

Profit Maximizing in Auctions of Public Goods ^{*†}

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Abstract

A profit-maximizing auctioneer can provide a public good to a group of agents. Each group member has a private value for the good being provided to the group. We investigate an auction mechanism where the auctioneer provides the good to the group, only if the sum of their bids exceeds a reserve price declared previously by the auctioneer. For the two-bidder case with private values drawn from a uniform distribution we characterize the continuously differentiable symmetric equilibrium bidding functions for the agents, and find the optimal reserve price for the auctioneer when such functions are used by the bidders. We also examine another interesting family of equilibrium bidding functions for this case, with a discrete number of possible bids, and show the relation (in the limit) to the differentiable bidding functions.

Keywords: public goods, auctions, externalities.

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1 Introduction

As a motivating example consider a group of people living on an island. The island is connected to the mainland by a ferry service. Each of the islanders stands to gain if a bridge is built over the water, and each has a private valuation of how much such a bridge is worth to her. Assume that a profit maximizing monopolist (the provider) offers to build a bridge if she is paid at least a minimum (predeclared) amount. Each islander is requested to submit a sealed bid. If the sum of bids reaches at least the announced minimum, each bidder pays her bid and the bridge is built. If the bids are insufficient, no payments are made and the bridge is not built. If the bridge is built it is a public good for the islanders (the agents), and we assume that no one can be prevented from using it. This no-exclusion assumption is reasonable if the costs of monitoring are relatively high.

Given such a situation, two interesting questions are the following. What minimum price should be declared by the provider, and how should the agents bid, as a function of their information.

We assume that each agent does not know the valuations of other agents. They know only the probability distribution from which these valuations were drawn. This distribution is also known to the provider. If the provider knew the private values, she could of course offer to provide the good at the sum of values, obtaining an efficient outcome while extracting the entire surplus of the agents. However, since the values are not known to the provider, other methods must be used. In our setting, profit maximizing and (ex post) efficiency are incompatible, as demonstrated in Section 6.

To summarize, we examine an auction situation with the following procedure. The provider states a minimum price at which the good will be provided. Every agent bids (simultaneously) how much she is willing to pay for the good to be provided. If the sum of the bids of the group exceeds the minimum price, then the good is provided to the group, and each agent pays her bid. Each agent may be tempted to free-ride, as the good may be provided if the other agents make sufficiently high bids.

The equilibrium analysis of this situation is mathematically complex. This is a result of the fact that the sum of bids is the relevant value when determining the probability of provision with a given bid. The distribution of the sum of a number of random variables

is less tractable than that of the maximum, which is used in most auction analyses. Since this paper is a first attempt to solve such models, we restrict most of the analysis to the two-bidder case, with independent uniform distributions¹. Even with this restriction the problem is not trivial. We describe a slightly more general model, with n bidders, but our analysis of the solution is restricted to two bidders (agents). Each agent has a private value independently drawn from a uniform distribution over the unit interval.

Our model is related both to the theory of public goods and to that of auctions. However, it seems that it has not been previously addressed by either stream of literature. Auctions with incomplete information where the object is awarded according to the sum of bids are a novelty; similarly the provision (or non-provision) of a public good with the aim of profit maximizing replacing the quest for efficiency.

There are many related works containing some aspects of our model, and most do not assume incomplete information or profit-maximizing providers. We first address the public goods literature. Much is known about provision of public goods to agents with private values, and the consequent problem of free riding. This problem was first mentioned in Samuelson (1954). Other theoretical papers on free riding are Olson (1965), Stigler (1974), Brubaker (1975) and Cornes and Sandler (1984). Some experimental works on free riding are Schneider and Pommerehne (1981), Marwell and Ames (1981), Isaac, Walker and Thomas (1984) and Isaac, McCue and Plott (1985). A mechanism such as the Groves-Ledyard (1977) mechanism can elicit truthful declarations of private values and therefore achieve efficient allocations, but has a number of limitations. The most problematic of these is that to achieve budget balance the Groves-Ledyard strategies depend on the actions of other players, thus necessitating much information about the preferences of others. As they note, their mechanism is not practical to implement. For our assumption that the provider wishes to maximize her expected income, and does not care directly about efficiency, this class of mechanism is completely unsuitable. Lindahl equilibria (Lindahl, 1919) have profit maximizing firms, but among other things, are not compatible with our assumption of non-excludability of the public good.

As in standard auctions, the bidders in our setting must take into account that the bids of other members of the group also count in determining whether the group will

¹These are similar to the assumptions made in Landsberger et al (1996), who investigate a novel auction situation and find technical difficulties even in the two-bidder case.

get the good. However, contrary to the situation of auctions of private goods, if other members of the group make higher bids, this increases the bidder's chances of profiting from the good. The externalities the good provides are thus positive. The auction literature does not contain much work on auctions with positive externalities.

Auctions with externalities have been dealt with by Jehiel, Moldovanu and Stacchetti (1994), Jehiel and Moldovanu (1996) and Jehiel, Moldovanu and Stacchetti (1996). These papers emphasize negative externalities, and the first of one is restricted to complete information. Jehiel and Moldovanu (1997) treat second price auctions with externalities that can be positive or negative. They allow complex payoff possibilities including dependence of a loser's payoff on the private valuation of the winner (which is unknown to the loser *ex ante*). For such complex cases, with a reserve price and a second price auction, the existence of pure strategy Nash equilibrium remains an open question. McAfee and McMillan (1992) model collusion in bidding cartels, which can be viewed as a case of auctions with positive externalities, as the profits from collusion are distributed among the cartel members. The bidders in their model can reach efficient outcomes, but they require punishments to enforce collusive agreements, and the collusion decreases the auctioneer's profits.

Even with our restriction to two bidders, we cannot provide a unique prediction or prescription of a solution. There exist multiple equilibria, and it is not immediately obvious that one of these would serve as a focal point. We describe an outcome in which the provider chooses a specific minimum price and the agents then use symmetric equilibrium functions (from private values to bids). These equilibrium bidding strategies are drawn from a family of continuously differentiable functions and are uniquely determined by the minimum price. These functions are in general quite complicated, but at the optimal minimum price for the provider the function reduces to a simple linear function of the private value². We justify this emphasis on differentiable bidding strategies in Section 6. We also examine other symmetric equilibrium bidding functions, including a family of functions with discrete bids (a reasonable assumption when there is a smallest monetary unit), which tend to the differentiable functions when the number of possible bids tends to infinity.

²Notably, such *multiplicative* functions are commonly used in analyses of offshore oil lease auctions, as in Dougherty and Nozaki (1975). However, they may not be optimal for these situations, as Engelbrecht-Wiggans (1978) contains a case of disequilibrium of such strategies in a federal offshore lease sale.

The structure of the paper is as follows. In Section 2 we present the model. In Section 3 we present the conditions for Nash equilibria of the auction situation, and give properties of such equilibria for the two-bidder case. Section 4 characterizes differentiable symmetric equilibrium bidding functions, and Section 5 analyzes a family of discrete ones. We conclude with final remarks in Section 6.

2 The Model

We assume a public good which can be provided by a provider to a group of agents $N = \{1, 2, \dots, n\}$. Each agent i has a private value v_i which is independently drawn from a distribution F_i over $[0, \bar{v}]$.

The procedure of the auction is as follows: At stage 1, the provider of the good, the auctioneer, states a minimum price $a \in \mathbb{R}_+$. Then, at stage 2, after hearing the value of a , each agent i submits a bid $b_i \in [0, \bar{v}]$ (this restriction is without loss of generality, as any higher bid is dominated by a bid in this range). The bids are made simultaneously. If $\sum_{i \in N} b_i \geq a$ then the good is provided. If the sum of bids is less than a , then the good is not provided.

The payoffs for the players, given their private values, are as follows: The auctioneer receives a payoff of $\sum_{i \in N} b_i$ if the good is provided, and zero if the good is not provided. Each agent i receives a payoff of $v_i - b_i$ if the good is provided, and zero if the good is not provided. We assume risk neutrality of the agents.

