Efficiency Does Not Imply Immediate Agreement[*]

Sergiu Hart[†] Zohar Levy[‡]

July 1997

Abstract

Gul [1989] introduces a non-cooperative bargaining procedure and claims that the payoffs of the resulting efficient stationary subgame perfect equilibria are close to the Shapley value of the underlying transferable utility game (when the discount factor is close to 1). We exhibit here an example showing that efficiency, even for strictly super-additive games, does *not* imply that all meetings end in agreement. Thus efficiency does not suffice to get Gul's result. *Journal of Economic Literature* Classification Numbers: C71, C72, C78, D4.

In a most interesting and important paper, Gul [1989] provides a bargaining procedure leading to the Shapley [1953] value for games with transferable utility.[1] The main result there (Theorem 1) states that the payoffs of efficient stationary subgame perfect Nash equilibria are close to the Shapley value, when the discount factor is close to 1.

The non-cooperative game proceeds as follows: At every period, there is a meeting between two randomly chosen players: a "proposer" i and a "responder" j. The proposer makes an offer to "buy out" the responder (for a certain amount);

---

[*]Research conducted in 1993-1994.

[†]Department of Economics; Department of Mathematics; and Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem. *e-mail:* hart@math.huji.ac.il. Research partially supported by grants of the U.S.-Israel Binational Science Foundation and the Israel Academy of Sciences and Humanities.

[‡]Department of Computer Science, The Hebrew University of Jerusalem. *e-mail:* zohar@cs.huji.ac.il.

[1]For other bargaining models generating the Shapley value, see: Harsanyi [1981]; Mas-Colell [1988]; O. Hart and Moore [1990]; Winter [1994]; S. Hart and Mas-Colell [1996].

the responder may either accept or reject the offer. In the former case, j "leaves" the game and i "represents" j in all subsequent periods. At the end of each period, each player which is still in the game gets a payoff equal to the worth of the coalition of players that he represents. Let Q_t denote the resulting partition of the set of players at time t (i.e., $Q_t := \{M_t^i : i \text{ is in the game at time } t\}$, where M_t^i is the set of all players represented by i at stage t), then the sum of payoffs of all players is

$$W \equiv W(N,v;\delta) := (1-\delta)\sum_{t=0}^{\infty}\sum_{M_t^i \in Q_t} \delta^t v(M_t^i),$$

where (N,v) is the given coalitional game and δ is the (common to all players) discount factor.

A strategy profile (i.e., an $N-$tuple of strategies) σ is *efficient* if there is no other strategy profile τ such that the expected sum of payoffs of all players is higher for τ than for σ; that is, $E_\sigma(W) = \max_\tau E_\tau(W)$.

The underlying coalitional game (N,v) is assumed to be *strictly super-additive*, i.e.

$$v(L) + v(M) < v(L \cup M)$$

for all non-empty $L, M \subset N$ with $L \cap M = \emptyset$.

Efficiency is used in Gul's proof in order to guarantee that every meeting ends in agreement. The intuition is that, by strict super-additivity, there will otherwise be a loss in total utility.[2]

However, this is not so. Consider the following example (N, v_1): There are 8 players, $N := \{1,2,3,4,5,6,7,8\}$. Each one of the last six players (3, ..., 8) initially owns one unit of a raw material (that yields no direct utility to any player). As for the first two players, 1 and 2, each one owns a constant returns to scale technology that transforms the raw material into utils, in the ratio 1-to-1. The coalitional function v_1 is thus[3]

$$v_1(S) = \begin{cases} 0, & \text{if } S \cap \{1,2\} = \emptyset; \\ |S| - 1, & \text{if } |S \cap \{1,2\}| = 1; \text{ and} \\ |S| - 2, & \text{if } S \supset \{1,2\}; \end{cases}$$

[2]This is claimed without proof in Gul's paper; it was considered to be clear and trivially true (F. Gul, private communication, 1993).

[3]The game v_1 is super-additive but not *strictly* super-additive; we will modify it slightly below. Notation: $|S|$ is the number of elements of the finite set S.

for all $S \subset N$. Clearly, 1 and 2 are perfect substitutes. Moreover, putting them together in the same coalition is the same as eliminating one of them: $v_1(S \cup \{1,2\}) = v_1(S \cup \{1\}) = v_1(S \cup \{2\})$ for any S. Therefore, in a "meeting procedure" where, at each step, two existing coalitions are chosen at random, it is easier for the inputs (of players $3, ..., 8$) to find a matching "technology" (i.e., players $1, 2$) if 1 and 2 are kept apart as long as possible. This way each one of the two has a chance to form a separate productive coalition. In short, it is more efficient to have two separate "factories" rather than one.

