutation Q is given by the square of the $\mathcal{L}^2$-distance of the corresponding distribution functions:

$$E(P^\star, Q) - E(Q,Q) = (\|P^\star - Q\|_2)^2 \tag{12}$$

where

$$\|\mu\|_2 := \left(\int_0^M \mu([s,\infty))^2 \, ds \right)^{\frac{1}{2}}$$

denotes the $\mathcal{L}^2$-norm on the space of distribution functions. Note that since the right hand side of (12) is strictly positive for all $Q \neq P^\star$, it follows that $P^\star$ is strongly uninvadable.

In light of (12), it seems natural to use the $\mathcal{L}^2$-topology in dynamic considerations. On the space of populations Δ, this topology is equivalent to the weak topology as shown by Hindy, Huang, and Kreps [10]. The following theorem demonstrates that the replicator dynamics converge globally to $P^\star$ from all initial states $Q(0)$ which have finite Kullback–Leiber distance (or cross–entropy) with respect to $P^\star$.

Theorem 6 *Assume $Q(0)$ dominates $P^\star$ and*

$$\int \log \frac{dP^\star}{dQ(0)} dP^\star < \infty \,. \tag{13}$$

Then the replicator dynamics with initial condition $Q(0)$ converge weakly to the equilibrium distribution $P^\star$.

Proof. Because of Theorem 3, $Q(t)$ and $Q(0)$ are uniformly equivalent. Thus, the Kullback–Leibler distance

$$K(Q(t), P^\star) := \int \log \frac{dP^\star}{dQ(t)} dP^\star$$

is well defined, finite, and nonnegative. With the use of the representation for the density of $Q(t)$ with respect to $Q(0)$ obtained in Theorem 3 it follows that

$$\begin{aligned} 0 &\leq K(Q(t), P^\star) \\ &= \int \log \left(\frac{dQ(0)}{dQ(t)} \frac{dP^\star}{dQ(0)} \right) dP^\star \\ &= K(Q(0), P^\star) - \int \int_0^t \sigma(x, Q(s)) ds dP^\star \\ &= K(Q(0), P^\star) - \int_0^t \left(E(P^\star, Q(s)) - E(Q(s), Q(s)) \right) ds \, . \end{aligned}$$

(12) yields

$$0 \leq K(Q(0), P^\star) - \int_0^t \left(\|P^\star - Q(s)\|_2 \right)^2 ds \, .$$

Hence, the integral

$$\int_0^\infty \left(\|P^\star - Q(s)\|_2 \right)^2 ds < \infty \tag{14}$$

exists. It follows that the distance $\|P^\star - Q(s)\|_2$ tends to zero as $s \to \infty$ since the map $s \mapsto \|P^\star - Q(s)\|_2$ is equicontinuous, which can be seen as follows. By the triangular inequality, $\left| \|P^\star - Q(s)\|_2 - \|P^\star - Q(t)\|_2 \right| \leq \|Q(s) - Q(t)\|_2$. The $\mathcal{L}^2$-norm is dominated by the variational norm,

$$\begin{aligned} \|Q(s) - Q(t)\|_2 &= \left(\int_0^M \left(Q(s)\left([x, \infty)\right) - Q(t)\left([x, \infty)\right) \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{M^{\frac{1}{2}}}{2} \|Q(s) - Q(t)\| \, , \end{aligned}$$

and the trajectory $(Q(t))$ is Lipschitz, $\|Q(s) - Q(t)\| \leq 2\sigma_\infty |s - t|$, compare the proof of Corollary 3. ■

5.3 Cournot duopoly

Another example where the strategy space is generally modelled as continuous is a duopoly model where firms choose quantities, i.e. $S = [0, M]$, for some M large enough. Let $p(x+y) = a - b(x+y)$ denote the linear inverse demand function with $a, b > 0$. A representative firm's profit function is given by

$$f(x, y) = p(x + y)x - C(x),$$

where $C(x)$ is a twice continuously differentiable and convex cost function. Given this assumption there exists a unique symmetric Nash equilibrium of the duopoly game in which both firms choose quantities x^c.

To study the stability properties of x^c we need the following useful fact, which is particular to the linear structure of the Cournot setup.