A strategy profile of the auction procedure consists of a value a that the auctioneer declares, and a bidding function for each bidder i , as a function of the declared value a and the player's private value. This bidding function is of the form $b_i : \mathbb{R}_+ \times [0, \bar{v}] \rightarrow [0, \bar{v}]$ which specifies her bid for providing the object for any value of a and any private value v_i .

3 Nash Equilibria of the Auction

We are interested in strategy profiles that are Nash equilibria. In this section we describe the conditions for Nash equilibrium and some basic results for the two-bidder case. In

the next section we use these results when focusing on equilibria using symmetric differentiable bidding functions. Denote by S the event that the good is provided.

The conditions for a profile $(a, (b_i)_{i \in N})$ to be a Nash equilibrium are the following:

1. Each bidder i 's bidding function b_i maximizes her expected profit for every possibility of her private value (for the given a and bidding functions of the other players). Formally, (we denote by b_{-i} and v_{-i} the bidding functions and private values, respectively, of the other agents)

$$Pr(S|a, b_{-i}, b_i(a, v_i))(v_i - b_i(a, v_i)) \geq Pr(S|a, b_{-i}, \hat{b}_i)(v_i - \hat{b}_i) \quad (1)$$

for all $i \in N$, for all $v_i \in [0, \bar{v}]$, and for all $\hat{b}_i \in [0, \bar{v}]$. The probabilities in the equation are the expected probabilities as viewed by agent i with respect to v_{-i} .

2. The auctioneer's declaration of a maximizes her expected profit, given the bidding functions of the bidders. Formally,

$$Pr(S|a, (b_i)_{i \in N})E\left(\sum_{i \in N} b_i(a, v_i) | S\right) \geq Pr(S|\hat{a}, (b_i)_{i \in N})E\left(\sum_{i \in N} b_i(\hat{a}, v_i) | S\right) \quad (2)$$

for all $\hat{a} \in \mathbb{R}_+$. The probabilities are the expected probabilities with respect to the distribution of the private values.

From this point, we restrict our analysis to the two-bidder case. This enables us to continue with a more tractable model. Even with this assumption the problem is not trivial. We assume therefore that there are two agents, and that the private values of the good to the agents are independently uniformly distributed over the interval $[0, 1]$.

A strategy profile is now a triple (a, b_1, b_2) , where

$$b_i : \mathbb{R}_+ \times [0, 1] \longrightarrow [0, 1]$$

is agent i 's bidding function, and $b_i(a, v_i)$ denotes agent i 's bid for the auctioneer's declaration of a and her private value v_i .

A strategy profile (a, b_1, b_2) is a Nash equilibrium if for each agent i and each private value v_i , the bid $b_i(a, v_i)$ is optimal, i.e.

$$b_i(a, v_i) \in \arg \max_x (v_i - x) Pr(b_{-i}(a, v_{-i}) + x \geq a),$$

and $a \in \arg \max_{a' \in \mathbb{R}_+} \Pi_0(a', b_1, b_2)$, where

$$\Pi_0(a, b_1, b_2) = \int_0^1 \int_0^1 (b_1(a, v_1) + b_2(a, v_2)) 1_{\{b_1(a, v_1) + b_2(a, v_2) \geq a\}} dv_2 dv_1.$$

A remark on the notation. The bidding functions are used after a is announced. Therefore, we first concentrate on the properties of the bidding functions for a given, fixed a , and abuse notation by referring to $b_i(v_i)$ instead of $b_i(a, v_i)$. We later return to the question of finding the optimal value of a , when given a family of bidding functions, one for each possible value of a . Note that even if $b_1(a, v_1)$ and $b_2(a, v_2)$ are in equilibrium only for one fixed a^* , then there exists a pair of bidding functions b'_1 and b'_2 such that (a^*, b'_1, b'_2) is an equilibrium profile, by defining

$$b'_i(a, v_i) = \begin{cases} b_i(a, v_i) & \text{if } a = a^* \\ 0 & \text{otherwise} \end{cases}.$$

The following lemmas show some general properties that equilibrium bidding profiles must satisfy. The first lemma deals with cases where a bidder cannot make a positive profit with any bid, given her private value and the other bidder's bidding function.

Lemma 1 *Assume that (a, b_1, b_2) is an equilibrium, and b_1, b_2 are nondecreasing. If $b_2(1) < a$ then for $0 \leq v \leq a - b_2(1)$, any bid in the interval $[0, a - b_2(1))$ is a best response for agent 1.*

Proof: A player with v in the above range can never receive a positive payoff. Any bid in the interval $[0, a - b_2(1))$ will give her a payoff of zero, and any higher bid will give an expected payoff of at most zero. Therefore any bid in this interval is a best response.

■(Lemma 1)

Denote by

$$P_w(x, b) = \int_0^1 1_{\{b(v) \geq a-x\}} dv$$

the probability of winning with a bid of x if the other player is using the bidding function b .

The next lemma shows that equilibrium bidding functions must be non-decreasing. The restriction that the following lemma applies only for private values at which it is possible to receive a positive payoff is implied by Lemma 1.

Lemma 2 *If (a, b_1, b_2) is an equilibrium, and for some x_1 and x_2 such that $x_1 < x_2$ we have $P_w(b_1(x_1), b_2) > 0$ and $P_w(b_1(x_2), b_2) > 0$, then $b_1(x_1) \leq b_1(x_2)$.*

Proof: Denote $p_1 = P_w(b_1(x_1), b_2)$ and $p_2 = P_w(b_1(x_2), b_2)$. Since $p_1 > 0$ it is true that $b_1(x_1) \leq x_1$, otherwise the functions are not in equilibrium. From the equilibrium assumption we have

$$p_1 \cdot (x_1 - b_1(x_1)) \geq p_2 \cdot (x_1 - b_1(x_2)) \quad (3)$$

and

$$p_2 \cdot (x_2 - b_1(x_2)) \geq p_1 \cdot (x_2 - b_1(x_1)). \quad (4)$$

Multiplying both sides of (3) by $(x_2 - b_1(x_1))$ (which is positive since $x_2 > x_1 \geq b_1(x_1)$), we get

$$p_1 \cdot (x_1 - b_1(x_1))(x_2 - b_1(x_1)) \geq p_2 \cdot (x_1 - b_1(x_2))(x_2 - b_1(x_1)). \quad (5)$$

Multiplying both sides of (4) by $(x_1 - b_1(x_1))$, which is non-negative, we get

$$p_2 \cdot (x_2 - b_1(x_2))(x_1 - b_1(x_1)) \geq p_1 \cdot (x_2 - b_1(x_1))(x_1 - b_1(x_1)). \quad (6)$$

Combining (5) and (6), dividing by $p_2 > 0$, multiplying out and cancelling equal terms gives us:

$$b_1(x_2)(x_2 - x_1) \geq b_1(x_1)(x_2 - x_1), \quad (7)$$

and as $x_2 > x_1$ the conclusion of the lemma holds. ■(Lemma 2)

We are especially interested in the case of symmetric equilibria, where the two agents use the same bidding function. For a fixed value of a , a bidding function $b : [0, 1] \rightarrow [0, a]$ is in equilibrium if for each player i , for any private value v_i , bidding $b(v_i)$ is a best response (in expectation) to the other player using the same function b . Formally, b is a symmetric equilibrium bidding function, given a , if

$$b(v) \in \arg \max_{x \in [0, a]} (v - x) \Pr(b(v_{-i}) + x \geq a) \quad \forall v \in [0, 1], \text{ for } i \in \{1, 2\}. \quad (8)$$

Denote $d = \lim_{v \rightarrow 1-0} b(v)$. Thus, for any $\varepsilon > 0$, there is positive probability that $b(v)$ will be greater than $d - \varepsilon$. The following lemmas hold for symmetric equilibria. The first one states that in a symmetric equilibrium, any bid higher than $a - d$ has a positive probability of winning. Thus, the interval not “covered” by Lemma 1 is $[a - d, 1]$, and this is the interval where Condition 8 is non-trivial.

