Communication, Credible Improvements and the Core of an Economy with Asymmetric Information

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Abstract: We analyze an economy with asymmetric information and endogenize the possibilities for information transmission between members of a coalition. We then define a concept of the Core that takes into account these communication possibilities. The internal consistency of the improvements is considered and an Internally Consistent Core is introduced, that requires credibility from the improvements.

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1 Introduction

In his seminal paper Wilson (1978) analyzes an economy with asymmetric information and proposes a definition of the Core. The difficulty that economies with asymmetric information raise stems from the fact that agents evaluate bundles after they have received their information and that not all of them have the same information. Otherwise, either if agents evaluated bundles before their private information is revealed to them or if all agents received the same information, we would be facing a special case of Arrow-Debreu economies and hence no new definition of the Core would be needed. Wilson (1978) notes that when information is asymmetric, it is not enough for each member of a coalition to know that he prefers one allocation to another in order for a coalition to improve upon the latter. It must be *commonly known* by all members of the coalition that this is so. The requirement that the improvement be common knowledge is needed because agents necessarily learn that they are improving upon an allocation when they are doing so, and they must be willing to transact after they have learned everything they learn. The important question is: with respect to what information should an improvement be common knowledge?

When there are no opportunities for communication, the answer is: with respect to the initial information of the agents. For if there is no information transmission, the agents cannot refine their initial information and consequently a coalition can improve upon an allocation only if it is self-evident that they can enforce something better. Wilson's Coarse Core takes these circumstances into account.

When opportunities for communication are allowed, on the other hand, the relevant information should be the initial information refined by the information transmission that has taken place. Therefore a coalition improves upon an allocation when it becomes common knowledge among its members that they can enforce something better, after the permitted communication has taken place. Wilson's Fine Core takes this into account and allows for unlimited communication among agents.

In the definition of both the Coarse and the Fine core the possibilities of communication are exogenously given. In this paper we want to make the information transmission endogenous. The idea is that while it may not be common knowledge that a given allocation is strictly preferred by the members of some coalition, it may *become* common knowledge after all of them express their willingness to trade. In other words, if agents are allowed to repeatedly express their willingness to carry out a trade, the common desirability of an allocation over another may become common knowledge after a sequence of public announcements. The information transmission is endogenous in the sense that it follows from the comparison of two given allocations, and different allocations will give rise to different information transmission. The way the information is transmitted and common knowledge is thus reached, is similar to the processes by which posteriors converge in Geanakoplos and Polemarchakis (1982) and Bacharach (1985).[1] When we allow for the kind of endogenous information transmission described above we get a definition of the Core that is intermediate between the Coarse Core and the Fine Core.

As Wilson (1978) notes, the opportunities for communication may disrupt arrangements for mutual insurance causing the emptiness of the Core. This is equally true for the Fine Core as for our Core with endogenous communication. But there is another disruption that the opportunities for communication may cause and it is related to the credibility of the improvements. With an argument similar to the one used in Ray (1989) for TU games with perfect information, we show that allocations that are not in the Coarse Core can, not only be improved upon by a coalition S, but they can be improved upon by means of allocations that cannot themselves be improved upon in an internally consistent manner. This means that the improvements can be chosen to be stable in a well-defined sense. The same property does not hold when communication is allowed. Specifically, there are allocations in the Core with communication that cannot be improved upon in a credible way. Namely, whenever they are improved upon by a coalition S by means of an allocation y and after some information transmission has taken place, there is a subcoalition of S that can use their new information and improve upon y by means of some other assignment of bundles z. This problem raises the question of what are the allocations that cannot be improved upon in an internally consistent way. When the sort of endogenous information transmission described above is allowed, the answer is what we call the Internally Consistent Core.

Few things can be said at this point about the Internally Consistent Core. It clearly contains the Core with endogenous communication but there is no inclusion relation with the Coarse Core. The fact that it is not a subset of it may be surprising at first sight but it is a natural consequence of allowing communication: an allocation y that is commonly known to be strictly preferred to another x when they are evaluated with the initial information may not be credible, because a subcoalition may improve upon the former using some communication which may be insufficient to improve upon the latter.

Although the Coarse Core, the Core with communication and the Internally Consistent Core are different concepts, the improvements on which they are based essentially coincide when we restrict attention to the grand coalition and to one-person coalitions. In other words, the basic

[1] See also Hart and Tauman (1996) for an economic application.

notions of efficiency and of individual rationality remain unchanged independently of the notion of improvements we use. This allows us to prove a modified version of the No Trade Theorem (see Milgrom and Stokey (1982) for an early version of the theorem) which says that if an allocation is (interim) efficient, it cannot become common knowledge, even after allowing for communication, that there is a mutually beneficial trade.

In this paper, as in Wilson (1978), we abstract from incentive problems. Implicit in the definition of an allocation is the assumption that the states are verifiable. We do this because we want to focus on the topic of endogenous information transmission and the appropriate concept of improvements. For a paper that takes into account incentive problems and thus restricts attention to the allocations that are incentive compatible, see Vohra (1997). A more related paper is Dutta and Vohra (1997) where, restricting attention to incentive compatible allocations, it is attempted to endogenize the information transmission. Holmstrom and Myerson (1983) and Krasa (1997) deal with endogenous communication, but they restrict attention to the grand coalition.