Lemma 4 *$E(\delta_{x^c}, Q) > E(Q, Q)$ for all $Q \neq \delta_{x^c}$.*

Proof. Let $\bar{q} := \int_0^\infty xQ(dx)$ denote average output. The Lemma follows from the following chain of (in)equalities.

$$E(\delta_{x^c}, Q) = f(x^c, \bar{q}) > f(\bar{q}, \bar{q}) \geq E(Q, \delta_{\bar{q}}) = E(Q, Q).$$

The first equality follows from the linearity of the profit function $f(x, y)$ in y.

$$\begin{aligned} E(\delta_{x^c}, Q) &= \int_0^\infty [x^c p(x^c + y) - C(x^c)]\, Q(dy) \\ &= x^c p(x^c + \bar{q}) - C(x^c) = f(x^c, \bar{q}). \end{aligned}$$

To verify the first inequality suppose $\bar{q} < x^c$ and let $r(y) := \arg\max_x f(x, y)$ denote a firm's reaction function. Since $\frac{\partial^2 f(x,y)}{\partial x \partial y} < 0$, $r(y)' < 0$. Thus, $r(\bar{q}) > r(x^c) = x^c$. By definition of $r(y)$ and $\bar{q} < x^c$

$$f(r(\bar{q}), \bar{q}) > f(\bar{q}, \bar{q}).$$

Concavity of $f(x, y)$ in x implies that

$$f(x^c, \bar{q}) > f(\bar{q}, \bar{q}).$$

A similar argument holds for $\bar{q} > x^c$.

The second inequality follows directly from concavity of $f(x, y)$ in x and Jensen's inequality.

$$f(\bar{q}, \bar{q}) \geq \int_0^\infty f(x, \bar{q}) Q(dx) = E(Q, \delta_{\bar{q}}).$$

Finally, the last equality follows again from linearity of $f(x,y)$ in y. ■

Lemma 4 implies in particular that δ_{x^c} is strongly uninvadable. Theorem 5, therefore, yields that the Cournot equilibrium is Lyapunov stable. We show next that replicator dynamics converge globally to the Cournot equilibrium from all initial states which put positive weight on the equilibrium.

Proposition 2 *The replicator dynamics converge weakly to the Cournot equilibrium from any initial state with $Q(0)(\{x^c\}) > 0$. In particular, the Cournot equilibrium is weakly attracting.*

Proof. Since the profit difference $\sigma(x^c, Q) > 0$ for all populations Q, the weight on x^c increases with time for every initial state $Q(0)$, which puts positive probability on x^c. By Corollary 3 the fitness differential $\sigma(x^c, Q(t))$ vanishes. The mean payoff is

$$\begin{aligned} E(Q,Q) &= \int_0^\infty f(x,\bar{q})Q(dx) \\ &= \int_0^\infty \left[ax - bx^2 - bx\bar{q} - C(x)\right] Q(dx) \\ &\leq \bar{q}\,(a - 2b\bar{q}) - C(\bar{q}) - bVar(Q) \\ &= f(\bar{q},\bar{q}) - bVar(Q). \end{aligned}$$

Thus

$$\begin{aligned} \sigma(x^c, Q) &= E(\delta_{x^c}, Q) - E(Q,Q) \\ &\geq f(x^c,\bar{q}) - f(\bar{q},\bar{q}) + bVar(Q). \end{aligned}$$

Since $f(x^c,\bar{q}) > f(\bar{q},\bar{q})$ by the proof of Lemma 4, $\sigma(x^c, Q(t)) \to 0$ implies that $Var(Q(t)) \to 0$ and $\bar{q} \to x^c$. Thus, $Q(t) \to \delta_{x^c}$ in $\mathcal{L}^2$, which implies weak convergence . ■

5.4 Bertrand duopoly

In Bertrand models of price competition with homogenous goods a continuous strategy space is particularly appropriate given the usual "undercutting" argument. Strategies for both firms are now prices $x, y \in \mathbb{R}_+$. Market demand is given by a continuous and bounded function $q(x)$ which is strictly decreasing for all x with $q(x) > 0$. Furthermore, we assume that there exists an $\bar{x} < \infty$ such that for all $x \geq \bar{x}$, $q(x) = 0$. Marginal costs, c, are constant,

with $c > 0$. A representative firm's profit function is then given by

$$f(x,y) = \begin{cases} (x-c)q(x) & x < y \\ (x-c)q(x)/2 & x = y \\ 0 & x > y \end{cases}$$

where we assume as usual that both firms get half of demand if they choose equal prices.

