Log-concave Probability Distributions: Theory and Statistical Testing [1]

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Abstract

This paper studies the broad class of *log-concave probability distributions* that arise in economics of uncertainty and information. For univariate, continuous, and log-concave random variables we prove useful properties without imposing the differentiability of density functions. Discrete and multivariate distributions are also discussed. We propose simple non-parametric testing procedures for log-concavity. The test statistics are constructed to test one of the two implications of log-concavity: increasing hazard rates and *new-is-better-than-used* (NBU) property. The tests for increasing hazard rates are based on normalized spacing of the sample order statistics. The tests for NBU property fall into the category of Hoeffding's U-statistics.

Key Words: Log-concave Distributions, Increasing Hazard Rate, NBU Property, Non-parametric Testing, U-statistic.

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Contents

1 Introduction

In economics of uncertainty and information, it is usually necessary to make explicit assumptions on the probability distributions of the random variables in question. However, worrying about the robustness of their results, economists are compelled to specify, whenever possible, a class of distributions rather than a single particular parametric family. The most important concept that arises in this context is the class of log-concave distributions. [2]

A random vector is log-concavely distributed if the logarithm of its probability density function is concave on its support. Log-concave probability distributions constitute a broad class. At the same time they share very important properties. Surprisingly systematic treatment of this class is still elusive. The papers by Bagnoli and Bergstrom (1989, 1990) and by Burdett and Muus (1989) are, to the author's knowledge, the only attempt on this subject. While they document clearly and convincingly the importance of the concept of log-concavity, they deal primarily with univariate random variables whose density functions are continuously differentiable. The first goal of this paper is to extend the previous work in three directions. First, for the continuous univariate case, we prove some of the useful properties without imposing the differentiability of the density functions. Second, we discuss the generalizablility of those properties to the multivariate case. Lastly we extend the concept of log-concavity to the discrete distributions case. The treatment hence brings one step further toward a complete theory of log-concavity.

Log-concavity, as an important concept, is appealing not only to economic theorists, but also to empiricists. When taking the theoretical implications of the log-concavity assumption to field data, applied economists will naturally question the validity of the log-concavity assumption. Since one is unwilling to adopt any parametric form for the distribution, the test statistic has to be non-parametric in nature. The second goal of the paper is to develop such non-parametric testing procedures for log-concavity.

[2] See Bagnoli and Bergstrom (1989) for a clear account of some of the applications in economics.

Non-parametric testing for log-concavity remains to be an important open question for fruitful future research. In this paper we propose to test one of the two necessary conditions of log-concavity. First, log-concavity of a density function implies monotonicly increasing hazard rates. Second, monotonic increasing hazard rates implies sub-additive logarithm of the survivor function which, in turn, is equivalent to the property of *new-is-better-than-used* (NBU) in reliability theory. To test for log-concavity one needs to simply construct a test for increasing hazard and/or NBU.

This paper is structured as follows. In the next section we study the properties of log-concave and log-convex distributions for the continuous random variables. We investigate thoroughly the univariate case and extend the results to the cases of multivariate random variables. Section 3 extends the concept of log-concavity for discrete random variables. In section 4 we introduce a number of non-parametric test statistics. We conclude in section 5 and provide some bibliographical notes in section 6.

2 Theory: Continuous Distributions

This section deals only with continuous random variables. Throughout the paper we denote by X, Y, etc. random variables, by $F(x) \equiv Pr(X \leq x) : \mathbf{R}^k \mapsto [0,1]$, the distribution function, by $f(x) : \mathbf{R}^k \mapsto \mathbf{R}_+$, the probability density function, [3] and by $\Omega \equiv \{x \in \mathbf{R}^k : f(x) > 0\}$, the distribution support.

2.1 Definition and (Parametric) Examples

Definition 1 *A vector of random variables, X, is* log-concavely distributed *if, for any $x_1, x_2 \in \Omega$ and any $\lambda \in [0,1]$,*

$$f(\lambda x_1 + (1-\lambda)x_2) \quad \geq \quad [f(x_1)]^\lambda \, [f(x_2)]^{1-\lambda}. \tag{1}$$

X is log-convexly distributed *if the inequality (1) holds in opposite direction.*

Remark 1 *It follows from elementary calculus that if $f(x)$ is log-concave (or log-convex, then, (1) the distribution support Ω is a convex subset of*

[3] f is a Radon-Nikodym density with respect to the Lebesgue measure.

$\mathbf{R}^k$, *(2) $f(x)$ is continuous in Ω, and (3), if $f(x)$ is differentiable, then it is continuously differentiable. See, for example, Rockafellar (1970).*

Even though this paper is not confined to parametric distributions, it is, however, helpful to examine log-concavity or log-convexity of the commonly used parametric families.

Remark 2

(i) The following distributions are log-concave:

- *(1) the K-dimensional uniform distribution $U(\theta_1, \theta_2)$,*
- *(2) the K-dimensional normal distribution $N(\mu, \Sigma)$,*
- *(3) the one-dimensional negative exponential distribution $E(\lambda)$,*
- *(4) the one-dimensional logistic distribution,*
- *(5) the one-dimensional Gamma distribution $\Gamma(\alpha, \beta)$ with the shape parameter $\alpha \geq 1$ (hence the χ^2 distribution with more than two degrees of freedom),*
- *(6) the one-dimensional Beta distribution $B(a, b)$ with $a \geq 1$ and $b \geq 1$,*
- *(7) the one-dimensional Weibull distribution $W(\gamma, \alpha)$ with the shape parameter $\alpha \geq 1$.*

(ii) The following distributions are log-convex:

- *(1) the negative exponential distribution,*
- *(2) the Perato distribution,*
- *(3) the Gamma distribution with the shape parameter $\alpha \leq 1$,*
- *(4) the Beta distribution with $a < 1$ and $b < 1$,*
- *(5) the Weibull distribution with the shape parameter $\alpha \leq 1$,*
- *(6) the F-distribution $F(m_1, m_2)$ with the first degree of freedom $m_1 \leq 2$.*

(iii) There are distributions which are both log-concave and log-convex. For example, the negative exponential distributions are such cases. In fact, since linear functions are the only functions which are both concave and convex, the only distributions which are both log-concave and log-convex are exponential or truncated exponential.