Lemma 3 *If (a, b, b) is an equilibrium profile, then for all $\varepsilon > 0$, $P_w(a - d + \varepsilon, b) > 0$.*

Proof: If for all $v \in [0, 1]$, $b(v) \leq d - \varepsilon$, then using Lemma 2, $\lim_{v \rightarrow 1} b(v) < d$, contradicting the definition. Therefore, there exists $v_0 < 1$ such that $b(v_0) > d - \varepsilon$, and, using Lemma 2 again, $b(v) > d - \varepsilon$ for all $v \in [v_0, 1]$. This implies that $P_w(a - d + \varepsilon, b) \geq 1 - v_0 > 0$. ■(Lemma 3)

Lemma 4 states that for any private value above $a - d$, an equilibrium bid will be *strictly* lower than the private value.

Lemma 4 *If (a, b, b) is an equilibrium profile, then for all $v \in (a - d, 1]$, $b(v) < v$.*

Proof: For any such v , bidding a value greater to or equal to v gives a non-positive expected profit, so such a bid is dominated by bidding $\frac{v+(a-d)}{2}$, which gives positive gain, with positive probability (from Lemma 3). Therefore, $b(v) < v$ for any $v \in (a - d, 1]$. ■(Lemma 4)

We are now ready for the final lemma, which allows us to conclude that if a symmetric equilibrium bidding function is continuous, then it is strictly increasing for values of v in $(a - d, 1]$.

Lemma 5 *Assume (a, b, b) is an equilibrium profile. If there exist $x'' > x'$ such that $y = b(x'') = b(x') > a - d$, then there exists $\varepsilon > 0$ such that $b(v) \notin [a - y - \varepsilon, a - y) \forall v \in [0, 1]$.*

Proof: Assume the claim is not true, i.e. there exist such x, x' and y , but no such ε . Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ and a sequence $\{v_n\}_{n=1}^{\infty}$, such that $b(v_n) = t_n$ for all n , $t_n < a - y$ for all n , and $\lim_{n \rightarrow \infty} t_n = a - y$. Define, for all n , $b^{-1}(t_n) = \inf\{x | b(x) = t_n\}$.

For all n , we have

$$(v_n - t_n)(1 - x'') \geq (v_n - t_n)P_w(t_n, b), \quad (9)$$

since $t_n < a - y$ implies $P_w(t_n, b) \leq 1 - x''$, and

$$(v_n - t_n)P_w(t_n, b) \geq (v_n - t_n)(1 - x'), \quad (10)$$

since b is in equilibrium. Combining (9) and (10) we get

$$(v_n - t_n)(1 - x'') \geq (v_n - t_n)(1 - x'), \quad (11)$$

which implies

$$(a - y) - t_n \geq v_n(x'' - x') + (a - y)x' - t_n x''. \quad (12)$$

Taking the limit $n \rightarrow \infty$, we have

$$0 \geq (x'' - x')(\lim_{n \rightarrow \infty} v_n - (a - y)), \quad (13)$$

and since $x'' > x'$, this implies

$$\lim_{n \rightarrow \infty} b^{-1}(t_n) \leq a - y. \quad (14)$$

Since b is increasing, so is b^{-1} , and the bid at the limit is at least as high as the private value there, in contradiction to Lemma 4. ■(Lemma 5)

We have thus shown that for symmetric equilibrium bidding functions, existence of a plateau in the “relevant” area (above $a - d$) implies existence of a jump in the function. Therefore, if such a bidding function is continuous (has no jumps), it is strictly increasing for values in the range $[a - d, 1]$.

4 Differentiable Symmetric Equilibria

In this section we characterize the Nash equilibria for the two agents, given the minimum selling price a announced by the auctioneer, with the restriction that the two agents use the same bidding function, and the bidding function is differentiable over its entire range. This assumption of differentiability can actually be considerably weakened, and assuming continuity allows us to derive differentiability as a result. Given a continuous symmetric equilibrium bidding function, we can divide the interval of possible private values $[0, 1]$ into two subintervals. Denoting as before $d = \lim_{v \rightarrow 1-0} b(v)$, the first interval is $[0, a - d]$, in which no bid can give expected positive profit (assuming the other player uses b). In the interval $(a - d, 1]$ a positive profit is possible, and is indeed achieved with b . A symmetric equilibrium bidding function b that is continuous in the interval $(a - d, 1]$ is also differentiable on $(a - d, 1)$.³ Thus, for the functions dealt with in Section 4.1 (the case $b(1) = a$), it is enough to assume that the function is continuous over $[0, 1]$ to get differentiability on $(0, 1)$, and for those of Section 4.3 (the case $b(1) = d < a$), continuity over $(a - d, 1]$ implies differentiability over $(a - d, 1)$. Combining this result with Lemma 5 we have that a continuous symmetric equilibrium function satisfying $b(1) = a$ is both strictly increasing and differentiable on $(0, 1)$, and an analogous result holds on $(a - b(1), 1)$ when $b(1) < a$.

4.1 Differentiable Symmetric Equilibria with $b(1) = a$

We consider first the case where $b(1) = a$ (the case $b(1) < a$ will be dealt with in Section 4.3). Assuming that b is differentiable, it is strictly increasing from Lemma 5. Therefore, there exists a unique inverse $b^{-1} : [0, a] \rightarrow [0, 1]$. We start with Condition (8), which is a necessary condition for any symmetrical equilibrium.

Note that since v_i is uniformly distributed over the unit interval, $Pr(b(v_{-i}) + x \geq a) = Pr(v_{-i} \geq b^{-1}(a - x)) = 1 - b^{-1}(a - x)$, hence differentiable with respect to x . Since we assume that b is differentiable, a necessary condition for Condition (8) to hold is that the derivative of the right hand side of the condition is equal to zero for any x in the

³A proof can be obtained from the authors.

argmax. Equating this derivative to zero gives

$$b^{-1'}(a-x) = \frac{1-b^{-1}(a-x)}{v-x}. \quad (15)$$

and since $x = b(v)$ must satisfy this for equilibrium, it must satisfy

$$b^{-1'}(a-b(v)) = \frac{1-b^{-1}(a-b(v))}{v-b(v)} \quad (16)$$

for all $v \in (0, 1)$. Substituting $t = a - b(v)$ and therefore $v = b^{-1}(a - t)$ we have the following necessary condition:

$$b^{-1'}(t) = \frac{1-b^{-1}(t)}{b^{-1}(a-t)-(a-t)} \quad (17)$$

for $t \in (0, a)$.

Thus, replacing b^{-1} by c , we seek a function c which is differentiable on $(0, a)$ and continuous on $[0, a]$, satisfying

$$c'(t) = \frac{1-c(t)}{c(a-t)-(a-t)} \quad (18)$$

for all $t \in (0, a)$, and $c(0) = 0$.

We now give the main theorem, which contains the characterization of all symmetric equilibrium bidding functions that are differentiable and satisfy $b(1) = a$.

Theorem 1 *Equation (18) has a solution if and only if $0 < a < 1$. The solution is unique, it is strictly increasing and satisfies $0 \leq c(t) \leq c(a) = 1$ on $[0, a]$. Moreover one has that c is continuously differentiable on $[0, a)$ and is continuously differentiable at a if and only if $a \leq \frac{1}{2}$. The solution can be explicitly given by*

$$c(t) = 1 + \frac{1-a}{a-1-t-t^{1/a}(a-t)^{1-1/a}} \quad (19)$$

The proof of this theorem requires a thorough knowledge of calculus, and non-mathematically inclined readers are encouraged to skip the proof on a first reading.