As is well known the unique Nash equilibrium of this game is for both firms to choose $p^B = c$. Note, however, that the game does not possess an ESS. In particular, δ_{p^B} is not an ESS since

$$E(\delta_{p^B}, (1-\varepsilon)\delta_{p^B} + \varepsilon Q) < E(Q, (1-\varepsilon)\delta_{p^B} + \varepsilon Q)$$

for all Q with $Q((c,\bar{x})) = 1$. For the same reason, δ_{p^B} is not Lyapunov stable.

For a discrete strategy space Hehenkamp [8] demonstrates that the smallest price above c is globally stable. We can derive a similar result if we exclude from the strategy space some small open interval (c,b), $b > c$.

Proposition 3 *Let $S = \mathbb{R}_+ \backslash (c,b)$ for some $b > c$. Then δ_b is uninvadable, and, therefore, Lyapunov stable with respect to the replicator dynamics. Moreover, δ_b is strongly attracting.*

Proof. Let $\pi := (b-c)q(b)$ and let K be an upper bound for the payoffs. Choose

$$\eta < \left(1 + \frac{2K}{\pi}\right)^{-1}. \tag{15}$$

We show first that $\sigma(b,R) > 0$ for every population $R \neq \delta_b$ with $\|R - \delta_b\| \leq \eta$. Every such R can be written as $\alpha P^- + (1-\alpha-\beta)\delta_b + \beta P^+$ with $P^-([b,\infty)) = P^+([0,b]) = 0$ and $\alpha + \beta \leq \eta$, $\alpha, \beta \geq 0$. Then $E(\delta_b, R) = (1-\alpha+\beta)\,\pi/2$ and

$$\begin{aligned} E(R,R) &= \alpha E(P^-,R) + (1-\alpha-\beta)E(\delta_b,R) + \beta^2 E(P^+,P^+) \\ &\leq (1-\alpha-\beta)E(\delta_b,R) + \beta^2 K. \end{aligned}$$

Hence,

$$\begin{aligned} \sigma(b,R) &\geq (\alpha+\beta)\left((1-\alpha+\beta)\,\frac{\pi}{2} - \beta^2 K\right) \\ &\geq (\alpha+\beta)\left((1-\alpha+\beta)\,\frac{\pi}{2} - \beta K\right) \\ &\geq (\alpha+\beta)\left((1-\eta)\,\frac{\pi}{2} - \eta K\right) > 0, \end{aligned}$$

by (15). Thus, δ_b is uninvadable and, by Theorem 5, Lyapunov stable. Now assume $1 > Q(0)(\{b\}) \geq 1 - \eta$. Define $\alpha(t)$ and $\beta(t)$ via

$$Q(t) = \alpha(t)P^-(t) + (1 - \alpha(t) - \beta(t))\,\delta_b + \beta(t)P^+(t)$$

as above. Then, again

$$\sigma(b, Q(t)) \geq (\alpha(t) + \beta(t)) \left((1 - \eta)\,\frac{\pi}{2} - \eta K \right) > 0.$$

By Corollary 3 $\alpha(t) + \beta(t)$ must go to zero. ■

The outcome for the unrestricted strategy set, however, is still an open question. We can show that all prices $p \leq c - \delta$ will vanish, which does not exclude the possibility that prices $p = c$ and $p \geq \bar{x}$ will survive in the long run.

Proposition 4 *For all initial populations $Q(0)$ with $Q(0)((c, \bar{x})) > 0$ and arbitrary small $\delta > 0$, all strategies below $c - \delta$ die out: $Q(t)([0, c - \delta]) \to 0$ as $t \to \infty$.*

Proof. See appendix. ■

5.5 Mixed strategies

Replicator dynamics are usually defined on the (finite) space of pure strategies. In contrast, the static analog of an evolutionary stable strategy (ESS) allows for individuals playing mixed strategies. This is the reason why the set of ESS cannot be characterized by the stable rest points of the replicator dynamics. Every ESS is an asymptotically stable state of the finite replicator dynamics but not vice versa (see e.g. Weibull, [21], or Vega–Redondo, [20]). The divergence of these concepts disappears if one defines replicator dynamics on the set of mixed strategies.