(iv) There are distributions which are neither log-concave nor log-convex over the entire support. Examples include the log-normal distribution, the Beta distribution with $a > 1$ and $b < 1$ and the F-distribution with the first degree of freedom $m_1 > 2$.

2.2 The Univariate Case

We will now present the theory of log-concavity for the case of univariate continuous distributions. Notice that throughout this section we do not impose differentiability of the probability density function. For univariate distributions with continuously differentiable densities, the proofs are typically shorter.

The following lemma provides alternative characterizations of log-concavity.

Lemma 1 *Let $f(x) : \mathbf{R} \mapsto \mathbf{R}_+$. Suppose $\{x : f(x) > 0\} = (a, b)$. The following statements are equivalent.*

(i) $f(x)$ is log-concave.

(ii) For all $x \in (a, b)$, all $\beta \geq 0$ and all $\gamma \geq 0$,

$$f(x + \beta)f(x + \gamma) \geq f(x)f(x + \beta + \gamma) \tag{2}$$

(iii) For all $a < x_1 < x_2 < b$ and all $\delta \geq 0$,

$$f(x_1 + \delta)f(x_2) \geq f(x_2 + \delta)f(x_1) \tag{3}$$

(iv) For all $a < y_1 < y_2 < b$ and all $\alpha \geq 0$,

$$f(y_1 - \alpha)f(y_2) \leq f(y_2 - \alpha)f(y_1) \tag{4}$$

(v) *$f(x)$ is Polỳa Frequency of order 2, that is for all $a < u_1 < u_2 < b$ and all $a < v_1 < v_2 < b$,*

$$f(u_1 - v_1)f(u_2 - v_2) \geq f(u_1 - v_2)f(u_2 - v_1) \tag{5}$$

PROOF. Each of the statements (ii) through (v) in the lemma compares the function values of f evaluated at four arbitrarily chosen points on the real line. They state roughly that the product of the function values evaluated at the inner two points is no less than the product of function values at the outer two points (See Figure 1). Statement (ii) is simply an alternative definition of concavity (in terms of log $f(x)$). Statement (iii) is equivalent to (ii) by the following one-to-one mapping,

$$\begin{cases} x = x_1 \\ \gamma = x_2 - x_1 \\ \beta = \delta \end{cases} .$$

Statement (iv) is equivalent to (ii) by the following one-to-one mapping

$$\begin{cases} x = x_1 - \alpha \\ \gamma = x_2 - x_1 \\ \beta = \alpha \end{cases} .$$

Statement (v) is equivalent to (ii) by the following one-to-one mapping,

$$\begin{cases} x = u_1 - v_2 \\ \gamma = u_2 - u_1 \\ \beta = v_2 - v_1 \end{cases} .$$

Q.E.D.

For log-convex functions f, similar results hold. However, extra conditions are needed to take care of the possible problems on the boundary of the interval (a,b), beyond which the function f vanishes.

Remark 3 *Let $f(x) : \mathbf{R} \mapsto \mathbf{R}_+$. Suppose $\{x : f(x) > 0\} = (a, b)$. The following statements are equivalent*

(i) *$f(x)$ is log-convex.*

(ii) For all $x \in (a,b)$*, all* $\beta \geq 0$ *and all* $\gamma > 0$ *such that* $x + \beta + \gamma \in (a,b)$*,*

$$f(x+\beta)f(\gamma) \leq f(x)f(x+\beta+\gamma) \tag{6}$$

(iii) For all $a < x_1 < x_2 < b$ *and all* $\delta \geq 0$ *such that* $x_2 + \delta \in (a,b)$*,*

$$f(x_1+\delta)f(x_2) \leq f(x_2+\delta)f(x_1) \tag{7}$$

(iv) For all $a < y_1 < y_2 < b$ *and all* $\alpha \geq 0$ *such that* $y_1 - \alpha \in (a,b)$*,*

$$f(y_1-\alpha)f(y_2) \geq f(y_2-\alpha)f(y_1) \tag{8}$$

(v) For all $a < u_1 < u_2 < b$ *and all* $a < v_1 < v_2 < b$ *such that* $u_1 - v_2 \in (a,b)$ *and* $u_2 - v_1 \in (a,b)$*,*

$$f(u_1-v_1)f(u_2-v_2) \leq f(u_1-v_2)f(u_2-v_1) \tag{9}$$

2.2.1 CDF, Survivor, Hazard Rate and Side Integrals

In many economic applications, researchers find it necessary to make assumptions on the *CDF, survivor function, hazard rates, or left or right side integrals* of a random variable in question. Log-concavity of the density function has important implications on these functions. First we need to slightly extend the conventional definition.

Definition 2 *Let* X *be a continuous random variable with density* $f(x)$ *and CDF* $F(x)$ *whose support* Ω *is an open interval* $(a,b) \subset \mathbf{R}$*. Define in the interval (a,b),*

$S(x) \equiv 1 - F(x)$ *as its survivor function,*

$h(x) \equiv f(x)/S(x)$ *as its hazard function,*

$G(x) \equiv \int_a^x F(u)\,du$ *as its left side integral,*

$H(x) \equiv \int_x^b S(u)\,du$ *as its right side integral.* [4]

[4] The conventional uses of these terms are defined only for non-negative valued random variables. We adopt these terms here simply because of lack of better alternatives.

Proposition 1 *For the functions defined above, the following logical implications hold.*

$$
\begin{aligned}
f(x) \text{ is log-concave} &\Longrightarrow_A \; h(x) \text{ is non-decreasing in } x\\
&\Longleftrightarrow_B \; S(x) \text{ is log-concave}\\
&\Longrightarrow_C \; H(x) \text{ is log-concave.}
\end{aligned}
$$

$$
\begin{aligned}
f(x) \text{ is log-concave} &\Longrightarrow_D \; F(x) \text{ is log-concave}\\
&\Longrightarrow_E \; G(x) \text{ is log-concave.}
\end{aligned}
$$

PROOF. A version of the proposition under the assumption that the density function is continuously differentiable is proved in Bagnoli and Bergstrom (1989). There the proof for Implications C and E, however, does not require the differentiability of the density function. The main trick is the fact that for a twice differentiable, strictly monotone function, $g(.)$ defined in (a,b) with either $g(a) = 0$ or $g(b) = 0$, if the ratio $g''(x)/g'(x)$ is decreasing in x then so is the ratio $g'(x)/g(x)$. Letting $g(x) = H(x)$ proves Implication C. Letting $g(x) = G(x)$ proves Implication E.