Proof: We start with the “only if”-part. Assume c is a solution of (18) and let $m := \frac{a}{2}$. For $t \in [0, m]$ define

$$u_1(t) := c(m-t) - m + t, \quad u_2(t) := c(m+t) - m - t. \quad (20)$$

Then

$$u_1'(t) = 1 - c'(m-t) = 1 - \frac{1 - c(m-t)}{c(m+t) - m - t} = 1 + \frac{u_1(t) + m - t - 1}{u_2(t)} \quad (21)$$

$$u_2'(t) = -1 + c'(m+t) = -1 + \frac{1 - c(m+t)}{c(m-t) - m + t} = -1 - \frac{u_2(t) + m + t - 1}{u_1(t)} \quad (22)$$

and $u_0 := u_1(0) = c(m) - m = u_2(0)$, $u_1(m) = 0$. Next observe that

$$u_1'(t)u_2(t) + u_1(t)u_2'(t) = -2t \quad \forall t \in [0, m]. \quad (23)$$

Integrating (23), we have $u_1(t)u_2(t) = K - t^2$. Substituting $t = 0$ gives $K = u_0^2$, therefore $u_1(t)u_2(t) = u_0^2 - t^2$. Since $0 = u_1(m)u_2(m) = u_0^2 - m^2$, we see that $|u_0| = m$. In particular, $u_2(t) \neq 0$ on $[0, m)$ and

$$\frac{1}{u_2(t)} = \frac{u_1(t)}{m^2 - t^2} \quad \text{for } 0 \leq t < m$$

and by (21) we obtain a Riccati type equation for u_1

$$u_1'(t) = 1 + \frac{u_1(t)(u_1(t) + \alpha - t)}{m^2 - t^2} \quad (24)$$

with $\alpha = m - 1$ and initial condition $u_1(0) = u_0$ and $u_1(m) = 0$. To solve (24) we first perform a “velocity transform” by $w(t) := u_1(\phi(t))$, $\phi(0) = 0$ such that $\phi'(t) = m^2 - \phi(t)^2$. Hence $\phi(t) = m \tanh(mt)$, $\phi^{-1}(x) = \frac{1}{m} \operatorname{arctanh}(\frac{x}{m})$ and

$$w'(t) = \phi'(t)u_1'(\phi(t)) = u_1(\phi(t))^2 + (\alpha - \phi(t))u_1(\phi(t)) + \phi'(t) \quad (25)$$

The transform $y(t) = \exp(-\int_0^t w(s)ds)$ now yields

$$y'(t) = -w(t)y(t), \quad y''(t) = -(w'(t) - w^2(t))y(t) \quad (26)$$

Combining (25) and (26) one obtains

$$y''(t) = -(w'(t) - w^2(t))y(t) = \quad (27)$$

$$= ((\alpha - \phi(t))u_1(\phi(t)) + \phi'(t))\frac{y'(t)}{w(t)} = \quad (28)$$

$$= (\alpha - \phi(t))y'(t) - \phi'(t)y(t) = \quad (29)$$

$$= \alpha y'(t) - (\phi(t)y(t))' \quad (30)$$

and so, integrating,

$$y'(t) = \alpha y(t) - \phi(t)y(t) + c_0, \quad (31)$$

with $c_0 = y'(0) - \alpha y(0) + \phi(0)y(0) = -u_0 - \alpha$, since $y(0) = 1$ by definition. Equation (31) is a linear differential equation of first order. It is solved using the variation-of-parameter formula

$$y(t) = \exp\left(\alpha t - \int_0^t \phi(s)ds\right) + c_0 \int_0^t \exp\left(\alpha(t-s) - \int_s^t \phi(\sigma)d\sigma\right)ds,$$

since the solution of the homogenous equation (i.e. for the case $c_0 = 0$) is given by $\exp\left(\alpha t - \int_0^t \phi(s)ds\right)$. Now since

$$\int_s^t \phi(\sigma)d\sigma = \int_s^t m \tanh(m\sigma) d\sigma = \ln\left(\frac{\cosh(mt)}{\cosh(ms)}\right)$$

and

$$\int_0^t \exp\left(\alpha(t-s) - \int_s^t \phi(\sigma)d\sigma\right)ds = \frac{e^{\alpha t}}{\cosh(mt)} \int_0^t e^{-\alpha s} \cosh(ms) ds \quad (32)$$

$$= \int_0^t e^{\alpha t} e^{-\alpha s} \frac{\cosh(ms)}{\cosh(mt)} ds \quad (33)$$

$$= \frac{e^{\alpha t}}{\cosh(mt)} \left(\frac{e^{-\alpha t}}{m^2 - \alpha^2} [\alpha \cosh(mt) + m \sinh(mt)] - \frac{\alpha}{m^2 - \alpha^2} \right) \quad (34)$$

$$= \frac{\alpha e^{\alpha t}}{(\alpha^2 - m^2) \cosh(mt)} - \frac{\alpha}{\alpha^2 - m^2} - \frac{m \tanh(mt)}{\alpha^2 - m^2}. \quad (35)$$

Using the fact that $\int \tanh x \, dx = \ln \cosh x$ to show that

$$\exp\left(\alpha t - \int_0^t \phi(s) ds\right) = \frac{e^{\alpha t}}{\cosh mt} \quad (36)$$

and (32)-(35), we obtain

$$y(t) = \frac{m^2 + u_0 \alpha}{m^2 - \alpha^2} \frac{e^{\alpha t}}{\cosh(mt)} - \frac{m(u_0 + \alpha)}{m^2 - \alpha^2} \tanh(mt) - \frac{\alpha(u_0 + \alpha)}{m^2 - \alpha^2}.$$

Now, using the definition of w , (26) and (31) gives

$$u_1(t) = w(\phi^{-1}(t)) = -\frac{y'(\phi^{-1}(t))}{y(\phi^{-1}(t))} = \phi(\phi^{-1}(t)) - \alpha - \frac{c_0}{y(\phi^{-1}(t))}.$$

Since the equation

$$\operatorname{arctanh}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \quad (37)$$

and the definition of ϕ^{-1} imply that

$$e^{\alpha \phi^{-1}(t)} = \left(\frac{m+t}{m-t} \right)^{\frac{\alpha}{2m}}, \quad (38)$$

and since the equation

$$\cosh x = \frac{1}{\sqrt{1 - \tanh(x)^2}} \quad (39)$$

implies that

$$\cosh\left(\operatorname{arctanh}\left(\frac{t}{m}\right)\right) = \frac{m}{\sqrt{m^2 - t^2}}, \quad (40)$$

therefore, if $0 \leq t < m$ and $u_0 + \alpha \neq 0$, then

$$\begin{aligned} u_1(t) &= t - \alpha - \frac{c_0}{y(\phi^{-1}(t))} = t - \alpha - \frac{m^2 - \alpha^2}{t + \alpha - \frac{m^2 + \alpha u_0}{m(\alpha + u_0)} (m+t)^{\frac{1}{2} + \frac{\alpha}{2m}} (m-t)^{\frac{1}{2} - \frac{\alpha}{2m}}} \\ &= t - \frac{a}{2} + 1 - \frac{a-1}{\frac{a}{2} - 1 + t - \operatorname{sgn}(u_0) \left(\frac{a}{2} + t\right)^{1-1/a} \left(\frac{a}{2} - t\right)^{1/a}} \end{aligned}$$

using $\alpha = \frac{a}{2} - 1$ and $|u_0| = \frac{a}{2}$ for the last equation. If $2m = a = 1$ we have $u_1(t) = t - \alpha = t - m + 1$. In this case, however, $u_1(m) = 1$, contradicting the condition $u_1(m) = 0$. Hence $a \neq 1$.

Now for u_2 we obtain instead of (24) the following Riccati equation

$$u_2'(t) = -1 - \frac{u_2(t)(u_2(t) + \alpha + t)}{m^2 - t^2} \quad (41)$$

Since $u_2(0) = u_1(0)$, it can be verified that $u_2(t) = u_1(-t)$ gives the unique solution of (41) if $0 \leq t < m$.