2 Economies

Let Ω be a finite set with generic element ω. The set Ω represents the set of possible *states of the world* and ω represents one such state. Subsets of Ω are called events. A commodity vector is an element of $\mathbb{R}^l_+$. A *bundle* is a function $x_i : \Omega \to \mathbb{R}^l_+$ that assigns a commodity vector to each state of the world. We denote the set of bundles by $\mathbb{B}$. An *agent* i is a fourtuple $(\mathcal{P}_i, u_i, e_i, \mu_i)$ where

$\mathcal{P}_i$ is a partition of Ω that represents i's information

$u_i : \mathbb{R}^l_+ \times \Omega \to \mathbb{R}$ is agent i's state contingent utility function

$e_i : \Omega \to \mathbb{R}^l_+$ is agent i's state contingent initial endowment of commodities

μ_i is a strictly positive probability measure on Ω that represents agent i's prior beliefs.

For each partition Π of Ω, we denote by $\Pi(\omega)$ the element of Π that contains ω. For each bundle $x_i \in \mathbb{B}$ we denote by $u_i(x_i)$ the function $u_i(x_i) : \Omega \to \mathbb{R} | u_i(x_i)(\omega) = u_i(x_i(\omega), \omega)$. Let Π_i be a partition of Ω and let $x_i : \Omega \to \mathbb{R}$ be a bundle. Agent i's conditional expected utility of x_i relative to Π_i is the function $\mathrm{E}_{\mu_i}(u_i(x_i)|\Pi_i) : \Omega \to \mathbb{R}$ that assigns to each state ω, agent i's conditional expectation of $u_i(x_i)$ given the event $\Pi_i(\omega)$, namely

$$\mathrm{E}_{\mu_i}(u_i(x_i)|\Pi_i)(\omega) = \mathrm{E}_{\mu_i}(u_i(x_i)|\Pi_i(\omega)).$$

Definition 1 An *economy with asymmetric information* $\mathcal{E} = \langle N, (\mathcal{P}_i, u_i, e_i, \mu_i)_{i\in N}\rangle$, is a finite collection of agents.

Non-empty subsets of N are called coalitions.

Definition 2 Let S be a coalition. An *information structure* for S is a collection $(\Pi_i)_{i\in S}$ of partitions of Ω, one for each agent in S.

Information structures for N are called simply information structures.

Let $(\Pi_i)_{i\in S}$ be an information structure for S. The *meet* of the partitions $(\Pi_i)_{i\in S}$ is the finest partition of Ω that is coarser than each Π_i, $i \in S$, and is denoted by $\Pi_S = \wedge_{i\in S}\Pi_i$. Similarly, the *join* of the partitions $(\Pi_i)_{i\in S}$ is the coarsest partition that is finer than each Π_i, $i \in S$ and is denoted by $\vee_{i\in S}\Pi_i$. An event E is said to be *common knowledge among the members of S at ω with respect to information structure* $(\Pi_i)_{i\in S}$ if $\Pi_S(\omega) \subseteq E$. If an information structure is not specified, it should be understood that we mean the initial information partitions $\mathcal{P}_i$. The formal definition of common knowledge was first introduced by Aumann (1976).

Definition 3 Let S be a coalition. An *assignment of bundles to agents in S* is a function $y : S \to \mathbb{B}$.

Definition 4 Let $\mathcal{E} = \langle N, (\mathcal{P}_i, u_i, e_i, \mu_i)_{i\in N}\rangle$ be an economy and let S be a coalition. An *S-allocation* is an assignment of bundles to agents in S such that $\sum_{i\in S} y(i) = \sum_{i\in S} e_i$.

S-allocations represent all those redistributions of their endowments that coalition S can carry out without the consent of others. N-allocations are simply called allocations. Denote by $\mathcal{A}(S)$ the set of all S-allocations and by $\mathcal{A}$ the set of allocations. If y is an assignment of bundles to agents in S and $T \subseteq S$, we shall write y_T for the projection of y on T and, in particular, y_i for $y(i)$.

Definition 5 Let $\mathcal{E} = \langle N, (\mathcal{P}_i, u_i, e_i, \mu_i)_{i\in N}\rangle$ be an economy. We say that allocation x is *individually rational* if

$$\mathrm{E}_{\mu_i}(u_i(x_i)|\mathcal{P}_i) \geq \mathrm{E}_{\mu_i}(u_i(e_i)|\mathcal{P}_i) \qquad \forall i \in N.$$

Definition 6 Let $\mathcal{E} = \langle N, (\mathcal{P}_i, u_i, e_i, \mu_i)_{i\in N}\rangle$ be an economy. We say that allocation x is *strongly efficient* if there is no allocation $y \in \mathcal{A}$ and state $\omega^* \in \Omega$ at which it is common knowledge that

$$\mathrm{E}_{\mu_i}(u_i(y_i)|\mathcal{P}_i) \geq \mathrm{E}_{\mu_i}(u_i(x_i)|\mathcal{P}_i) \qquad \forall i \in N,$$

with strict inequality for some $i \in N$ and some ω.
We say that allocation x is *efficient* if there is no allocation $y \in \mathcal{A}$ and state $\omega^* \in \Omega$ at which it is common knowledge that

$$\mathrm{E}_{\mu_i}(u_i(y_i)|\mathcal{P}_i) \geq \mathrm{E}_{\mu_i}(u_i(x_i)|\mathcal{P}_i) \qquad \forall i \in N$$

with strict inequality at some ω for all $i \in N$.
We say that allocation x is *weakly efficient* if there is no allocation $y \in \mathcal{A}$ and state $\omega \in \Omega$ at which it is common knowledge that

$$\mathrm{E}_{\mu_i}(u_i(y_i)|\mathcal{P}_i) > \mathrm{E}_{\mu_i}(u_i(x_i)|\mathcal{P}_i) \qquad \forall i \in N.$$

The existence of individually rational and strongly efficient allocations is guaranteed if the individuals' utility functions are continuous (see Wilson (1978) for details).