We now prove Implications A, B, D for the general case.

Implication A: By the Definition 2, $h(x)$ is non-decreasing if and only if for any $a < x_1 < x_2 < b$,

$$
\begin{aligned}
& f(x_2)S(x_1) - f(x_1)S(x_2) && (10)\\
= & \int_0^\infty [f(x_1 + u)f(x_2) - f(x_2 + u)f(x_1)]\, du \geq 0. && (11)
\end{aligned}
$$

The last inequality holds since the integrand is non-negative by lemma 1 vis-à-vis (3).

Implication B: Since

$$(\log S(x))' = -\frac{f(x)}{S(x)} = -h(x),$$

$h(x)$ is non-decreasing in x if and only if the first derivative of the logarithm of $S(x)$ is non-increasing in x if and only if $\log(S(x))$ is concave.

Implication D: Since

$$(\log F(x))' = \frac{f(x)}{F(x)},$$

it suffices to show that the above expression is non-increasing, that is, for any $a < x_1 < x_2 < b$,

$$\frac{f(x_1)}{F(x_1)} \geq \frac{f(x_2)}{F(x_2)},$$

which holds if and only if

$$\begin{aligned} & f(x_1)F(x_2) - f(x_2)F(x_1) \\ = & \int_{-\infty}^{0} [f(x_2 + u)f(x_1) - f(x_1 + u)f(x_2)] \, du \\ = & \int_{0}^{\infty} [f(x_2 - u)f(x_1) - f(x_1 - u)f(x_2)] du \geq 0. \end{aligned}$$

The above integrand is non-negative due to Lemma 1 vis-à-vis (4).

Q.E.D.

Remark 4 *For log-convex distributions, similar results hold under further conditions on the distribution support (cf. Remark 3).*

i. If $b = \infty$ then log-convexity of $f(x)$ implies that $h(x)$ is non-increasing, the latter is equivalent to that $S(x)$ is log-convex.

ii. If $a = -\infty$ then log-convexity of $f(x)$ implies that $F(x)$ is log-convex.

2.2.2 Modality, Tails and Moments

The next result establishes the relationship between log-concavity and unimodality.

Definition 3 *A probability density $f(x)$ is said to be* unimodal *if there exists a mode $m \in \Omega$ such that $f(x) \leq f(y)$ for all $x \leq y \leq m$ or for all $m \geq y \geq x$. $f(x)$ is* strongly unimodal *if f is unimodal and if the convolution of f with any unimodal g is unimodal.*

Notice the definition of unimodality does not require the uniqueness of the modes. The definition of strong unimodality is very hard to verify by definition. In searching for analytical tools, Ibragimov (1956) proves the following surprising result.

Proposition 2 *A random variable X is log-concavely distributed if and only if its density function $f(x)$ is strongly unimodal.*

It is intuitive that all log-concave densities are unimodal. It is also easy to imagine that not all unimodal densities are log-concave. For example, a Perato distribution is unimodal and log-convex. It is very hard, however, to sense why the set of strong unimodal densities coincide with the set of log-concave densities. A direct implication of the result is that the class of log-concave distributions are closed under convolution by the definition of strong unimodality.

Corollary 1 *If $f(x)$ is log-concave, then*

(i) It at most has an exponential tail, i.e.

$$f(x) = o(e^{-\mu x}) \qquad \text{for some } \mu > 0, \text{ as } x \to \infty.$$

(ii) All the moments exist.

Proof. Let f be log-concave. By Lemma 1, $f(x+1)/f(x)$ in non-increasing in x and hence converges to some number $r \geq 0$. $r \leq 1$ by unimodality. $r \neq 1$ since $f(x)$ is integrable. Hence, it must be the case that $0 \leq r < 1$. Part (i) is now proved. Part (ii) follows from Part (i) directly by the bounded convergence theorem, since exponential distributions have moments of all orders. Q.E.D.

2.2.3 Transformations

Given a log-concavely distributed random variable, what about certain functions of it? We first look at linear transformation.

Proposition 3 *Let X be a random variable whose density function, $f(x)$, is log-concave. Than for any $\alpha \neq 0$, the random variable $Y = \alpha X + \beta$ is log-concave.*

PROOF. By change of variables, the density of Y is

$$g(y) = |\alpha|^{-1} f\left(\alpha^{-1}(y-\beta)\right), \qquad a < \alpha y + \beta < b. \tag{12}$$

For any $y_1 < y_2$ in the distribution support, and any $\delta \geq 0$,

$$\begin{aligned}
& \alpha^2 g(y_1+\delta) g(y_2) \\
= \; & f\left(\alpha^{-1}(y_1+\delta-\beta)\right) f\left(\alpha^{-1}(y_2-\beta)\right) \\
= \; & f(x_1+\alpha^{-1}\delta) f(x_2) \\
\geq \; & f(x_2+\alpha^{-1}\delta) f(x_1) \\
= \; & f\left(\alpha^{-1}(y_2+\delta-\beta)\right) f\left(\alpha^{-1}(y_1-\beta)\right) \\
= \; & \alpha^2 g(y_2+\delta) g(y_1)
\end{aligned}$$

The inequality in the above proof follows from equations (3) or (4) in Lemma 1, depending on whether α is positive or negative. Q.E.D.

For nonlinear transformations, we need stronger conditions.

Proposition 4 *Let X be a random variable whose density function, $f(x)$, is log-concave and monotonic decreasing. Consider a function $l(.)$ satisfying*

(i) $x = l(y)$ is strictly increasing, differentiable and convex,

(ii) $l'(y)$ is logconcave.

Then the random variable $Y = l^{-1}(X)$ is log-concave.