Now, since c solves (18), u_1 and u_2 are defined on all of $[0, m)$. This implies that the denominator of u_1 ,

$$\eta(t) := \frac{a}{2} - 1 - t - \operatorname{sgn}(u_0) \left(\frac{a}{2} + t\right)^{1-1/a} \left(\frac{a}{2} - t\right)^{1/a}$$

does not equal zero in $(-\frac{a}{2}, \frac{a}{2})$. Now

$$\eta(m) = a - 1 \quad \text{and} \quad \lim_{t \downarrow -m} \eta(t) = \begin{cases} -\operatorname{sgn}(u_0) \infty & , \quad \text{if } a < 1, \\ -1 & , \quad \text{if } a > 1, \end{cases}$$

so from the mean value theorem we rule out the case $a > 1$ and the case $a < 1$, $u_0 = -m$, as these cases would cause η to be zero at some point in $(-\frac{a}{2}, \frac{a}{2})$. Hence we obtain

$$c(t) = 1 + \frac{1 - a}{a - 1 - t - t^{1/a}(a - t)^{1-1/a}},$$

which shows that c is uniquely determined.

On the other hand, one immediately checks that c given by (19) satisfies the equation (18) and so the “if”-part is proven, too.

One can verify that c is strictly increasing and continuously differentiable on $[0, a)$. Finally,

$$c'(t) = (1 - a) \frac{(a - t)^{1/a} + (1 - t) t^{\frac{1}{a}-1}}{\left((a - 1 - t)(a - t)^{\frac{1}{2a}} - t^{1/a}(a - t)^{1-\frac{1}{2a}}\right)^2},$$

which shows that $\lim_{t \uparrow a} c'(t)$ exists if and only if $a \leq \frac{1}{2}$. By continuity c is (left-) differentiable at a if and only if $a \leq \frac{1}{2}$. ■(Theorem 1)

4.2 Equilibrium Payoffs

Examples of differentiable symmetric equilibrium bidding functions, with $b(1) = a$, are given in Figure 1. Note that for $a = \frac{1}{2}$, the differentiable symmetric equilibrium bidding function is $b(v) = \frac{v}{2}$, a simple linear function of the private value. Such a multiplicative bidding function is extremely easy to use. The following calculations show that this is not the only advantage of having $a = \frac{1}{2}$: it is also the *optimal* choice by the auctioneer, if the bidders use the differentiable bidding functions characterized by Theorem 1.

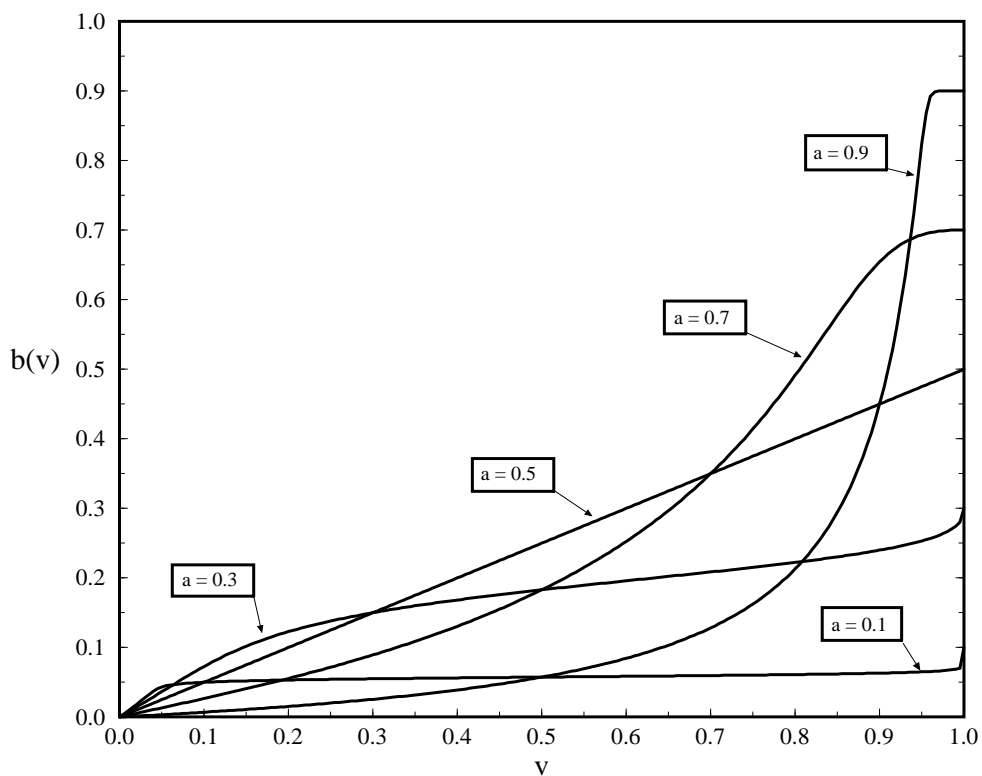


Figure 1: Differentiable symmetric equilibrium bidding functions for five different values of a .

We now investigate the expected payoff the auctioneer receives as a function of the announced price a . If player 1 and player 2 use bidding functions $b_1(a, \cdot)$ and $b_2(a, \cdot)$ respectively, then the payoff function (relating the private values of the players to the

auctioneer's payoff) for the auctioneer reads as follows

$$\begin{aligned} \Pi_0 : [0, 1] \times [0, 1] &\rightarrow \mathbf{R} \\ (v_1, v_2) &\mapsto \begin{cases} b_1(a, v_1) + b_2(a, v_2) & , \text{ if } b_1(a, v_1) + b_2(a, v_2) \geq a \\ 0 & , \text{ else.} \end{cases} \end{aligned}$$

Put $c_i(a, \cdot) := (b_i(a, \cdot))^{-1}$, ($i = 1, 2$), then the expected value for the auctioneer's payoff is

$$\begin{aligned} \mathbf{E}\Pi_0 &= \int_0^1 \int_0^1 (b_1(a, v_1) + b_2(a, v_2)) \cdot 1_{\{(v_1, v_2) | b_1(a, v_1) + b_2(a, v_2) \geq a\}}(v_1, v_2) dv_2 dv_1 \\ &= \int_0^1 \int_{a-b_1(a, v_1)}^a (b_1(a, v_1) + t_2) \partial_2 c_2(a, t_2) dt_2 dv_1 \\ &= \int_0^a (t_1 (c_2(a, a) - c_2(a, a - t_1)) \\ &\quad + \int_{a-t_1}^a t_2 \partial_2 c_2(a, t_2) dt_2) \partial_2 c_1(a, t_1) dt_1. \end{aligned} \tag{42}$$

(Here ∂_2 denotes the partial derivative w.r.t. the second variable.) For the case of an equilibrium bidding function, where $c = c_1(a, \cdot) = c_2(a, \cdot)$ this becomes

$$\begin{aligned} \mathbf{E}\Pi_0 &= c(a) \int_0^a t c'(t) dt - \int_0^a t c(a - t) c'(t) dt + \int_0^a \int_{a-t_1}^a t_2 c'(t_2) c'(t_1) dt_2 dt_1 \\ &= 2a c(a)^2 - a c(0) c(a) - c(a) \int_0^a c(t) dt - a \int_0^a c(a - t) c'(t) dt \\ &\quad - \int_0^a \int_{a-t_2}^a c'(t_1) dt_1 c(t_2) dt_2 \\ &= 2a c(a)^2 - a c(0) c(a) - 2c(a) \int_0^a c(t) dt - a \int_0^a c(a - t) c'(t) dt \\ &\quad + \int_0^a c(a - t) c(t) dt \end{aligned}$$

using partial integration and by interchanging the integrals in the last step.

Now for the equilibrium bidding function obtained in Theorem 1 we have $c(0) = 0$, $c(a) = 1$ and from the differential equation (18)

$$\int_0^a c(a - t) c'(t) dt = \int_0^a ((a - t) c'(t) + 1 - c(t)) dt = a \tag{43}$$

where we again have performed partial integration. This yields

$$\mathbf{E}\Pi_0 = (2 - a)a - \int_0^a (2 - c(a - t))c(t) dt. \quad (44)$$

Now we use the special structure of the equilibrium bidding function (19) by writing $c(t) = 1 + (1 - a)c_0(t)$ and obtain

$$\begin{aligned} \int_0^a (2 - c(a - t))c(t) dt &= a - (1 - a) \int_0^a c_0(a - t) dt + (1 - a) \int_0^a c_0(t) dt \\ &\quad - (1 - a)^2 \int_0^a c_0(a - t) c_0(t) dt. \end{aligned}$$

Observe that the first and the second integral on the right hand side are equal and that the integrand of the third one is symmetric around $\frac{a}{2}$ (which is basically of numerical interest), we thus obtain

$$\mathbf{E}\Pi_0 = a(1 - a) + (1 - a)^2 2 \int_0^{\frac{a}{2}} c_0(a - t) c_0(t) dt \quad (45)$$

Numerical evaluation shows that $\mathbf{E}\Pi_0$ assumes its maximum at $a = \frac{1}{2}$ with a value of $\frac{1}{3}$. Note that for this case the bidding functions of the bidders are the simple linear functions $b(v) = \frac{v}{2}$. The graph of the expected payoff of the auctioneer as a function of a is given in Figure 2.