3 Improvements

In this section we present several ways a coalition can improve upon an allocation. They differ in the degree of the communication that is allowed. The first notion is due to Wilson (1978) and allows no communication between the agents. In order for a coalition to improve upon an allocation it must be common knowledge among the member of the coalition, with respect to their initial information partitions, that they can enforce an S-allocation that is strictly preferred by all of them.

Definition 7 Let T be a coalition, let x be an assignment of bundles to agents in T, and let $(\Pi_i)_{i\in T}$ be an information structure for T. Let $S \subseteq T$ be another coalition. We say that S-allocation $y \in \mathcal{A}(S)$ is a *coarse improvement of S upon x with respect to information structure $(\Pi_i)_{i\in T}$ at ω^** if

$$\mathrm{E}_{\mu_i}[u_i(y_i)|\Pi_i](\omega) > \mathrm{E}_{\mu_i}[u_i(x_i)|\Pi_i](\omega) \qquad \forall i \in S, \quad \forall \omega \in \Pi_S(\omega^*).$$

When y is a coarse improvement of S upon x with respect to $(\Pi_i)_{i \in T}$ at some $\omega \in \Omega$, we say that x is improved upon by S with respect to $(\Pi_i)_{i \in T}$.

Associated with the notion of coarse improvements, we have the definition of the Coarse Core.

Definition 8 The *Coarse Core* of the economy $\mathcal{E}$ with respect to information structure $(\Pi_i)_{i \in N}$ is the set of its allocations that can be improved upon with respect to $(\Pi_i)_{i \in N}$ by no coalition.

The Coarse Core of the economy $\mathcal{E}$ with respect to information structure $(\mathcal{P}_i)_{i \in N}$ is simply called the Coarse Core of $\mathcal{E}$.

The second notion is appropriate when agents have unlimited communication possibilities.

Definition 9 Let $\mathcal{E} = \langle N, (\mathcal{P}_i, u_i, e_i, \mu_i)_{i \in N} \rangle$ be an economy and let x be an allocation in it. We say that S-allocation y is a *fine improvement of S upon x* if there is an information structure $(\Pi_k)_{k \in S}$ for S such that

for all $i \in S$, Π_i is not coarser than $\mathcal{P}_i$

for all $i \in S$, Π_i is not finer than $\vee_{k \in S} \mathcal{P}_k$

and with respect to which y is a coarse improvement of S upon x_S.

Associated with fine improvements, we have the notion of the Fine Core, found in Wilson (1978).

Definition 10 The *Fine Core* of the economy $\mathcal{E}$ is the set of all its allocations that can be finely improved upon by no coalition.

We want to introduce a notion of improvement, that allows for some but not unlimited communication. To understand our notion, note that for an S-allocation y to be a coarse improvement upon x, it must be common knowledge without communication that y is strictly preferred to x by all agents in S. Also, for y to be a fine improvement upon x, it should be possible to redistribute the information initially held by the members of the coalition in a way that nobody learns more than what can be learned by pooling all the information, nobody forgets what he knows, and that makes it common knowledge that y is strictly preferred to x. Our notion lies somewhere in the middle since it does not allow unlimited information transmission. We allow only the transmission of information that takes place when the agents repeatedly agree to carry out the transaction. The fact that y is strictly preferred to x by all members of S should *become* common knowledge after a long handshake during which the agents ratify their willingness to take y instead of x. Only after this long handshake can we say that it became common knowledge that y is better than x. In order to formalize the idea we need some piece of notation.

Let f and g be two functions on Ω. We denote by $[f > g]$ the event at which $f > g$ namely,

$$[f > g] = \{w \in \Omega : f(w) > g(w)\}.$$

Similarly $[f \leq g] = [f > g]^c$. Also, we denote by $\mathcal{P}(f > g)$ the partition generated by $[f > g]$, namely

$$\mathcal{P}(f > g) = \{[f > g], [f \leq g]\}.$$

Definition 11 Let T be a coalition, let x be an assignment of bundles to agents in T, let $(\Pi_i)_{i \in T}$ be an information structure for T, and let $S \subseteq T$ be a coalition. Let $y \in \mathcal{A}(S)$ be an S-allocation and define the information structures for S, $(\Pi_i^S(n))_{i \in S}$, for $n = 0, 1, 2, \ldots$, recursively as follows:

$$\Pi_i^S(0) = \Pi_i \qquad \forall i \in S,$$

for $n \in \mathbb{N}$,

$$\Pi_i^S(n+1) = \Pi_i^S(n) \vee \Big[\bigvee_{k \in S} \mathcal{P}\Big(\mathrm{E}(u_k(y_k)|\Pi_k^S(n)) > \mathrm{E}(u_k(x_k)|\Pi_k^S(n))\Big)\Big] \qquad \forall i \in S$$

and

$$\Pi_i^S = \lim_{n \to \infty} \Pi_i^S(n).$$

We say that $\langle y, (\Pi_i^S)_{i \in S}\rangle$ *is an improvement of* S *upon* $\langle x, (\Pi_i)_{i \in T}\rangle$ *at* $\omega \in \Omega$ if

$$\mathrm{E}[u_i(y_i)|\Pi_i^S(n)](\omega) > \mathrm{E}[u_i(x_i)|\Pi_i^S(n)](\omega) \qquad \forall i \in T, \quad \forall n \in \mathbb{N}.$$

When $\langle y, (\Pi_i^S)_{i \in S}\rangle$ is an improvement of S upon $\langle x, (\Pi_i)_{i \in N}\rangle$ at some ω, we say that $\langle x, (\Pi_i)_{i \in N}\rangle$ is improved upon by S.

Implicit in the above definition, there is a "dialogue" in the sense of Bacharach (1985). In this dialogue agents repeatedly communicate their willingness to perform a trade. In order for $\langle y, (\Pi_i^S)_{i \in S}\rangle$ to be an improvement of S upon $\langle x, (\Pi_i)_{i \in N}\rangle$ at ω, the following should hold:

1. All agents in S should prefer y to x at ω when they evaluate these assignments according to the information structure Π_i

2. All agents in S should prefer y to x at ω when they evaluate these assignments according to the information structure that results from the refinement of Π_i with the fact that all the previous items hold

3. The same as above, and so on ad infinitum.

Note that in order for y to be part of an improvement upon x, it must be the case that the agents prefer y to x along the whole dialogue. An alternative definition would require only that the agents prefer y to x in the limit, namely after the refinement process converged, and independently of the answers along the dialogue.

Clearly, since the individual partitions are assumed to be finite, the refinement process just described ends after a finite number of rounds. More formally, by definition of the partitions $\Pi_i^S(n)$, the sequence $\{\Pi_i^S(n)\}_{n \in I\!N}$ is non-decreasing. Since the state space is finite, there must be a step $M \in I\!N$ such that for all $n > M$ we have $\Pi_i^S(n) = \Pi_i^S(n+1)$ for all $i \in S$. This shows that the partitions Π_i^S are well-defined and satisfy

$$\Pi_i^S = \Pi_i^S \vee \Big[\bigvee_{k \in S} \mathcal{P}\Big(\mathrm{E}(u_k(y_k)|\Pi_i^S) > \mathrm{E}(u_k(x_k)|\Pi_i^S) \Big) \Big]. \tag{1}$$

Remark. Assume $\langle y, (\Pi_i^S)_{i \in S} \rangle$ is an improvement of S upon $\langle x, (\Pi_i)_{i \in N} \rangle$ at some ω, then y is a coarse improvement of S upon x_S with respect to the information structure $(\Pi_i^S)_{i \in S}$. Namely it *became* common knowledge among the members of S that x is strictly preferred to y.

Proof : Since $\langle y, (\Pi_i^S)_{i \in S} \rangle$ is an improvement upon $\langle x, (\Pi_i)_{i \in S} \rangle$ at ω we have that

$$\mathrm{E}(u_i(y_i)|\Pi_i^S(n))(\omega) > \mathrm{E}(u_i(x_i)|\Pi_i^S(n))(\omega) \qquad \forall i \in S \quad \forall n \in I\!N.$$

Since $\Pi_i^S(n) = \Pi_i^S$ for some $n \in I\!N$ we have

$$\mathrm{E}(u_i(y_i)|\Pi_i^S)(\omega) > \mathrm{E}(u_i(x_i)|\Pi_i^S)(\omega) \qquad \forall i \in S \quad .$$

This, together with equality (1) implies that

$$\Pi_i^S(\omega) \subseteq \bigcap_{k \in S} [\mathrm{E}(u_k(y_k)|\Pi_k^S) > \mathrm{E}(u_k(x_k)|\Pi_k^S)] \qquad \forall i \in S.$$

Consequently,

$$\Pi_S^S(\omega) \subseteq \bigcap_{k \in S} [\mathrm{E}(u_k(y_k)|\Pi_k^S) > \mathrm{E}(u_k(x_k)|\Pi_k^S)]$$

which means that it is common knowledge with respect to $(\Pi_k^S)_{k\in S}$ at ω that all members of S prefer y to x_S. $\square$

The previous remark allow us to say that if $\langle y, (\Pi_i^S)_{i\in S}\rangle$ is an improvement of S upon $\langle x, (\Pi_i)_{i\in N}\rangle$ at some ω, then *it can become common knowledge among the members of S that x is strictly preferred to y.*

In order to understand the definition of an improvement, consider the following example.

Example 1 Consider the following three-agent four-state economy with a common prior given by $\mu = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Agents' utility functions are constant across states and given by $u_i(a,b) = ab$, for $i = 1, 2, 3$. The initial endowments and information partitions are described in the following table, which also contains a feasible allocation y.