PROOF. By change of variables, the density of Y is

$$g(y) = l'(y) f(l(y)). \tag{13}$$

It suffices to show that $\bar{g}(y) = f(l(y))$ is log-concave. For any $y_1 < y_2$ in the distribution support, and any $\delta \geq 0$,

$$\begin{aligned}
& \bar{g}(y_1+\delta)\bar{g}(y_2) \\
= \; & f(l(y_1+\delta)) f(l(y_2))
\end{aligned}$$

$$\begin{aligned}
&= f\left(l(y_1)+\delta l'(y^*)\right) f\left(l(y_2)\right), \qquad \text{for some } y^* \in [y_1, y_1+\delta] \\
&\geq f\left(l(y_2)+\delta l'(y^*)\right) f(l(y_1)) \\
&= f\left(l(y_2+\delta)+\delta[l'(y^*)-l'(y^{**})]\right) f\left(l(y_1)\right), \qquad \text{for some } y^{**} \in [y_2, y_2+\delta] \\
&\geq \left(l(y_2+\delta)\right) f\left(l(y_1)\right) \\
&= \bar{g}(y_2+\delta)\bar{g}(y_1)
\end{aligned}$$

Q.E.D.

2.2.4 Mixture of Densities

Mixture models are popular structures both in dealing with unobserved heterogeneity in economics and in hierarchical models in Bayesian statistics. As will be shown below, log-convexity is preserved by mixing. Log-concavity, however, is not.

Proposition 5 *Let $g(x;v)$ be the conditional density function of X given $V=v$ in B. Let V be distributed with (marginal) density of $\pi(v)$. Then the marginal density of X,*

$$f(x) \equiv E_\pi\{g(x;V)\} = \int_B g(x;v)\pi(v)\,dv$$

is log-convex if $g(x;v)$ is log-convex in x.

Proof . For any $x_1, x_2 \in \Omega$ and for any $\lambda \in [0,1]$,

$$\begin{aligned}
& f\left(\lambda x_1+(1-\lambda)x_2\right) \\
=\; & E_\pi\left\{g(\lambda x_1+(1-\lambda)x_2;V)\right\} \\
\leq\; & E_\pi\left\{[g(x_1;V)]^\lambda \cdot [g(x_2;V)]^{1-\lambda}\right\} \\
\leq\; & \left\{E_\pi[g(x_1;V)]\right\}^\lambda \cdot \left\{E_\pi[g(x_2;V)]\right\}^{1-\lambda} \\
=\; & [f(x_1)]^\lambda \cdot [f(x_2)]^{1-\lambda}
\end{aligned}$$

The second inequality follows from an application of the Hölder's inequality, which states that for any $\alpha \in (0,1)$ and any two random variables X and Y, $E[|XY|] \leq \left\{E[|X|^{1/\alpha}]\right\}^\alpha \left\{E[|Y|^{1/(1-\alpha)}]\right\}^{(1-\alpha)}$. *Q.E.D.*

Proposition 5 does not impose any conditions on the mixing distribution. The setting of Proposition 5 can be viewed as a mixture structure in numerous economic models, where $\pi(v)$ is called the mixing distribution for some unobserved heterogeneity. The conditional density $g(x; v)$ is the fundamental quantity of interest. But due to the unobservability of V, the marginal density, $f(x)$, has to be used for inference.

2.2.5 Truncated Distribution

Probably the most popular property of the log-concave distributions is associated with the truncation of the random variable. Consider a random variable X which is truncated at$X \geq c$ for some constant $c \in \Omega$. We are interested in the shape of the truncated distribution $f^{c}(x)$, the effect of c on the truncated means $m(c)$ and on the truncated variance $v(c)$. Define

$$f^c(x) \equiv \frac{f(x)}{S(c)} 1_{(x \geq c)}, \qquad m(c) \equiv E[X|X \geq c], \qquad v(c) \equiv V[X|X \geq c] \tag{14}$$

Proposition 6 *If X is a log-concave random variable then*

(i) $f^c(x)$ *is log-concave.*

(i) $m'(c) \in (0, 1]$.

(ii) $V'(c) \leq 0$.

PROOF. Part (i) is trivial. All the known proofs of (ii) and (iii) are very involved. See, for example, Heckman and Honoré (1990). Q.E.D.

2.3 The Multivariate Case

For multivariate random vectors, the two theorems of Prékopa (1971, 1973) on log-concave probability measures are fundamental. The proof of the Prékopa theorems follows a general integral inequality, which will not be given here.

Lemma 2 (Prékopa Theorem 1) *Suppose a random vector X is log-concavely distributed. Then for any two convex sets A and B of $\mathbf{R}^k$, and any $\lambda \in (0,1)$,*

$$P[X \in \lambda A + (1-\lambda)B] \geq [P(X \in A)]^{\lambda}[P(X \in B)]^{1-\lambda}, \tag{15}$$

where the sign + means the Minkowski addition of sets:

$$\lambda A+(1-\lambda)B \equiv \left\{x \in \mathbf{R}^k : x = \lambda x_1 + (1-\lambda)x_2 \text{ for some } x_1 \in A \text{ and } x_2 \in B\right\}.$$

Lemma 3 (Prékopa Theorem 2) *Let $f(x,y)$ be the density function of $k_1 + k_2$ random variables (X,Y). Suppose $f(x,y)$ is log-concave and let A be a convex subset of $\mathbf{R}^{k_2}$. Then the function of the x defined as,*

$$g(x) = \int_A f(x,y)dy \tag{16}$$

is log-concave in $\mathbf{R}^{k_1}$.

2.3.1 CDF and Survivor

Lemma 2 provides a direct implication of the following result.

Proposition 7 *Let X be a k-dimensional continuous random vector with log-concave pdf $f(x)$. Let $F(x)$ be its CDF. Define $S(x) = P(X > x)$. Then both $F(x)$ and $S(x)$ are log-concave. (cf. Proposition 1.)*

Proof. For any x in the distribution support define the convex set

$$A_x \equiv \{w \in R^k : w \leq x\}.$$

Then $F(x) = P(A_x)$. A direct application of Lemma 2 shows that $F(x)$ is log-concave. The log-concavity of $S(x)$ follows the same line of reasoning. Q.E.D.

2.3.2 Marginal Densities

With multivariate distributions, one additional aspect is of interest: what is the relationship between log-concave joint densities and log-concave marginal densities?

Proposition 8

(i) Let X be a k-dimensional continuous random vector with log-concave pdf $f(x)$. Then the marginal densities of any sub-vector are also log-concave.