Next we compute the expected payoff each agent receives for different values of the announced price a , if they both use the symmetrical differentiable bidding function derived in this section. In general, the payoff function for each agent is

$$\Pi_1 : [0, 1] \times [0, 1] \rightarrow \mathbf{R} \quad (46)$$

$$(v_1, v_2) \mapsto \begin{cases} v_1 - b_1(a, v_1) & , \text{ if } b_1(a, v_1) + b_2(a, v_2) \geq a \\ 0 & , \text{ else.} \end{cases} \quad (47)$$

Hence the expected payoff for our case is

$$\mathbf{E}\Pi_1 = \int_0^1 \int_0^1 (v_1 - b_1(a, v_1)) \cdot \mathbf{1}_{\{(v_1, v_2) | b_1(a, v_1) + b_2(a, v_2) \geq a\}} dv_1 dv_2 \quad (48)$$

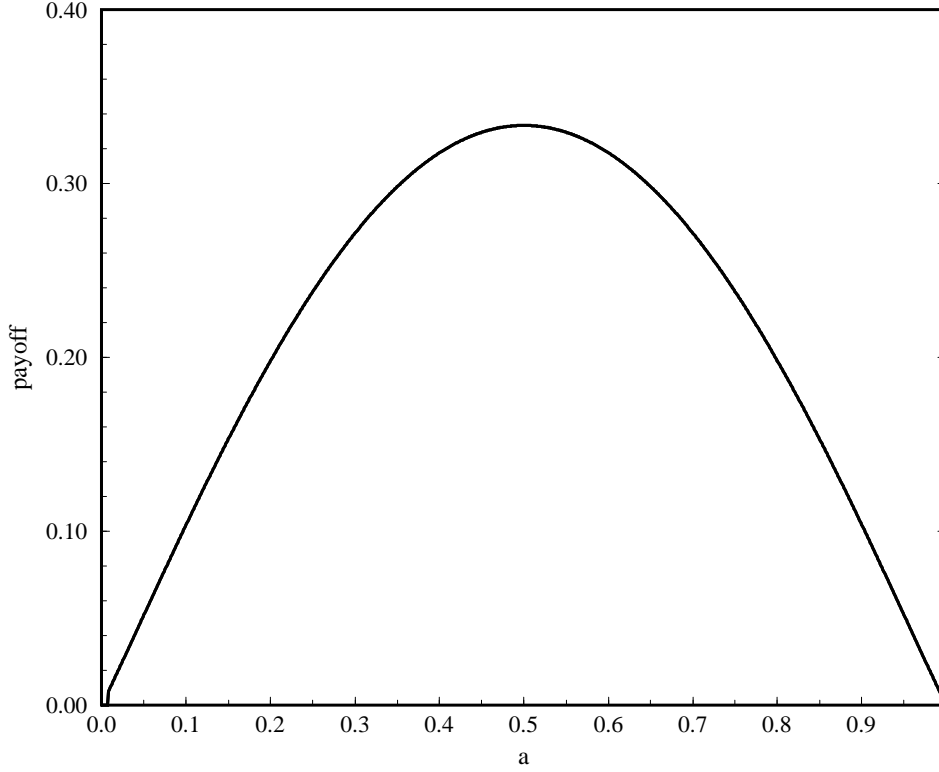


Figure 2: The auctioneer's expected profit as a function of a , when the bidders use the differentiable symmetric equilibrium bidding function with $b(1) = a$.

$$= \int_0^a \int_{a-t_2}^a (c_1(a, t_1) - t_1) c_1'(a, t_1) c_2'(a, t_2) dt_1 dt_2 \quad (49)$$

$$= \int_0^a \frac{1}{2} (c_1(a, a)^2 - c_1(a, a-t)^2) c_2'(a, t) dt \\ - \int_0^a (a c_1(a, a) - (a-t) c_1(a, a-t)) c_2'(a, t) dt \\ + \int_0^a \int_{a-t_2}^a c_1(a, t_1) dt_1 c_2'(a, t_2) dt_2 \quad (50)$$

$$= \frac{1}{2} - \frac{1}{2} \int_0^a c_1(a, a-t)^2 c_2'(a, t) dt - a + \int_0^a (a-t) c_1(a, a-t) c_2'(a, t) dt \\ + \int_0^a \int_{a-t_2}^a c_1(a, t_1) dt_1 c_2'(a, t_2) dt_2 \quad (51)$$

where c_i is b_i^{-1} , and we used also $c_i(a, a) = 1, c_i(a, 0) = 0$. Now we put $c := c_1(a, \cdot) =$

$c_2(a, \cdot)$, use (43), and

$$\int_0^a t c(a-t) c'(t) dt = - \int_0^a (c(a-t) - t c'(a-t)) c(t) dt \quad (52)$$

$$= - \int_0^a (c(a-t) c(t) dt + a^2 - \int_0^a t c(a-t) c'(t) dt \quad (53)$$

(by partial integration and (43)), which yields

$$\int_0^a t c(a-t) c'(t) dt = -\frac{1}{2} \int_0^a c(a-t) c(t) dt + \frac{1}{2} a^2. \quad (54)$$

This yields

$$\mathbf{E}\Pi_1 = \frac{1}{2} - \frac{1}{2} \int_0^a c(a-t)(c(a-t) c'(t) + c(t)) dt - a + \frac{1}{2} a^2 + \int_0^a c(t) dt \quad (55)$$

(where we have performed partial integration on the fourth term and changed the order of integration of the fifth term in (51)). Now from the differential equation

$$c(a-t)c'(t) + c(t) = (a-t)c'(t) + 1 \quad (56)$$

we obtain by (52)

$$\int_0^a c(a-t)(c(a-t) c'(t) + c(t)) dt = \frac{a^2}{2} + \frac{1}{2} \int_0^a c(a-t) c(t) dt + \int_0^a c(t) dt, \quad (57)$$

which finally shows

$$\mathbf{E}\Pi_1 = \frac{1}{2} - a + \frac{a^2}{4} - \frac{1}{4} \int_0^a c(a-t) c(t) dt + \frac{1}{2} \int_0^a c(t) dt \quad (58)$$

A graph of the expected payoff for each agent when both use the symmetrical differentiable bidding function is given in Figure 3.

When the auctioneer chooses $a = 0.5$, the bidders have an (ex-ante) expected profit of $\frac{1}{6}$, using their equilibrium bidding functions $b(v) = \frac{v}{2}$.