		Endowment (e_i)				Allocation y			
Agent	$\mathcal{P}_i$	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4
1	$\{\{\omega_1,\omega_2,\omega_3\},\{\omega_4\}\}$	$(2,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$	$(0,1)$
2	$\{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\}$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$	$(1,1)$	$(1,1)$	$(0,0)$	$(1,2)$
3	$\{\{\omega_1\},\{\omega_2,\omega_3,\omega_4\}\}$	$(0,3)$	$(1,1)$	$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,2)$	$(0,0)$

The expected utilities of each agent, derived from the above allocations are given by:

	ω_1	ω_2	ω_3	ω_4
$\mathrm{E}(u_1(e_1)\|\mathcal{P}_1)$	$\{1/3$	$1/3$	$1/3\}$	$\{1\}$
$\mathrm{E}(u_1(y_1)\|\mathcal{P}_1)$	$\{2/3$	$2/3$	$2/3\}$	$\{0\}$
$\mathrm{E}(u_2(e_2)\|\mathcal{P}_2)$	$\{0$	$0\}$	$\{1/2$	$1/2\}$
$\mathrm{E}(u_2(y_2)\|\mathcal{P}_2)$	$\{1$	$1\}$	$\{1$	$1\}$
$\mathrm{E}(u_3(e_3)\|\mathcal{P}_3)$	$\{0\}$	$\{1/3$	$1/3$	$1/3\}$
$\mathrm{E}(u_3(y_3)\|\mathcal{P}_3)$	$\{1\}$	$\{2/3$	$2/3$	$2/3\}$

At no state of the world is it common knowledge that all agents prefer the assignment y to the assignment x, because at state ω_4 agent 1 strictly prefers x to y. Consequently, y is not a coarse improvement of N upon the endowment.

Assume, however, that the true state is ω_1. At this state all three agents prefer y to x. Since this is true for all states except for ω_4, after y not having been rejected by any agent, it becomes common knowledge that state w_4 is not the true state.

This gives rise to the following information structure:

$$\begin{aligned}\Pi_1(1) &= \{\{\omega_1,\omega_2,\omega_3\},\{\omega_4\}\}\\ \Pi_2(1) &= \{\{\omega_1,\omega_2\},\{\omega_3\},\{\omega_4\}\}\\ \Pi_3(1) &= \{\{\omega_1\},\{\omega_2,\omega_3\},\{\omega_4\}\}\end{aligned}$$

with the corresponding new expected utilities for each agent:

	ω_1	ω_2	ω_3	ω_4
$\mathrm{E}(u_1(x_1)\vert\Pi_1(1))$	{1/3	1/3	1/3}	{1}
$\mathrm{E}(u_1(y_1)\vert\Pi_1(1))$	{2/3	2/3	2/3}	{0}
$\mathrm{E}(u_2(x_1)\vert\Pi_2(1))$	{0	0}	{1}	{0}
$\mathrm{E}(u_2(y_2)\vert\Pi_2(1))$	{1	1}	{0}	{2}
$\mathrm{E}(u_3(x_3)\vert\Pi_3(1))$	{0}	{1/2	1/2}	{0}
$\mathrm{E}(u_3(y_3)\vert\Pi_3(1))$	{1}	{1	1}	{0}

Note that all agents prefer y to x except for agent 2 at state ω_3 (ω_4 is irrelevant at this stage, since it became common knowledge that it did not occur). Consequently, it becomes apparent at this stage (do not forget that we are at ω_1) that ω_3 is not the true state. This gives rise to the following information structure:

$$\begin{aligned}\Pi_1(2) &= \{\{\omega_1,\omega_2\},\{\omega_3\},\{\omega_4\}\}\\ \Pi_2(2) &= \{\{\omega_1,\omega_2\},\{\omega_3\},\{\omega_4\}\}\\ \Pi_3(2) &= \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\}\end{aligned}$$

The corresponding expected utilities, with respect to the updated information structure is given by:

	ω_1	ω_2	ω_3	ω_4
$\mathrm{E}(u_1(x_1)\vert\Pi_1(2))$	{1/2	1/2}	{0}	{1}
$\mathrm{E}(u_1(y_1)\vert\Pi_1(2))$	{1	1}	{0}	{ 0}
$\mathrm{E}(u_2(x_2)\vert\Pi_2(2))$	{0	0}	{1}	{0}
$\mathrm{E}(u_2(y_2)\vert\Pi_2(2))$	{1	1}	{0}	{2}
$\mathrm{E}(u_3(x_3)\vert\Pi_3(2))$	{0}	{1}	{0}	{0}
$\mathrm{E}(u_3(y_3)\vert\Pi_3(2))$	{1}	{0}	{2}	{0}

At this point it becomes common knowledge that the true state is ω_1 because if the true state were ω_2, agent 3 would prefer x to y, while at ω_1 all three agents prefer y to x. As a conclusion we can say that:

Allocation y together with the limit information structure

$$\Pi_i(3) = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}, \qquad i = 1, 2, 3$$

is an improvement of N upon $\langle x, (\mathcal{P}_i)_{i \in N} \rangle$.

The above definition of an improvement leads to the following definition of a core with endogenous communication.