(ii) Let $X = (X_1, ..., X_k)$. If the X_i's are independent and each has a log-concave density function, then their joint density is log-concave.

Proof. Part (i) follows from a direct application of Lemma 3. Part (ii) is trivial. Q.E.D.

Remark 5 *Similar result holds for log-convex joint densities. The proof, though, is different. Consider, without loss of generality, the bivariate case. Suppose a bivariate density function $f(x, y)$ is log-convex. Then for any $\lambda \in [0, 1]$ and for any x_1, x_2, y such that both (x_1, y) and (x_2, y) are in Ω,*

$$f(\lambda x_1 + (1 - \lambda)x_2,\ y) \le [f(x_1, y)]^{\lambda}[f(x_2, y)]^{1-\lambda}.$$

Therefore,

$$\begin{aligned}
f_X(\lambda x_1 + (1-\lambda)x_2) &= \int f(\lambda x_1 + (1-\lambda)x_2,\ y)dy \\
&\le \int [f(x_1, y)]^{\lambda}[f(x_2, y)]^{1-\lambda} dy \\
&= \int \left[\frac{f(x_1, y)}{f_Y(y)}\right]^{\lambda} \left[\frac{f(x_2, y)}{f_Y(y)}\right]^{1-\lambda} f_Y(y)dy \\
&= E_Y\left\{\left[\frac{f(x_1, y)}{f_Y(y)}\right]^{\lambda} \left[\frac{f(x_2, y)}{f_Y(y)}\right]^{1-\lambda}\right\} \\
&\le \left\{E_Y\left[\frac{f(x_1, y)}{f_Y(y)}\right]\right\}^{\lambda} \left\{E_Y\left[\frac{f(x_2, y)}{f_Y(y)}\right]\right\}^{1-\lambda} \\
&= [f_X(x_1)]^{\lambda}[f_X(x_2)]^{1-\lambda}
\end{aligned}$$

The last inequality is due to the Hölder's inequality.

2.3.3 Order Statistics

Let $X_1, ..., X_n$ be an iid sample from a log-concave distribution with common density $f(x)$. Let $Y_1, ..., Y_n$ be the sample order statistics.

Proposition 9 *For the sample statistics defined above,*

(i) The joint density of the Y_i's is log-concave.

(ii) For all r=1, ..., n, Y_r is log-concave.

Proof. The joint density of the $Y_i's$ is

$$f_O(y_1, ..., y_n) = n! \prod_{i=1}^{n} f(y_i) 1_{(0 \leq y_1 \leq ... \leq y_n)}. \tag{17}$$

which is truncated $f(x)$. Part (i) follows. To prove Part (ii), take any $1 \leq r \leq n$. The marginal density of Y_r is

$$f_r(y) = \frac{n!}{(r-1)!(n-r)!} [F(y)]^{r-1} [1 - F(y)]^{n-r} f(y). \tag{18}$$

$f_r(y)$ is log-concave since each of the three components on the right hand side is log-concave. Q.E.D.

3 Theory: Discrete Distributions

Recall, from Remark 1, that the definition of log-concavity in Definition 1 precludes the case of discrete distributions. This section discusses the discrete analog.

Let the sample space $\Omega = \{x_1, x_2, ...\}$ be a countable subset of the Euclidean space R, with $x_1 < x_2 < ...$. Let $p_i = P(X = x_i)$ be the probability function for X.

Definition 4 *A discrete random variable X is log-concave if*

$$p_{i+1}^2 \geq p_i p_{i+2} \qquad \textit{for all } i \tag{19}$$

Parametric examples of discrete log-concave distributions include the Bernoulli trials, binomial distributions, Poisson distributions, geometric distributions, and negative binomial distributions.

With this definition it is easy to check that virtually all the results discussed in section 2.2 hold. In particular

Proposition 10 *Suppose a discrete random variable X is log-concavely distributed, then*

(i) For any integer m, and any integers $i < j$ (cf. Lemma 1),

$$\frac{p_{i+m}}{p_i} \geq \frac{p_{j+m}}{p_j}$$

(ii) $h_i \equiv \frac{p_i}{\sum_{k \geq i} p_k}$ is non-decreasing(cf. Proposition 1).

(iii) p is strongly unimodal (cf proposition 2).

(iv) p has at most an exponential tail (cf . Corollary 1).

(v) X has all its moments (cf . Corollary 1).

(vi) The random variable $Y = l(X)$ is log-concave, provided that $y = l(x)$ is strictly increasing (cf. Proposition 3).

(vii) Log-concavity is preserved within convolution operations (cf. Proposition 2).

(viii) The truncated distribution

$$p_i^m = \frac{p_i}{\sum_{k \geq m} p_k}, \qquad i \geq m$$

satisfy (19) (cf. Proposition 6).

Proof. The proof is quite similar to its continuous counterpart. Denote for $i = 1, 2, ...$

$$q_i = \log p_{i+1} - \log p_i \tag{20}$$

then in terms of the q_i's, the log-concavity in Definition 5 is equivalent to $q_i \geq q_j$ for all $i < j$.

Part (i) follows easily, since for any positive integer m, and $i < j$,

$$\log \frac{p_{i+m}}{p_i} = q_i + q_{i+1} + ... + q_{i+m_1} \tag{21}$$

$$\log \frac{p_{j+m}}{p_j} = q_j + q_{j+1} + ... + q_{j+m_1}. \tag{22}$$

Part (ii) follows from part (i), since

$$p_j(\sum_{m \geq i} p_m) - p_i(\sum_{m \geq j} p_m) = \sum_{m \geq 0} [p_j p_{i+m} - p_i p_{j+m}] \geq 0.$$

Parts (vi), (vii) and (viii) are trivial. Parts (iv) and (v) are implications of part (iii), the proof of which is omitted here. Q.E.D.

4 Statistical Tests for Log-concavity

We now turn to the second task of the paper: the construction of statistical tests for log-concavity. We do so by focusing on positive-valued, univariate random variables. Throughout this section, let $X_1, X_2, ..., X_n$ be an iid sample from some distribution $F(x)$ with density $f(x)$. Assume $F(0) = 0$. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the sample order statistics. Take $X_{(0)} \equiv 0$ whenever it is called for.