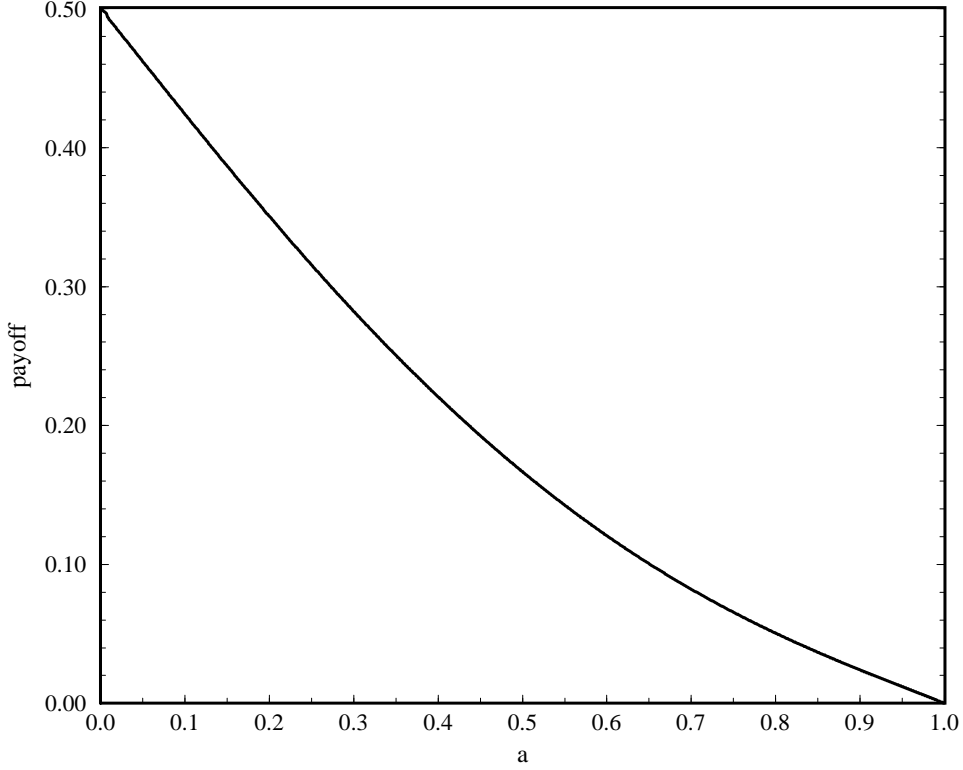


Figure 3: Agents' expected payoffs when using the differentiable symmetric equilibrium bidding function with $b(1) = a$.

4.3 Differentiable Symmetric Equilibria with $b(1) < a$

In this section we seek symmetric equilibrium bidding functions when $\frac{a}{2} < b(1) < a$ (since if $b(1) < \frac{a}{2}$ the good is never provided, we do not need to investigate this case for characterization of the symmetric equilibrium bidding functions). Denote $d = b(1)$. We seek functions that are differentiable. Thus, they are strictly increasing for values in the range $[a - b(1), 1]$ (from Lemma 5).

For b to be such a symmetric equilibrium bidding function, it must satisfy (8) for all $v \in [0, 1]$. Since any bid of less than $a - d$ gives probability 0 of the good being provided, for $v \in [0, a - d)$ no bid gives a positive expected profit, and therefore bids for such v will be between 0 and $[a - d]$. For any higher value of v , i.e. $v \in [a - d, 1)$, Equation (15) must

be satisfied. Since b is strictly increasing, it has a unique inverse. Denoting the inverse by c , we derive the requirement that Equation (18) is satisfied for $t \in (a - d, d)$, by c which is differentiable on $(a - d, d)$, continuous on $[a - d, d]$, and that $c(a - d) = a - d, c(d) = 1$.

Using the same method as in Theorem 1, it can be shown that the unique function c satisfying the above requirements is given by

$$c(t) = 1 + \frac{1 - a + ad - d^2}{a - 1 - t - (d - t)^{\frac{d-1}{2d-a}}(d - a + t)^{\frac{d-a+1}{2d-a}}}. \quad (59)$$

This gives us (by inversion) a function b from $[a - d, 1]$ to $[a - d, d]$. Note that if $d = a$, then (59) reduces to (19). We now need to determine possible values of b for private values in $[0, a - d)$. From Lemma 1 we know that for this range, $b(v) \leq a - d$. The binding constraint for equilibrium is that such a bid should not cause the other bidder to gain more by bidding more than d when she has a private value of 1. This constraint is satisfied if

$$(1 - (a - b(v)))(1 - v) \leq (1 - d)(1 - a + d) \quad \forall v \in [0, a - d], \quad (60)$$

which is equivalent to

$$b(v) \leq \frac{d(a - d) + v(1 - a)}{1 - v} \quad \forall v \in [0, a - d]. \quad (61)$$

Noting that $b(0) = 0$ and $b(a - d) = a - d$ satisfy (61), and that the left derivative of b at $a - d$ is bounded, it is obvious that many differentiable functions can be found that will satisfy (61) and will have a right derivative at $a - d$ equal to the left derivative of b there, giving us, together with the inverse of c for values of v in $[a - d, d]$, a differentiable symmetric equilibrium bidding function.

To summarize, we have

Theorem 2 *For any auctioneer's declaration $a \in (0, 1)$, if $b : [0, 1] \rightarrow [0, a]$ is a differentiable function, then b is a symmetric equilibrium bidding function if and only if (denoting $d = b(1)$ and $c = b^{-1}$):*

1. $\frac{a}{2} \leq d \leq a$.

2. For $t \in [a - d, d]$, c satisfies (59) and $c(a - d) = a - d$.
3. b satisfies (61), is differentiable for such values, and the right derivative at $(a - d)$ is equal to the left derivative of c^{-1} there.

5 Step Equilibria

In actual auctions, bids are not real numbers but amounts of money. Each bid must be an integer multiple of a smallest unit. This motivates the approach in this section. We now examine a family of non-differentiable equilibria parameterized by the number of discontinuities. The limit, as the number of discontinuities goes to infinity, is a differentiable bidding function as described in the previous section. Therefore, as long as the basic unit of money is small enough, the results of the previous section are approximately valid. A step-function equilibria has bidding functions of the following form for $a \leq 1$ and integer $s \geq 1$

$$b^s(v) = \begin{cases} 0 & \text{if } 0 \leq v < c_1 \\ \frac{1}{s} \cdot a & \text{if } c_1 \leq v < c_2 \\ \vdots & \vdots \\ \frac{k}{s} \cdot a & \text{if } c_k \leq v < c_{k+1} \\ \vdots & \vdots \\ a & \text{if } c_s \leq v \leq 1 \end{cases} \quad (62)$$

Necessary conditions for such a function to be in equilibrium are that when the true value is equal to c_k for some $1 \leq k \leq s$ the agent is indifferent between bidding $\frac{(k-1)a}{s}$ and bidding $\frac{ka}{s}$. Formally, defining $c_0 = 0$,

$$\left(c_k - \frac{(k-1)a}{s} \right) (1 - c_{s-k+1}) = \left(c_k - \frac{ka}{s} \right) (1 - c_{s-k}) \quad 1 \leq k \leq s \quad (63)$$

The following theorem will show that if Equation (63) is satisfied for $0 = c_0 < c_1 < \dots < c_k < \dots < c_s \leq 1$, then b^s is an equilibrium bidding function.

Theorem 3 *If Equation (63) is satisfied for $0 = c_0 < c_1 < \dots < c_k < \dots < c_s \leq 1$, then $b^s(v)$ given by (62) is an equilibrium bidding function.*

Proof: Assume $c_k \leq v \leq c_{k+1}$. Thus, $b^s(v) = \frac{ka}{s}$. It is obvious that under the assumption that the other agent is using the bidding function b^s , any bid which is not a multiple of $\frac{a}{s}$ is dominated by a bid that is such a multiple. We need to show that the expected payoff from bidding $b^s(v)$ is at least as good as the expected gain from bidding any other multiple of $\frac{a}{s}$. The proof will be by induction. First we show that $\frac{ka}{s}$ is no worse an action than $\frac{(k-1)a}{s}$, i.e.

$$\left(v - \frac{(k-1)a}{s}\right) (1 - c_{s-k+1}) \leq \left(v - \frac{ka}{s}\right) (1 - c_{s-k}).$$

This is equivalent to

$$\begin{aligned} (v - c_k)(1 - c_{s-k+1}) + \left(c_k - \frac{(k-1)a}{s}\right) (1 - c_{s-k+1}) &\leq \\ (v - c_k)(1 - c_{s-k}) + \left(c_k - \frac{ka}{s}\right) (1 - c_{s-k}). \end{aligned}$$

The second terms of each side are equal from Equation (63), so we need to show that $(1 - c_{s-k+1}) \leq (1 - c_{s-k})$, which is true from the assumption that $c_i \leq c_j$ for $i < j$.