Let $< U, T >$ denote the underlying symmetric 2–player normal form game, where $U : T \times T \to \mathbb{R}$ denotes the payoff matrix and T is the finite set of pure strategies. Let n be the number of pure strategies. Let $\Delta(T) := \{x \in \mathbb{R}^n \,|x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ denote the $n-1$ dimensional simplex.

Since we are concerned with the evolution of mixed strategies, our strategy set S is given by $\Delta(T)$. The corresponding payoff function is simply

$$f(x, y) = xUy = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j U_{ij}.$$

Note that a mix over mixed strategies induces just another mixed strategy. Let

$$\mu := \int_{\Delta(T)} xQ(dx)$$

denote the mean mixed strategy in the population. Compare now the fitness of some subpopulation in which everyone mixes half–half between two pure strategies i and j with some other subpopulation in which half of the players choose i and the others j. Since their mean mixed strategy is the same, one cannot expect evolution to select among those subpopulations. At best one can hope that the mean mixed strategies possess some kind of stability. Such a result was proved by Hines [9] (see also Vega–Redondo [20, p. 60]). By Lemma 5 of Zeeman [22], the evolution of the mean strategy satisfies

$$\mu'(t) = C(Q(t))U\mu(t),$$

where $C(Q) = \int (x-\mu)(x-\mu)Q(dx)$ denotes the covariance matrix.

Proposition 5 *(Hines, [9]) Let $x^* >> 0$ be a completely mixed ESS. Then μ^* is asymptotically stable if and only if $\mu^* = x^*$.*

6 Conclusion

We have shown in this paper that the replicator dynamics can be applied to continuous strategy spaces without modification. The only condition is that the underlying payoff function must be bounded, which can often be achieved by imposing arbitrarily large bounds on the strategy space.

In games in which a continuous strategy space is more natural, e.g. when quantities, prices, timing etc. are strategies, it should be possible now to use replicator dynamics directly on the continuous strategy space rather than on a more or less suitable discretization.

We have applied the theory to a number of examples. In the Nash demand game, the War of Attrition, and a Cournot duopoly, the results for the discretization are (roughly) reproduced. However, in a Bertrand setting the results of the discrete model turned out to be somewhat misleading. While in the discrete model the price above marginal cost is asymptotically stable, in the continuous model no such stable outcome seems to exist. The shape of the limit distribution is still an open question.

Further applications of the model are replicator dynamics defined on mixed strategy spaces and games in which continuous preference parameters are selected for in an evolutionary game (see e.g. the analysis of the endowment effect in Huck, Kirchsteiger, and Oechssler [14]).

Appendix

Proof of Theorem 2. It suffices to prove the first formula because the second formula follows by taking $a = b = 1$.

For a measurable function f bounded by 1, one has by the monotonicity of the integral with respect to R

$$\left|\int f d\mu\right| = \left|\int f\xi dR\right| \leq \int |f\xi|\, dR \leq \int |\xi|\, dR,$$

hence $\|\mu\| \leq \int |\xi|\, dR$. To show equality, set $A = \{\xi > 0\}$ and $f = 1_A - 1_{A^c}$. Then f is bounded by 1, hence

$$\|\mu\| \geq \left|\int f d\mu\right| = \left|\int_A \xi dR - \int_{A^c} \xi dR\right| = \int |\xi|\, dR\,,$$

because $\xi = |\xi|$ on A and $-\xi = |\xi|$ on A^c. ■

Proof of Corollary 2. The discrete measures P^n and P are dominated by the counting measure $\zeta = \sum_{k=1}^n \delta_k$. The densities are

$$\frac{dP^n}{d\zeta}(i) = p_i^n\,.$$

By Theorem 2,

$$\|P^n - P\| = E\zeta\left|\frac{dP^n}{d\zeta} - \frac{dP}{d\zeta}\right| = \sum_i |p_i^n - p_i|\ .$$

The left–hand side goes to zero if and only if for every i the probabilities p_i^n tend to p^i. ■