Definition 12 The *Core with endogenous communication*, or simply the *Core*, of $\mathcal{E}$ is the set of all allocations $x \in \mathcal{A}$, such that $\langle x, (\mathcal{P}_i)_{i \in N} \rangle$ is improved upon by no coalition.

Given the motivation of the Core, the next two propositions can hardly be surprising.

Proposition 1 The core of an economy is a subset of its Coarse Core.

Proof : Assume that x is an allocation that is not in the Coarse Core of the economy. Then there exists an S-allocation y and a state ω at which it is common knowledge among the members of S that y is preferred by each of them to x. But then $\langle y, (\mathcal{P}_i)_{i \in S} \rangle$ is an improvement of S upon $\langle x, (\mathcal{P}_i)_{i \in S} \rangle$ at ω since nothing can be learned from the fact that all agents strictly prefer y to x. $\square$

Proposition 2 The Fine Core of an economy is a subset of its Core.

Proof : Let $\mathcal{E}$ be an economy with set of agents N. Assume that allocation x is not in the Core of $\mathcal{E}$. This means that there is a coalition $S \subseteq N$ with an improvement upon $\langle x, (\mathcal{P}_i)_{i \in N} \rangle$. Let $\langle y, (\Pi_i^S)_{i \in S} \rangle$ be that improvement. This means that y coarsely improves upon x_S with respect to $(\Pi_i^S)_{i \in S}$. Moreover, by construction, for all $i \in S$, Π_i^S is not coarser than $\mathcal{P}_i$ and not finer than $\vee_{k \in S} \mathcal{P}_k$. Consequently y is a fine improvement upon x. $\square$

4 Internally Consistent Improvements

One question that has been asked in the context of perfect information is whether the S-allocations that are used to improve upon other allocations are themselves immune to deviations of some of the members of S. Improvements that are immune to internal deviations are internally consistent. [2] Ray (1989) showed that in the context of perfect information, it is always possible to find internally consistent improvements upon allocations that are not in the Core. In this section we want to answer the same question but for the context of economies with asymmetric information. As we shall see, Ray's result still holds, as long as there is no communication. As soon as communication is allowed, we'll be able to find allocations in the core that are not improved upon by any internally consistent improvement.

Definition 13 Let $x \in \mathcal{A}(T)$ be a T-allocation, let $(\Pi_i)_{i \in T}$ be an information structure, and let $S \subseteq T$ be a coalition. We say that $y \in \mathcal{A}(S)$ is an *internally consistent coarse improvement of S upon x with respect to $(\Pi_i)_{i \in T}$ at $\omega \in \Omega$*

- when $|S| = 1$, if y is a coarse improvement upon x with respect to $(\Pi_i)_{i \in T}$ at ω
- when $|S| > 1$, if y is a coarse improvement upon x with respect to $(\Pi_i)_{i \in T}$ at ω such that there is no coalition $F \subsetneq S$ with an internally consistent coarse improvement upon y with respect to $(\Pi_i)_{i \in S}$ at ω.

The following result is the extension of Ray (1989) to economies with asymmetric information.

Theorem 1 Let $\mathcal{E}$ be an economy with asymmetric information, let x be an allocation in it and let $(\Pi_i)_{i \in N}$ be an information structure. The following statements are equivalent:

i. There is a coalition with a coarse improvement upon x with respect to $(\Pi_i)_{i \in N}$

ii. There is a coalition with an internally consistent coarse improvement upon x with respect to $(\Pi_i)_{i \in N}$.

Proof :

[2] The idea of internally consistent improvements is related to the non-cooperative solution concept of Coalition-Proof Nash Equilibrium defined in Bernheim, Peleg, and Whinston (1987)

$i \to ii$. Assume there is a coalition $S \subseteq N$ with a coarse improvement upon x at some ω^*. Namely, there is an S-allocation y and event $\Pi_S(\omega^*) \in \bigwedge_{k\in S}\Pi_k$ such that $\mathrm{E}(u_i(y_i)|\Pi_i)(\omega) > \mathrm{E}(u_i(x_i)|\Pi_i)(\omega)$ for all $\omega \in \Pi(\omega^*)$, for all $i \in S$. If y is an internally consistent coarse improvement of S upon x at ω^*, we are done. So assume that there is a coalition $T \subsetneq S$ and an internally consistent coarse improvement z of T upon y at ω^*. Namely there is a T-allocation z and an event $\Pi_T(\omega^*) \in \wedge_{k\in T}\Pi_k$ such that $\mathrm{E}(u_i(z_i)|\Pi_i)(\omega) > \mathrm{E}(u_i(y_i)|\Pi_i)(\omega)$ for all $\omega \in \Pi_T(\omega^*)$ and for all $i \in T$. But then, since $\Pi_T(\omega^*) \subseteq \Pi_S(\omega^*)$, we have that $\mathrm{E}(u_i(z_i)|\Pi_i)(\omega) > \mathrm{E}(u_i(x_i)|\Pi_i)(\omega)$ for all $\omega \in \Pi_T(\omega^*)$, which means that z is an internally consistent coarse improvement of T upon x as well.

$ii \to i$. An internally consistent coarse improvement is, in particular, a coarse improvement.

□

Next we want to extend the notion of internal consistency to the improvements that define the Core with endogenous communication.