For the induction step we assume that bidding $\frac{ka}{s}$ is no worse than bidding $\frac{(k-m)a}{s}$ for $m \geq 1$ and show that it is no worse than bidding $\frac{(k-m-1)a}{s}$. From Equation (63) it is true that

$$(1 - c_{s-k+m}) = \frac{\left(c_{k-m+1} - \frac{(k-m-1)a}{s}\right) (1 - c_{s-k+m+1})}{\left(c_{k-m+1} - \frac{(k-m)a}{s}\right)}. \quad (64)$$

Our induction assumption is that

$$\left(v - \frac{ka}{s}\right) (1 - c_{s-k}) \geq \left(v - \frac{(k-m)a}{s}\right) (1 - c_{s-k+m}),$$

and we need to show that

$$\left(v - \frac{(k-m)a}{s}\right) (1 - c_{s-k+m}) \geq \left(v - \frac{(k-m-1)a}{s}\right) (1 - c_{s-k+m+1}).$$

This is equivalent (substituting from Equation (63)) to

$$\frac{\left(v - \frac{(k-m)a}{s}\right) \left(c_{k-m+1} - \frac{(k-m-1)a}{s}\right) (1 - c_{s-k+m+1})}{\left(c_{k-m+1} - \frac{(k-m)a}{s}\right)} \geq \left(v - \frac{(k-m-1)a}{s}\right) (1 - c_{s-k+m+1}).$$

After simplifying, we are left with $v \geq c_{k-m+1}$ which is true from our assumptions. Thus we have shown that bidding $\frac{ka}{s}$ is no worse than anything smaller. Similarly, it can be shown that this bid is no worse than anything larger. \blacksquare (Theorem 3)

The following lemma shows that for a step bidding function that is a symmetric equilibrium, with an even number of steps s , the value of $c_{s/2}$ must be equal to a . Using this fact and a recursive algorithm, a computer program enabled us to calculate the values of the other c_i 's for any even s . We also calculated equilibria for low odd s 's by using brute force in solving Equation (63) for such s 's, with the constraints that the c_i 's are increasing and between 0 and 1.

Lemma 6 *If (a, b^s, b^s) , with b^s given by (62), is a Nash equilibrium, and if s is even, then $c_{s/2} = a$.*

Proof: If (a, b^s, b^s) is an equilibrium, then Equation (63) holds. Defining $u_k = c_k - \frac{k}{s} \cdot a$ for $1 \leq k \leq s$, Equation (63) is equivalent to

$$u_k(u_{s-k+1} - u_{s-k}) = \frac{a}{s} \left(-u_k + 1 - \frac{s-k+1}{s} \cdot a - u_{s-k+1} \right), \quad (65)$$

hence

$$u_k u_{s-k} - u_{k-1} u_{s-k+1} = (u_k - u_{k-1}) u_{s-k+1} - (u_{s-k+1} - u_{s-k}) u_k = \quad (66)$$

$$\frac{a^2}{s^2} (s - 2k + 1). \quad (67)$$

If $s = 2n$ then

$$u_n^2 = u_0 u_2 n + \sum_{k=1}^n u_k u_{2n-k} - u_{k-1} u_{2n-k+1} = \quad (68)$$

$$0 + \sum_{k=1}^n \frac{a^2}{s^2} (s - 2k + 1) = \quad (69)$$

$$\frac{a^2}{4}, \quad (70)$$

so $(c_{s/2} - \frac{a}{2})^2 = \frac{a^2}{4}$ and therefore $c_{s/2} = a$. ■(Lemma 6)

It is interesting to look at the limiting case when the steps are regular and the number of steps becomes large. As s goes to infinity, replacing $k\frac{a}{s}$ by t , c_k by $c(t)$, and $\frac{a}{s}$ by ε , (63) becomes

$$(c(t) - (t - \varepsilon))(1 - c(a - t + \varepsilon)) = (c(t) - t)(1 - c(a - t)). \quad (71)$$

Rearranging, we have

$$\frac{c(a - t + \varepsilon) - c(a - t)}{\varepsilon} = \frac{1 - c(a - t + \varepsilon)}{c(t) - t}. \quad (72)$$

Taking the limit as ε goes to zero, we get

$$c'(a - t) = \frac{1 - c(a - t)}{c(t) - t}, \quad (73)$$

which is equivalent to (18), the relation for the differentiable case when $b(1) = a$. This case was solved analytically in Section 4.1.

A similar analysis can be given, with the appropriate modifications, for the case where $\frac{a}{2} < b(1) < a$, the case dealt with in Section 4.3.

6 Concluding Remarks

We conclude with some remarks and directions for future research.

1. There are a number of justifications for our emphasis on differentiable bidding functions. Many analyses of standard auctions assumed that equilibrium bidding functions were differentiable to derive their results. For example see Rothkopf (1969), Wilson (1969), Oren and Williams (1975), Wilson (1977), Rothkopf (1977) and Reece (1978). For more details about auction theory see McAfee and McMillan (1987) and Milgrom (1987). Even though in our model there do exist symmetric equilibria with non-differentiable bidding functions, we emphasize the case of differentiable functions. Chatterjee and Samuelson (1983) call such functions “well-behaved”, and justify the emphasis placed on them with a number of arguments. They suggest that strictly increasing (monotonic) bidding functions are a focal point (as in Schelling, 1960).

In addition, it is possible to use the augmentation method from d’Aspremont, Crémer and Gérard-Varet (1998) to modify the auction so as to eliminate all equilibria except the differentiable one (and maybe others giving the same expected profit to the auctioneer). However, the use of this method depends on the distributions of private values being common knowledge among all the participants. This is a stronger assumption than needed for Nash equilibrium, and is probably not satisfied in many practical applications.

2. When the good is a public good for a group of participants, it seems reasonable in an auction setting to allow the whole group to pay their bids, as once the good is provided it benefits all members of the group⁴. This is the basis for the auction mechanism that we use.
3. Efficiency and profit maximizing are incompatible if we are restricted to incentive compatible schemes (which is a reasonable assumption in the auction setup), and the distribution of the private values for each agent includes zero. If the announced minimum price a is greater than zero (which must hold for profit maximization), then in any case where the sum of the private values is less than a (which occurs with positive probability) the good will definitely not be provided in any equilibrium. This is not an efficient outcome, as all players could be made strictly better off (e.g.

⁴In Bliss and Nalebuff’s (1984) model of public good provision, the public good can be paid for only by one agent (in equilibrium the one that values it most).

by providing the good and having each bidder pay half of her private value).

4. For $n > 2$ the problem of finding differentiable symmetric bidding functions is much harder. What one obtains is a differential equation involving convolutions of the solution with itself. A differential equation of this type belongs to the class of functional differential equations, about which very little is known.
5. An interesting generalization of the model is to allow more than one form of the public good, with each form benefitting a different group of agents. In this case, the good is provided to the group with the highest sum of bids, if this sum is at least the minimum price a . The different groups do not necessarily have empty intersections. Such a situation, in a complete information setting, is dealt with in Lerner (1998), which deals with the allocation of broadcasting licenses to groups of agents.

An example of a case where there is more than one group of agents, and the groups have a non-empty intersection is the following. There are three agents and two groups. The agents are $\{1, 2, 3\}$ and the groups are $\{1, 3\}$ and $\{2, 3\}$. Thus, the good can be provided either to 1 and 3 (and is a public good for them) or to 2 and 3, or it can be not provided. Each agent has a private value independently drawn from the uniform distribution over the unit interval. For agent 3, this is her value for both the groups to which she belongs. An example of such a situation is a playground that can be designed for young children or for older ones. Agent 1 has only a small child, agent 2 has only an older child, and agent 3 has both a young child and an older one. For this example, the auctioneer can make a positive profit even by announcing $a = 0$, as there is competition between agents 1 and 2. However, to induce agent 3 to make a positive bid, the auctioneer must announce a positive value for a , possibly leading to an inefficient outcome (and in some equilibria agent 3 always bids zero even for positive values of a). There appears to be scope for future research into situations like this, even though the mathematical complexity increases considerably.

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