Proof of Lemma 1. As a candidate for $\tilde{F}$, we propose $\tilde{F}(Q) = (2 - \|Q\|)^+\ F(Q)$. $\tilde{F}$ is zero for $\|Q\| \geq 2$, bounded and coincides with F on Δ because probability measures have norm 1. Let Q and R be measures with $\|Q\|, \|R\| \leq 2$. Choose $P = \frac{|Q|+|R|}{2}$ as the dominating measure. Here, we use the notation $|\mu| := \mu^+ + \mu^-$,where the nonnegative measures μ^+ and μ^- form the Hahn decomposition of μ, i.e. $\mu = \mu^+ - \mu^-$. Note that $F(Q)$ has the density $\sigma(\cdot, Q)$ with respect to Q and therefore

$$\frac{dF(Q)}{dP} = \sigma(\cdot, Q)\frac{dQ}{dP}\,.$$

With the use of Theorem 2, we obtain

$$\|F(Q) - F(R)\| = E^P\left|\frac{dF(Q)}{dP} - \frac{dF(R)}{dP}\right|$$

$$= \int_S \left| \sigma(x,Q)\frac{dQ}{dP}(x) - \sigma(x,R)\frac{dR}{dP}(x) \right| dP(x)$$

$$\leq \int |\sigma(x,Q) - \sigma(x,R)| \left|\frac{dQ}{dP}(x)\right| dP(x) + \int |\sigma(x,R)| \left|\frac{dQ}{dP}(x) - \frac{dR}{dP}(x)\right| dP(x)$$

$$\leq \int |\sigma(x,Q) - \sigma(x,R)|\, d|Q|(x) + \int |\sigma(x,R)| \left|\frac{dQ}{dP}(x) - \frac{dR}{dP}(x)\right| dP(x)\,.$$

Using the boundedness and Lipschitz conditions (3) and (2) in conjunction with Theorem 2, it follows that

$$\begin{aligned} \|F(Q) - F(R)\| &\leq L\,\|Q-R\|\,\|Q\| + 2\sigma_\infty\,\|Q-R\| \\ &\leq 2(L+\sigma_\infty)\,\|Q-R\|\,. \end{aligned} \tag{16}$$

Hence, F is Lipschitz continuous on the set of measures with variational norm less than 2.

To extend this property to $\tilde{F}$, we distinguish three cases. If both $\|Q\|, \|R\| \geq 2$, then $\tilde{F}(Q) = \tilde{F}(R) = 0$ and there is nothing to show. If $\|Q\| \geq 2 > \|R\|$, then

$$\begin{aligned} \left\|\tilde{F}(Q) - \tilde{F}(R)\right\| &= \left\|\tilde{F}(R)\right\| \\ &= (2 - \|R\|)\,\|F(R)\|\,. \end{aligned}$$

By Theorem 2 and the boundedness condition (3)

$$\|F(R)\| = \int_S |\sigma(x,R)| \left|\frac{dR}{dP}(x)\right| dP(x) \leq \sigma_\infty \|R\|\,. \tag{17}$$

Therefore,

$$\begin{aligned} \left\|\tilde{F}(Q) - \tilde{F}(R)\right\| &\leq (2 - \|R\|)\sigma_\infty\,\|R\| \\ &\leq 2(\|Q\| - \|R\|)\sigma_\infty \\ &\leq 2\sigma_\infty\,\|Q-R\|\,. \end{aligned}$$

If, finally, both $\|Q\|, \|R\| \leq 2$, then

$$\begin{aligned} \left\|\tilde{F}(Q) - \tilde{F}(R)\right\| &= \|(2-\|Q\|)F(Q) - (2-\|R\|)F(R)\| \\ &\leq (2-\|Q\|)\,\|F(Q) - F(R)\| + \|F(R)\|\,|\|Q\| - \|R\||\,. \end{aligned}$$

Now we use the Lipschitz continuity of F obtained in (16) and the upper bound for $\|F(R)\|$ of (17):

$$\begin{aligned}\left\|\tilde{F}(Q)-\tilde{F}(R)\right\| &\leq (2-\|Q\|)(2L+\sigma_\infty)\,\|Q-R\|+2\sigma_\infty\,\|Q-R\| \\ &\leq 4(L+\sigma_\infty)\,\|Q-R\|\ .\end{aligned}$$