4.1 Tests for Increasing Hazard Rate

From Proposition 1, a necessary condition of log-concavity is that the hazard rate is monotone increasing, which is equivalent to $\log(S(x))$ being concave. A natural way to test log-concavity is then to construct tests for increasing hazard rate or for concavity of $\log(S(x))$. There are several non-parametric such tests available in the reliability literature. These tests are based on measuring the normalized spacing of the order statistics. Define,

$$D_i = (n - i + 1)(X_{(i)} - X_{(i-1)}) \qquad i = 1, 2, ..., n. \tag{23}$$

as the normalized spacings, for which we have

Proposition 11 *Let $D = (D_1, D_2, ..., D_n)$ be the normalized spacing as defined above. Then*

(i) Under the null hypothesis that X_i are negative exponential with survivor $S_0(x) = e^{-\lambda x}$ (for some unknown parameter $\lambda > 0$), D_i are iid exponential (with the same parameter λ).

(ii) Under the alternative that $S(x)$ is log-concave, (but not exponential), the D_i's exhibit a downward trend, more precisely

$$P(D_i \leq D_j) < 1/2 < P(D_i > D_j) \qquad \text{for all} \qquad i < j. \tag{24}$$

Proof. Let $X_O = (X_{(1)}, ..., X_{(n)})$, $D = (D_1, ..., D_n)$. The joint density function of the order statistics can be expressed as in (17). Consider two cases.

(i). Take $f(x) = \lambda e^{-\lambda x}$. Conduct a transformation of random variables from X_O to D with

$$\begin{cases} y_1 = \frac{d_1}{n} \\ y_2 = \frac{d_1}{n} + \frac{d_2}{n-1} \\ \\ \dots \\ \\ y_n = \frac{d_1}{n} + \frac{d_2}{n-1} + ... + d_n. \end{cases} \tag{25}$$

The joint probability density of D is

$$f_D(d_1, ..., d_n) = \lambda^n e^{-\lambda \sum_i d_i} \prod_i 1_{(d_i \geq 0)}. \tag{26}$$

This proves Part (i).

(ii) The trick, suggested by Proschan and Pyke (1967), is to study the random variable $Z = s(X) = -\log S(X)$, where $S(x)$ is the survivor function of X. By construction $s(x)$ is differentiable, convex and strictly increasing in x with $s(0) = 0$.

Lemma 4 *Corresponding to the sample* $(X_1, ..., X_n)$ *from* $F(x)$, $(Z_1, ..., Z_n) = (s(X_1), ..., s(X_n))$ *constitutes an iid sample from the unit exponential distribution.*

Proof of the Lemma: It is obvious that the Z_i's are iid, since the X_i's are. To show that for any i, Z_i is unit exponential,

$$\begin{aligned} F_Z(z) &= P(Z_i \le z) \\ &= P[-\log S(X_i) \le z] \\ &= P[S(X_i) \ge e^{-z}] \\ &= P[F(X_i) \le 1 - e^{-z}] \\ &= P[X_i \le F^{-1}(1 - e^{-z})] \\ &= 1 - e^{-z}. \end{aligned}$$

The lemma is proved.

Proof of the Proposition: Let $Z_O = (Z_{(1)}, ..., Z_{(n)})$ be the order statistics of the Z's. By construction, $Z_{(i)} = s(X_{(i)})$ for all i. Let $B = (B_1, .., B_n)$ be the normalized spacing of the $Z_{(i)}$'s. From Part i and Lemma 4, the B_i's are iid (unit exponential). Hence for any $i \ne j$, $P(B_i > B_j) = 1/2 = P(B_i < B_j)$.

But since $s(x)$ is differentiable and convex, for any $i < j$, $B_i > B_j$ implies $D_i > D_j$, but not vise versa. Therefore $P(D_i > D_j) > P(B_i > B_j) = 1/2$. Q.E.D.

Without any distribution assumption of F, the evidence of downward trend in the D_i's should be in favor of increasing hazard rate, hence in favor of log-concavity of $f(x)$. Let R_i denote the rank of D_i among $D_1, D_2, ..., D_n$. The following non-parametric tests based on D_i or R_i are proposed.

(i) Proschan and Pyke (1967) Test: Reject H_0 for larger than critical value of

$$P_n = \sum_{i<j} 1_{(D_i > D_j)} \tag{27}$$

(ii) Epstein (1960) Test: Reject H_0 for larger than critical value of

$$E_n = \sum_{i=1}^{n-1} \sum_{j=1}^{i} \frac{D_j}{\sum D_k} \tag{28}$$

(iii) Bickel and Doksum (1969) Test: Reject H_0 for larger than critical value of

$$B_n = \sum_{i=1}^{n} i \log(1 - \frac{R_i}{n+1}) \tag{29}$$

Let $\{F_{\theta_n,\lambda}\}$ be a sequence of local alternatives with $\theta_n = \theta_0 + O(n^{-1/2})$ such that $F_{\theta_0,\lambda}(x) = 1 - e^{-\lambda x}$. As is shown in Bickel and Doksum (1969) and Bickel (1969), the tests mentioned above are all (1) unbiased against any increasing hazard rate (as a direct application of Proposition 10), (2) asymptotically normal under the null, and (3) asymptotically most powerful against the local alternatives $\{F_{\theta_n,\lambda}\}$.

4.2 Tests for NBU Property

Proposition 12 *Let X be a random variable with positive support. If its survivor function $S(x)$ is log-concave, then for any pair of non-negative numbers, x_1, x_2,*

$$S(x_1 + x_2) \leq S(x_1)S(x_2) \tag{30}$$

Proof. We need a lemma on the super-additivity of convex functions.

Lemma 5 *Let $s : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be a monotone increasing, convex function with $s(0) = 0$. Then for any pair of positive numbers, x, y,*

$$s(x + y) \geq s(x) + s(y) \tag{31}$$

Proof of the lemma: Consider any pair of positive numbers, x, y. Without loss of generality assume $x < y$. Express y as

$$y = \frac{x}{y} \cdot x + \left(1 - \frac{x}{y}\right) \cdot (x + y).$$

Since s is convex,

$$s(y) \leq \frac{x}{y} \cdot s(x) + \left(1 - \frac{x}{y}\right) \cdot s(x+y).$$

Therefore

$$\begin{aligned} s(x+y) - s(x) - s(y) &\geq \frac{1}{y-x}[xs(y) - ys(x)] \\ &= \frac{xy}{y-x}\left[\frac{s(y)}{y} - \frac{s(x)}{x}\right] \\ &\geq 0 \end{aligned}$$

The last step is true due to the assumptions on s.