This argument fits well Gul's pairwise meeting procedure. Indeed, it turns out that *if* 1 *and* 2 *meet in the first round* (which happens with probability $1/28$), then it is *more efficient* that this meeting should *not end in agreement*. It is better to wait for the next round, where there is a high probability (of $27/28$) that they will not meet again. This increases the chance that 1 and 2 will be able to form separate coalitions (with the other players) before they eventually meet again—which increases the total payoff. A precise computation[4] shows that:

- For any strategy profile σ where all meetings end in agreement

$$E_\sigma(W(N, v_1; \delta)) = \frac{3}{7}\delta + \frac{51}{98}\delta^2 + \frac{9}{14}\delta^3 + \frac{198}{245}\delta^4 + \frac{36}{35}\delta^5 + \frac{9}{7}\delta^6 + \frac{9}{7}\delta^7. \quad (1)$$

- For any strategy profile τ where all meetings end in agreement, except for first round meetings between 1 and 2

$$\begin{aligned} E_\tau(W(N, v_1; \delta)) &= \frac{3}{7}\delta + \frac{103}{196}\delta^2 + \frac{5333}{8232}\delta^3 + \frac{955}{1176}\delta^4 + \frac{3529}{3430}\delta^5 \quad (2) \\ &\quad + \frac{937}{735}\delta^6 + \frac{727}{588}\delta^7 + \frac{9}{196}\delta^8. \end{aligned}$$

It may now be checked that $E_\tau(W(N, v_1; \delta)) > E_\sigma(W(N, v_1; \delta))$ for all $0 < \delta < 1$. The strict inequality is preserved if we modify v_1 slightly so as to make it strictly super-additive. Specifically, let $\varepsilon > 0$ and define $v_2 := v_1 + \varepsilon v_0$, where $v_0(S) := |S|^2$ for all S. The game v_2 is strictly super-additive (since v_0 is such), and satisfies

$$E_\tau(W(N, v_2; \delta)) > E_\sigma(W(N, v_2; \delta)) \text{ for all } \delta \in (\delta_0, 1), \quad (3)$$

[4] See the Appendix.

whenever $\varepsilon > 0$ is chosen appropriately small.[5]

We conclude with a number of remarks:

1. The example was generated as follows: For any game with at most 4 players, it can be shown that σ Pareto dominates τ. For 5 players, the difference $\Delta := E_\tau(W) - E_\sigma(W)$ is a linear combination of the worths $v(S)$, with those coalitions S that contain exactly one of 1 and 2 having positive coefficients, and the other coalitions—that contain neither 1 nor 2, or contain both—having negative coefficients. In order to make the difference Δ positive, one therefore takes the worths of the first kind of coalitions as large as possible, and the worths of the others as small as possible. Together with superadditivity, it leads to the game v_1. It then turns out that τ Pareto dominates σ for all δ in a neighborhood of 0 (specifically: $\delta \in (0, \delta_0)$ where $\delta_0 \approx 0.37$). Finally, to get the same result also for δ close to 1, we need to increase the number of players to 8.

2. We do not claim that τ is efficient (only that σ is not). In particular, if 1 and 2 meet in the first as well as in the second round, then of course they should not agree in the second round either (actually, they should not agree as long as the partition is still the trivial partition into singletons).

3. The statement in Gul's paper refers to equilibrium points. The example above does not rule out the possibility that efficient stationary subgame perfect equilibria entail agreement at every meeting. (But at this point there is neither a proof nor any intuitive reason why this should be so.)

4. The results in Gul's paper thus need to be restated, replacing "efficient" by "such that all meetings end in agreement".

A Appendix

Let $N = \{1, 2, ..., n\}$ be the set of players. The computations proceed as follows: Denote by A the event that 1 and 2 meet in the first round, and write $E(W) =$

[5]Precisely: For every $\delta_0 > 0$ there exists $\varepsilon_0 \equiv \varepsilon_0(\delta_0) > 0$ such that (3) is satisfied by any game $v_2 = v_1 + \varepsilon v_0$ with $\varepsilon \in [0, \varepsilon_0)$; see the Appendix.

$E(W|A)P(A)+E(W|A^C)P(A^C)$, where A^C is the complement of A; note that the probability $P(A)$ of A is $1/\binom{n}{2}$. The second term is the same for both σ and τ. As for the first term,

$$
\begin{aligned}
E_\tau(W(N,v;\delta)|A) &= (1-\delta)\sum_{i\in N} v(i)+\delta E_\sigma(W(N,v;\delta)) \text{ and} \\
E_\sigma(W(N,v;\delta)|A) &= (1-\delta)\sum_{i\in N} v(i)+\delta E_\sigma(W(N_{12},v;\delta)),
\end{aligned}
$$

where $N_{12} := \{\{1,2\},3,4,...,n\}$. Thus, denoting $\Delta(v) := E_\tau(W(N,v;\delta)-E_\sigma(W(N,v;\delta)$,we get

$$\Delta(v)=\delta\frac{1}{\binom{n}{2}}\left(E_\sigma(W(N,v;\delta))-E_\sigma(W(N_{12},v;\delta))\right).$$