This completes the proof of the Lemma. ■

Proof of Lemma 2. We have to check that a bounded payoff function f implies the Lipschitz and boundedness conditions on σ assumed in Lemma 1. Let M be a bound for the payoff function f. Then

$$|E(Q,Q)| = \left|\int_S\int_S f(x,y)Q(dy)Q(dx)\right| \leq M\|Q\|^2$$

and

$$\left|\int_S f(x,y)Q(dy)\right| \leq M\|Q\|,$$

yield

$$|\sigma(x,Q)| \leq M\,\|Q\|\,(1+\|Q\|) \tag{18}$$

whence (3) follows with $\sigma_\infty = 6M$.

For the Lipschitz condition (2), note that

$$\begin{aligned}|\sigma(x,Q)-\sigma(x,R)| &= \left|\int f(x,y)(Q-R)(dy)+E(R,R)-E(Q,Q)\right| \\ &\leq M\ \|Q-R\|+|E(R,R)-E(Q,Q)|\ .\end{aligned}$$

The bilinearity of E allows to write

$$|E(R,R)-E(Q,Q)| \leq |E(R,R-Q)|+|E(R-Q,Q)|\ .$$

But

$$\begin{aligned}|E(R,R-Q)| &= \left|\int_S\int_S f(x,y)R(dx)(R-Q)(dy)\right| \\ &\leq M\|R\|\|R-Q\|\end{aligned}$$

implies

$$|E(R,R)-E(Q,Q)| \leq M\,(\|R\|+\|Q\|)\ \|R-Q\|$$

and (2) follows. ■

Proof of Theorem 3. Let $(Q(t))$ be the unique solution of the differential equation

$$Q'(t) = \tilde{F}(Q(t)), \quad Q(0) = P$$

with $P \in \Delta$. Define new nonnegative measures $(R(t))$ via

$$\frac{dR(t)}{dP}(x) = \exp\left(\int_0^t \sigma(x, Q(s))ds\right) .$$

The integral on the right-hand side is well defined because the integrand $\sigma(x, Q(s))$ is uniformly bounded in x on every interval $[0, t]$ by (18) and the continuity of $(Q(s))$. Hence, the density is bounded and bounded away from zero. It follows that R is finite, $R \in \mathcal{M}^e$, and uniformly equivalent with respect to P.

In a first step, we show that R is continuously differentiable with derivative

$$R'(t)(A) = \int_A \sigma(x, Q(t))R(t)(dx) .$$

For this we need to show that

$$\lim_{h \to 0} \frac{1}{h} \left\| R(t+h) - R(t) - h \int \sigma(x, Q(t))R(t)(dx) \right\| = 0.$$

Using Theorem 2 we have that

$$\begin{aligned} & \frac{1}{h} \left\| R(t+h) - R(t) - h \int \sigma(x, Q(t))R(t)(dx) \right\| \\ = & \frac{1}{h} \int \left| \frac{dR(t+h)}{dR(t)}(x) - 1 - h\sigma(x, Q(t)) \right| R(t)(dx) \\ = & \int \frac{1}{h} \left| \exp\left(\int_t^{t+h} \sigma(x, Q(s))ds\right) - 1 - h\sigma(x, Q(t)) \right| R(t)(dx) . \end{aligned}$$

But the integrand approaches zero as $h \to 0$ for every x, and the integrand is bounded in x because so is $\sigma(x, Q(s))$.

Now we show that $R = Q$ on any compact interval $[0, T]$. Let f be a measurable function bounded by 1. Then

$$\begin{aligned} \left| \int f d(R(t) - Q(t)) \right| &= \left| \int_0^t \int_S f d\left(R'_s - Q'_s\right) ds \right| \\ &= \left| \int_0^t \int_S f(x)\sigma(x, Q(s))(R(s) - Q(s))(dx)ds \right| \\ &\leq \int_0^t \left| \int_S f(x)\sigma(x, Q(s))(R(s) - Q(s))(dx) \right| ds \end{aligned}$$

By (18) we have $|f(x)\sigma(x,Q(s))| \leq M\|Q(s)\|\,(1+\|Q(s)\|) \leq Mc$ for some constant c because $\|Q(s)\|$ is continuous and therefore bounded on compact sets. Hence,

$$\left|\int f d(R(t)-Q(t))\right| \leq Mc\int_0^t \|R(s)-Q(s)\|\,ds$$

for all measurable functions f bounded by 1. It follows that

$$\|R(t)-Q(t)\| \leq Mc\int_0^t \|R(s)-Q(s)\|\,ds\,.$$

Gronwall's inequality (see e.g. [11]) gives $\|R(t)-Q(t)\|=0$.