Proof of the Proposition: $S(x)$ is log-concave on the support of X. Let $s(x) = -\log(S(x))$. Apply the above lemma. Q.E.D.

Equation (30) is called the NBU property, since, if X is the time to death of a physical object, the chance $S(x_1)$ that a new unit will survive to age x_1 is, according to (30), greater than the chance $S(x_1 + x_2)/S(x_2)$ that a survived unit of age x_2 will survive for an additional time x_1.

Recall from the discussion at the end of section 2 that the only positive-valued distribution which is both log-concave and log-convex is the exponential family for which the property (30) holds in equality. Therefore a test for log-concavity will be based on the estimate for the following quantity

$$\begin{aligned} \Delta(F) &\equiv \int_0^\infty \int_0^\infty [S(x)S(y) - S(x+y)]\, dF(x)dF(y) \\ &= \frac{1}{4} - \int_0^\infty \int_0^\infty S(x+y)dF(x)dF(y) \\ &= \frac{1}{4} - E_F\left[1_{(X_1 > X_2 + X_3)}\right]. \end{aligned}$$

4.2.1 The U-Statistics

In non-parametric statistical inference, the class of U-statistics, due to Hoeffding (1948), play an important role. [5] Let $X_1, X_2, ...X_n$ be an iid sample from some unknown distribution F.

Let $\gamma(F)$ be a quantity of interest. If, for some $m < n$, there exists an m-dimensional *symmetric kernel*, $k(x_1, x_2, ..., x_m)$ such that [6]

$$\gamma(F) = E_F k(X_1, X_2, ..., X_m) = \int k(x_1, x_2, ..., x_m) dF(x_1), ..., dF(x_m), \quad (32)$$

then the U-statistic for $\gamma(F)$ is defined as

$$U_n(\gamma) = \frac{1}{\binom{n}{m}} \sum_c k(X_{i_1}, ..., X_{i_m}) \quad (33)$$

where the summation is over all combinations. Among many of the well-known examples of U-statistics are the sample moments $n^{-1} \sum X_i^q$, for estimation of population moments $E_F X^q$, the empirical CDF $\hat{F}_n(x)$ for estimation of $F(x)$, and the Wilcoxon one-sample statistic for estimation of $P_F(X_1 + X_2 \leq 0)$.

The following result is well-known for the U-statistics. Let U_n be the U-statistic for γ based on an iid sample of size n. Suppose, for the corresponding kernel $k(x_1, x_2, ..., x_m)$, $E_F k^2 < \infty$. Then

(i) $E_F U_n = \gamma$.

(ii) U_n converges to γ almost surely.

(iii) $n^{1/2} U_n$ is asymptotically normal.

There are other optimal properties of U-statistics. First, since U_n is the conditional expectation of $k(X_1, X_2, ..., X_m)$ given the sample order statistics and since the order statistics are always sufficient, U_n achieves smallest

[5] See Chapter 5 in Serfling (1980) for a classic treatment of U-statistics.

[6] A m-dimensional *symmetric kernel*, $k(x_1, x_2, ..., x_m)$ is a real-valued function on $B \subset \mathbf{R}^m$ such that k is invariant over all the possible permutations of its arguments.

variance of all unbiased estimates of γ, regardless of the true distribution function F. Second, with respect to a suitably defined sequence of σ-fields, the sequence of U_n admits a reverse martingale representation (Berk 1970.)

4.2.2 Hollander and Proschan Test Statistic

We can now construct our test statistic based on (33). The only problem is that $1/4 - 1_{(x_1 > x_2 + x_3)}$ is not symmetric. This is corrected by using the kernel

$$k(x_1, x_2, x_3) = \frac{1}{3} 1_{(x_1 > x_2 + x_3)} + \frac{1}{3} 1_{(x_2 > x_1 + x_3)} + \frac{1}{3} 1_{(x_3 > x_2 + x_1)} \tag{34}$$

The test statistic is then

$$U_n = 1/4 - \frac{1}{\binom{n}{3}} \sum_c k(X_{i_1}, X_{i_2}, X_{i_3}). \tag{35}$$

Using U_n, one would reject H_0 for sample values of U_n larger than its critical value. Hollander and Proschan (1972) propose test be based on the second part of U_n,

$$J_n = \frac{2}{n(n-1)(n-2)} \sum_{c'} 1(X_{i_1} > X_{i_2} + X_{i_3}) \tag{36}$$

where the summation is over all triple of integers (i_1, i_2, i_3) such that $1 \leq i_k \leq n$, $i_1 \neq i_2$, $i_1 \neq i_3$, and $i_2 < i_3$. The test will reject H_0 for small values of J_n. Applying the general theory of the U-statistics, they show that J_n is unbiased, asymptotically normal under the null, and consistent under the alternative. More precisely,

Proposition 13 *Let $H_0 : F(x) = 1 - e^{-\lambda x}$ for some unknown λ. Let H_a : $F(x)$ satisfies (30) but is not exponential. Then*

(i) J_n is unbiased against H_a.

(ii) Under H_0, the limiting distribution of $n^{1/2}(J_n - 1/4)$ is normal with mean 0 and variance 5/432.

(iii) If $F(x)$ is continuous, then J_n is consistent.

Proof. See Hollander and Proschan (1972). Q.E.D.

4.3 The Multivariate Case

For multivariate random vectors, tests for the log-concavity of its joint density can be done in one of two ways. First, since log-concavity of the joint density implies the log-concavity of all the marginal densities (Proposition 8), one may tests the log-concavity of the univariate case using one of the test statistics presented earlier in this section.