At this point we use MAPLE:

```
####
# strategy profiles:
#   sigma = 'always agreement'
#   tau = 'no agreement between 1,2 in first round'
# EW(v,Q) := expected sum of payoffs starting from
#   partition Q for sigma
# total(v,Q) := contribution to EW of Q in the
#   current round, together with all future rounds
#   (recursively), up to (and not including)
#   the grand coalition
# sumv(v,Q) := sum of v(S) over S in partition Q
# n := number of players
# single(n) := {{1},{2}, ... , {n}}
# double(n) := {{1,2},{3},{4},...,{n}}
# DEW(n,v) := difference between tau and sigma
#   when positive: counterexample!
####
sumv := proc(v, Q) local S, result; option remember;
  result := 0;
  for S in Q do  result := result + v(S)  od;
  result;
```

```
end;
####
total := proc(v, Q) local result, i, j, c, Q1;
  option remember;
  if nops(Q) = 1 then RETURN(0) fi;
  result := sumv(v, Q);
  c := delta*2/(nops(Q)*(nops(Q) - 1));
  for i to nops(Q) - 1 do
  for j from i + 1 to nops(Q) do
    Q1 := (Q minus {op(i,Q),op(j,Q)})
            union {op(i,Q) union op(j,Q)};
    result := result + c*total(v, Q1);
  od; od;
  result;
end;
####
EW := (v, Q) -> (1 - delta)*(total(v, Q)
   + v(`union`(op(Q)))*sum(delta^t,t=nops(Q)-1..infinity));
####
single := n -> {{i} $ i=1..n};
double := n -> single(n) minus {{1},{2}} union {{1,2}};
DEW := (n, v) ->
     (EW(v,single(n)) - EW(v,double(n)))*delta*2/(n*(n-1));
####
# the games v1 and v0
####
v1 := proc(S) local m; option remember;
  m:=nops (S intersect {1,2});
  if m = 0 then 0 else nops(S)-m fi;
end;
v0 := S -> nops(S)^2;
####
for n from 3 to 8 do
```

```
  'E[sigma](W(v[1]))' = EW(v1, single(n));
  'E[tau](W(v[1]))' = EW(v1, single(n)) + DEW(n, v1);
  'E[tau](W(v[1]))-E[sigma](W(v[1]))' = DEW(n, v1);
  'E[tau](W(v[0]))-E[sigma](W(v[0]))' = DEW(n, v0);
od;
####
```

For $n = 8$, this yields the formulas (1) and (2). The difference $\Delta(v_1) = E_\tau(W(N, v_1; \delta)) - E_\sigma(W(N, v_1; \delta))$ equals

$$\Delta(v_1) = \delta^2(1-\delta)\left[\frac{1}{196} + \frac{83}{8332}\delta + \frac{24}{1715}\delta^2 + \frac{1}{70}\delta^3 + \frac{1}{294}\delta^4 - \frac{9}{196}\delta^5\right].$$

The polynomial $p(\delta)$ in the square brackets [ ] has one root between 1 (where it is positive) and 2 (where it is negative), which is moreover its unique non-negative root (by Descartes' Rule of Signs). Therefore there is $c > 0$ such that $p(\delta) \geq c$ for all[6] $\delta \in [0,1]$. Hence $\Delta(v_1) \geq c\delta^2(1-\delta) > 0$ for all $\delta \in (0,1)$.

Considering now v_0, we get $\Delta(v_0) = -\delta(1-\delta)(35 + 45\delta + 60\delta^2 + 84\delta^3 + 126\delta^4 + 210\delta^5 + 420\delta^6)/490$, which is $\geq -2\delta(1-\delta)$ for all $\delta \in [0,1]$. This yields for $v_2 := v_1 + \varepsilon v_0$

$$\Delta(v_2) \geq c\delta^2(1-\delta) - 2\varepsilon\delta(1-\delta) = \delta(1-\delta)\,(c\delta - 2\varepsilon)\,.$$

To make this difference positive for all $\delta \in (\delta_0, 1)$, take $\varepsilon < c\delta_0/2$.

[6] A precise computation shows that we may take $c = 13/13720$ (the minimal value of p in the interval $[0,1]$ is at 1).

References

Gul, F. [1989], “Bargaining Foundations of Shapley Value”, *Econometrica* 57, 81–95.

Harsanyi, J.C. [1985], “The Shapley Value and the Risk Dominance Solutions of Two Bargaining Models for Characteristic-Function Games”, *Essays in Game Theory and Mathematical Economics*, R.J. Aumann *et al* (eds.), Bibliographisches Institut Mannheim, 43–68.

Hart, O. and J. Moore [1990], “Property Rights and the Nature of the Firm”, *Journal of Political Economy* 98, 1119–1157.

Hart, S. and A. Mas-Colell [1996], “Bargaining and Value”, *Econometrica* 64, 357–380.

Mas-Colell, A. [1988], “Algunos Comentarios sobre la Teoria Cooperativa de los Juegos”, *Cuadernos Economicos* 40, 143–161.

Shapley, L.S. [1953], “A Value for *n*-Person Games”, *Contributions to the Theory of Games* II (*Annals of Mathematics Studies* 28), H.W. Kuhn and A.W. Tucker (eds.), Princeton University Press, 307–317.

Winter, E. [1994], “The Demand Commitment Bargaining and Snowballing Cooperation”, *Economic Theory* 4, 255–273.