Definition 14 Let $x \in \mathcal{A}(T)$ be a T-allocation, let $(\Pi_i)_{i\in T}$ be an information structure, and let $S \subseteq T$ be a coalition. We say that $\langle y, (\Pi_i^S)_{i\in S}\rangle$ is an *internally consistent improvement of S upon $\langle x, (\Pi_i)_{i\in T}\rangle$ at $\omega \in \Omega$*

- when $|S| = 1$, if $\langle y, (\Pi_i^S)_{i\in S}\rangle$ is an improvement upon $\langle x, (\Pi)_{i\in T}\rangle$ at ω
- when $|S| > 1$, if $\langle y, (\Pi_i^S)_{i\in S}\rangle$ is an improvement upon $\langle x, (\Pi)_{i\in N}\rangle$ such that there is no coalition $F \subsetneq S$ with an internally consistent improvement upon $\langle y, (\Pi_i^S)_{i\in S}\rangle$ at ω.

When $\langle y, (\Pi_i^S)_{i\in S}\rangle$ is an internally consistent improvement of S upon $\langle x, (\Pi_i)_{i\in N}\rangle$ at some ω, we say that x is improved upon by S in an internally consistent way.

Definition 15 The *Internally Consistent Core* of $\mathcal{E}$ is the set of all allocations $x \in \mathcal{A}$ such that $\langle x, (\mathcal{P}_i)_{i\in N}\rangle$ is improved upon by no coalition in an internally consistent way.

The following proposition shows that the previous definition is not superfluous.

Proposition 3 The Core of an economy is a subset of its internally consistent core and the reverse inclusion does not necessarily hold.

Proof : Since an internally consistent improvement is also an improvement, it is clear that the Core is a subset of the Internally Consistent Core. To see that the reverse inclusion does not necessarily hold, consider the following economy, taken from Wilson (1978). There are three agents, each with the same constant across states utility function, given by $u(a) = \sqrt{a}$. There is a common prior given by the uniform distribution over the states space. Consider allocation x in the next table.

			Endowment (e)			Allocation (x)		
Agent	$\mathcal{P}_i$	State (ω)	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
1	$\{\{\omega_1\},\{\omega_2,\omega_3\}\}$		5	1	3	5	2	2
2	$\{\{\omega_2\},\{\omega_1,\omega_3\}\}$		3	5	1	2	5	2
3	$\{\{\omega_3\},\{\omega_1,\omega_2\}\}$		1	3	5	2	2	5

Allocation x is not in the Core because coalition $S = \{1, 2\}$ has an improvement at ω_1 which is given by the S-allocation y and the information structure $(\Pi_i^S)_{i\in S}$ below:

			Allocation (y)		
Agent	Π_i^S	State	ω_1	ω_2	ω_3
1	$\{\{\omega_1\},\{\omega_2,\omega_3\}\}$		5.5	0	0
2	$\{\{\omega_2\},\{\omega_1\},\{\omega_3\}\}$		2.5	6	4

The improvement $\langle y, (\Pi_i^S)_{i\in S}\rangle$ is not internally consistent since it is improved upon by agent 2 at ω_1 in an internally consistent way by means of his own endowment.

The same kind of argument can be made for every improvement upon $\langle x, (\mathcal{P}_i)_{i\in N}\rangle$. This shows that x belongs to the Internally Consistent Core of the economy. □

The previous result raises the question of what is the appropriate concept of the core when communication is allowed.[3] Fortunately, however, the basic concepts of individual rationality and efficiency are not influenced by the possibility of communication as the following propositions show.

Proposition 4 The following statements are equivalent:

[3]The fact that allocations outside the Core are not improved upon in an internally consistent way is not due to special manner the information is transmitted. It follows from the fact that there is communication.

i. Allocation x is individually rational

ii. There is no agent with an improvement upon x

iii. There is no agent with an internally consistent improvement upon x.

Proof : Trivial $\square$

Proposition 5 The following statements are equivalent:

i. Allocation x is efficient;

ii. The grand coalition does not have an improvement upon x;

iii. The grand coalition does not have an internally consistent improvement upon x.

Proof :

$i \to ii$. If $\langle y, (\Pi_i)_{i \in N} \rangle$ is an improvement of N upon $\langle x, (\mathcal{P}_i)_{i \in N} \rangle$, then for some ω^* we must have

$$\mathrm{E}_{\mu_i}(u_i(y_i)|\Pi_i)(\omega^*) > \mathrm{E}_{\mu_i}(u_i(x_i)|\Pi_i)(\omega^*) \qquad \forall i \in N.$$

Define the allocation z by

$$z_i(\omega) = \begin{cases} y_i(\omega) & \text{if } \omega \in \Pi_i(\omega^*) \\ x_i(\omega) & \text{otherwise} \end{cases}$$

It is not difficult to see that z dominates x and as a result x cannot be efficient.

$ii \to iii$. If there is no improvement, you will never find an internally consistent one.

$iii \to i$. Assume that x is an inefficient allocation. Then there is an allocation y and state ω^* at which it is common knowledge that y is at least as good as x in the eyes of all the agents, and that everybody strictly prefers y to x at some state. Without loss of generality we can choose y to be efficient. By the previous steps then, y can be improved upon in a consistent way by no coalition which means that x is improved upon in an internally consistent way.