Therefore, $Q(t)=R(t)$ is a nonnegative, finite measure. Because of $Q'(t)(S)=0$ it is a population. ■

Proof of Proposition 4. We start by noting that the weight on the strategies in the open interval $(c,\bar{x})$ increases with time.

$$\begin{aligned}
Q'(t)((c,\bar{x})) &= \int_{(c,\bar{x})} E(\delta_x,Q(t))Q(t)(dx) - Q\,(t)\,((c,\bar{x}))\int_{\mathbb{R}_+} E(\delta_x,Q(t))Q(t)(dx)\\
&\geq \int_{(c,\bar{x})} E(\delta_x,Q(t))Q(t)(dx) - Q\,(t)\,((c,\bar{x}))\int_{(c,\infty)} E(\delta_x,Q(t))Q(t)(dx)\\
&\geq \int_{(c,\bar{x})} E(\delta_x,Q(t))Q(t)(dx) - \int_{(c,\infty)} E(\delta_x,Q(t))Q(t)(dx)\\
&= -\int_{[\bar{x},\infty)} E(\delta_x,Q(t))Q(t)(dx) = 0
\end{aligned}$$

because the strategies above $\bar{x}$ have zero payoff in any circumstances.

Fix a $\delta>0$ with $Q(0)(\{c-\delta\})=0$. We show now that the weight on the closed interval bounded away from c, $[0,c-\delta]$, vanishes exponentially. By the replicator dynamics,

$$Q'(t)([0,c-\delta]) = \int_{[0,c-\delta]} E(\delta_x,Q(t))Q(t)(dx) - Q(t)([0,c-\delta])E(Q(t),Q(t))$$

$$= \int_{[0,c-\delta]} (x-c)q(x)Q(t)((x,\bar{x}])Q(t)(dx) - Q(t)([0,c-\delta])E(Q(t),Q(t))\,.$$

On $[0,c-\delta]$, $x-c\leq-\delta$. Moreover, q is decreasing. Hence

$$Q'(t)([0,c-\delta]) \leq -\delta q(c-\delta)Q(t)([c-\delta,\bar{x}])Q(t)([0,c-\delta]) - Q(t)([0,c-\delta])E(Q(t),Q(t))\,.$$

On the other hand, the strategies above marginal cost have a positive payoff if they win against their opponent, that is

$$\begin{aligned} Q'(t)([c,\bar{x}]) &= \int_{[c,\bar{x}]} E(\delta_x, Q(t))Q(t)(dx) - Q(t)([c,\bar{x}])E(Q(t),Q(t)) \\ &\geq -Q(t)([c,\bar{x}])E(Q(t),Q(t))\,. \end{aligned}$$

Comparing the growth rates on the two sets yields

$$\begin{aligned} \frac{Q'(t)([0,c-\delta])}{Q(t)([0,c-\delta])} - \frac{Q'(t)([c,\bar{x}])}{Q(t)([c,\bar{x}])} &\leq -\delta q(c-\delta)Q(t)([c-\delta,\bar{x}]) \\ &\leq -\delta q(c-\delta)Q(t)((c,\bar{x})) \\ &\leq -\delta q(c-\delta)Q(0)((c,\bar{x})) =: -\varepsilon < 0\,, \end{aligned}$$

where we have used the fact that the weight on $(c,\bar{x})$ increases with time. Armed with the uniform lower bound ε, it follows

$$\frac{Q(t)([0,c-\delta])}{Q(t)([c,\bar{x}])} \leq \frac{Q(0)([0,c-\delta])}{Q(0)([c,\bar{x}])}\, e^{-\varepsilon t} \tag{19}$$

and the weight on $[0,c-\delta]$ decreases exponentially to 0 as desired. ■

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