The second way to test for log-concavity is to use the result of Proposition 7 that the survivor $S(x)$ is log-concave (Proposition 7). According to Lemma 1, log-concavity of $S(x)$ implies that for any $\delta \geq 0$, $S(x+d)/S(x)$ is non-increasing in x, where $d = (\delta, \delta)$. This is equivalent to

$$S(k_1x_1 + \delta, k_2x_2 + \delta) \cdot S(x_1, x_2) \geq S(k_1x_1, k_2x_2) \cdot S(x_1 + \delta, x_2 + \delta), \quad (37)$$

for all $(x_1, x_2, \delta) \in \mathbf{R}^3_+$. Similar to the motivation for the Hollander and Proschan test statistics, (37) can be used to measure the discrepancy of $F(x_1, x_2)$, from the bivariate exponential distribution, whose density function is

$$f(x_1, x_2) = \begin{cases} (\lambda_1 + \lambda_3)e^{-(\lambda_1+\lambda_3)x_1}(\lambda_2)e^{-\lambda_2 x_2}, & x_1 \geq x_2 \\ \\ (\lambda_2 + \lambda_3)e^{-(\lambda_2+\lambda_3)x_1}(\lambda_1)e^{-\lambda_1 x_1}, & x_1 < x_2 \end{cases} \quad (38)$$

which is of course not continuous for $x_1 = x_2$, unless $\lambda_1 = \lambda_2$. We measure the discrepancy as

$$\begin{aligned} \Delta(F; k_1k_2) \quad = \quad & \int_0^\infty \int_0^\infty \int_0^\infty \{S(k_1x_1 + \delta, k_2x_2 + \delta) \cdot S(x_1, x_2) \\ & -S(k_1x_1, k_2x_2) \cdot S(x_1 + \delta, x_2 + \delta)\}\, dF(x_1, x_2) dF(\delta, \delta). \end{aligned}$$

The corresponding symmetric kernel is

$$k((x_1^1, x_2^1), (x_1^2, x_2^2), (x_1^3, x_2^3), (x_1^4, x_2^4)) = \psi_1 - \psi_2$$

where

$$\psi_1((x_1^1, x_2^1), (x_1^2, x_2^2), (x_1^3, x_2^3), (x_1^4, x_2^4)) = 1_{(x_1^1 > k_1 x_1^3 + z^4)} 1_{(x_2^1 > k_2 x_2^3 + z^4)} 1_{(x_1^2 > x_1^3)} 1_{(x_2^2 > x_2^3)} \quad (39)$$

and

$$\psi_2((x_1^1, x_2^1), (x_1^2, x_2^2), (x_1^3, x_2^3), (x_1^4, x_2^4)) = 1_{(x_1^1 > k_1 x_1^3)} 1_{(x_2^1 > k_2 x_2^3)} 1_{(x_1^2 > x_1^3 + z^4)} 1_{(x_2^2 > x_2^3 + z^4)}. \tag{40}$$

The corresponding U-statistic proposed by Bandyopadhyay and Basu (1991) is

$$U_n(k_1, k_2) = \frac{(n-4)!}{n!} \sum_p k((x_1^{i_1}, x_2^{i_1}), (x_1^{i_2}, x_2^{i_2)}, (x_1^{i_3}, x_2^{i_3}), (x_1^{i_4}, x_2^{i_4})), \tag{41}$$

where the summation is over all the possible permutations of four distinct elements of the sample. Asymptotic normality of $U\quad_n(k_1, k_2)$ follows from the general theory of U-statistics.

5 Conclusions

In this paper we have examined more thoroughly various properties of the class of log-concave probability distributions. In particular, we were able to prove all the results without relying on the differentiability assumption of the density function. We also investigated multivariate continuous distributions. We extended the concept of the log-concavity to the case of the discrete random variables. Our discussion can be viewed as a one step further towards a formal theory of log-concavity.

As the second contribution of the paper, we presented simple non-parametric methods to test for different implications of log-concavity. We concentrated on seeking tests for log-concavity only for positive-valued random variables. We were able to recognize the important implications of log-concavity in terms of the monotonicity of the hazard functions and in terms of the new-is-better-than used property. Hence the tests can be done by performing tests for increasing hazard rate and for NBU property, methods readily available from the reliability literature. How to construct simple non-parametric procedure to directly test log-concavity remains to be a very important research area.

6 Bibliographical Notes

Log-concavity of probability density functions was first introduced by Ibragimov (1956) as a necessary and sufficient condition for *strong unimodality* (Proposition 2). In his seminal book on total positivity, Karlin (1968) proved the equivalence between log-concavity and Pólya Frequency of Order 2 (Part v of Lemma 1). The concept of log-concavity was revolutionalized by introduction of log-concave probability measures due to Prèkopa (1971, 1973) (Lemmas 2 and 3) . While for completeness we reproduced several key results on log-concavity from the previous literature (Lemmas 1, 2, 3 and Propositions 1, 2, 6, 7, 8), Propositions 3, 4, 5, 9, 10 and Remarks 3, 5 seem to be new.

The earliest application is in the reliability literature on optimal scheduling of inspections of an electronic system subject to failure. There the trade-off is between the possibility of forced failures due to infrequent inspections and the unwarranted cost due to very frequent inspections. If the probability density function of the system life time is log-concave, Barlow, Hunter and Proschan (1963) show that the optimal inspection intervals decrease with time. Recently, Parmigiani (1995) shows that log-concavity also bestows a fast algorithm in solving this dynamic decision problem. Other areas where log-concavity has found much usage include (1) the study of the passage time distribution for birth-death processes (Keilson, 1979), (2) the study of unbiased hypothesis statistical testing (Rinott, 1976), and (3) the proof that the log-likelihood function is globally concave in parameters (Pratt, 1981).

In economics, log-concavity has been viewed as a generalization of the normality. Heckman and Honoré (1990)'s generalization of the Roy's (1951) model of occupation choices is one of the most glamours examples. One of the earliest economic applications of log-concavity is in the job search literature, where under the assumption that the wage offer distribution is log-concave, one is able to sign many comparative dynamics derivatives (Burdett and Ondrich, 1980, Flinn and Heckman, 1983). Devine and Kiefer (1991) contains a nice survey. Recent application in controlled semi-Markovian processes is reported in An (1994). Other applications are numerous. For a partial account see Bagnoi and Bergstrom (1989).

The sources of the different test statistics reported in section 4 are listed in the text. Our contribution here is to adopt those procedures to test for log-concavity. Prositions 11 and 12 are previously known. Our proofs seem to be new and more intuitive.

7 References

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Figure 1. A Characterization of Concave Functions
(not included here)