$\square$

Corollary 1 Let $\mathcal{E}$ be a two person economy where agents have continuous utility functions. The Internally Consistent Core is a nonempty subset of the Coarse Core.

Proof : It follows from Propositions 4 and 5 and from the existence of individually rational and efficient allocations. □

Corollary 2 (Modified No-Trade Theorem) Assume x is an efficient allocation. Then there is no allocation y that can become commonly known to be strictly preferred to x by all agents.

Proof : Since x is efficient, by Proposition 5 there is no allocation y and state ω at which it can become common knowledge that y is preferred to x by all agents. □

The well-known No-Trade Theorem (see Milgrom and Stokey (1982), Holmstrom and Myerson (1983) or Rubinstein and Wolinsky (1990) for different versions) states that if an allocation is ex-ante efficient, it must be (interim) efficient as well. The interpretation is that if the agents traded in contingent commodities before they get their private information, and they reached an ex-ante efficient allocation, then after they get their private information, from wherever it may come, it is impossible to find a reallocation that is commonly known to be preferred by all agents. Our modified no-trade theorem is different in that the starting point is a stage at which the individuals already got their private information. The starting point is one at which the information is truly asymmetric. Our theorem says that if the allocation is (interim) efficient (and not necessarily ex-ante efficient), it is impossible for the agents to find an allocation that, after some dialogue takes place, becomes common knowledge to be strictly preferred to the former. In both theorems there is a comparison between an initial situation and a final situation where agents have more information. The difference is that while the initial situation in the no-trade theorem is one is one in which the information is symmetric and the allocation is ex-ante efficient, in the modified version the initial situation is one where the agents already have asymmetric information and the allocation is only (interim) efficient.

Proposition 5 cannot be generalized for the case where all coalitions are allowed. The only thing can be said is that if a coalition cannot find an improvement upon x, it will never find one that is internally consistent nor one that is coarse. The example that appears in the proof of Proposition 3 shows, on the other hand, that a coalition may have an improvement upon x without having an internally consistent one. The same example shows that a coalition may have an improvement upon x without having a coarse improvement upon x. As for the relation between the Coarse Core and the Internally Consistent Core, Corollary 1 shows that for two-person economies, the Internally Consistent Core is a subset of the Coarse Core. In general, however, there is no inclusion relation whatsoever, as the following example shows.

Example 2 Consider the following economy with three agents, two commodities, four states of the world, and a common prior $\mu = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Agents' utility functions are constant across states and given by $u_i(a, b) = \sqrt{a + b}$ for both $i = 1, 2$ and $u_3(a, b) = \sqrt{\min\{a, b\}}$.

		Endowment			
Agent	$\mathcal{P}_i$	ω_1	ω_2	ω_3	ω_4
1	$\{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}$	$(5, 0)$	$(7, 0)$	$(5, 0)$	$(7, 0)$
2	$\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$	$(0, 5)$	$(0, 7)$	$(0, 5)$	$(0, 7)$
3	$\{\{\omega_1, \omega_2, \omega_3, \omega_4\}\}$	$(2, 2)$	$(0, 0)$	$(2, 2)$	$(0, 0)$

The endowment e is not in the Coarse Core because the allocation y below is commonly known (at every state) to be strictly preferred to e by all agents.

		Allocation y			
Agent	$\mathcal{P}_i$	ω_1	ω_2	ω_3	ω_4
1	$\{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}$	$(6, 0)$	$(6, 0)$	$(6, 0)$	$(6, 0)$
2	$\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$	$(0, 6)$	$(0, 6)$	$(0, 6)$	$(0, 6)$
3	$\{\{\omega_1, \omega_2, \omega_3, \omega_4\}\}$	$(1, 1)$	$(1, 1)$	$(1, 1)$	$(1, 1)$

Moreover, it can be checked that y belongs to the Coarse Core. Allocation y, however, does not form part of an internally consistent improvement upon e because coalition $S = \{1, 2\}$ has the following internally consistent improvement upon it at ω_2.

		Allocation z			
Agent	Π_i^S	ω_1	ω_2	ω_3	ω_4
1	$\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$	$(3, 0)$	$(7, 0)$	$(5, 2)$	$(7, 0)$
2	$\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$	$(2, 5)$	$(0, 7)$	$(0, 3)$	$(0, 7)$

To see this, note that if ω_2 is the true state of the world, both agents prefer z to y (z provides them with an expected utility of 7). Since ω_2 is the only state at which both agents prefer z to y, they learn from their mutual agreement that the state is indeed ω_2, and it becomes common knowledge that z is preferred to y by both of them. Moreover, z is a "credile" S-allocation because it cannot be improved upon by any singleton at ω_2.

Consequently, y is not in the Internally Consistent Core. This shows that the Coarse Core is not a subset of the Internally Consistent Core. On the other hand, allocation z does not form part of an internally consistent improvement upon e. Moreover, the initial endowment belongs to the Internally Consistent Core. To see this, note that no pair of agents that contains agent 3 can improve upon the endowment since no agent can individually increase agent's 3 utility. It is not difficult to check that agents 1 and 2 cannot find an improvement upon the endowment. And finally all the improvements of the grand coalition are not internally consistent because an internally consistent improvement upon them similar to the one in the example can be found. This shows that the Internally Consistent Core is not a subset of the Coarse Core.

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