

The Nash- and Kalai-Smorodinsky Bargaining Solution for Decision Weight Utility Functions*

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1 Introduction

Almost all economic theories are based upon the axioms underlying expected utility preferences and many experiments have shown that the preference relation of individuals often violate at least one of these axioms. Therefore, these economic theories have to be extended to the domain of non-expected utility (NEU) preferences. For axiomatic bargaining theory this has been done in Rubinstein, Safra and Thomson (1992). In this article both the Nash bargaining solution (NBS), as proposed in Nash (1950), and the Kalai-Smorodinsky bargaining solution (KSBS), as proposed in Kalai and Smorodinsky (1975), are redefined in terms of NEU preference relations and for each bargaining solution a new axiomatization is given on the domain of NEU preferences. The interpretations of these new definitions for both bargaining solutions are very attractive.

Despite these nice positive and general results, there is a real possibility that the formulation of this NEU axiomatic bargaining theory in terms of preference relations will not be taken up by applied economists. The reason is that applied economists prefer to work with utility functions rather than preference relations. In order to encourage applied economists to use NEU preference relations economic theory has to provide NEU utility functions and formulas that are simple to apply. The popularity of the expected utility NBS and KSBS in many economic applications stems from the fact that both are easy to apply.

A significant part of the literature on NEU preferences (e.g. Gul, 1991, Tversky and Wakker, 1995, and the references therein) deals with decision weight utility function (DW) preferences, i.e. preferences that can be represented jointly by the utility function that measures the subjective value on the domain of outcomes (or prizes) and by the decision weights that capture the attitude toward risk

or uncertainty. The list of references in Tversky and Wakker (1995) of experimental studies in favour of the S-shaped decision weight function is impressive. It is the class of DW preferences that is considered in this note, although we will not impose the S-shape. By doing so, we also allow for the disappointment averse (DA) utility functions axiomatized in Gul (1991).

The two aims of this note are modest. First, we want to demonstrate how easily the new definitions of the NBS and KSBS in terms of NEU preference relations can be applied to the subclass of DW utility functions. Second, we want to fully characterize these two bargaining solutions for this subclass and to provide simple formulas. For each bargaining solution an elegant formula is derived which extends the standard expected utility formula in a straightforward manner. Especially, the new formula of the NBS is strikingly similar to the well known asymmetric NBS for expected utility functions (e.g. Roth, 1979), provided the bargaining weights are properly chosen. These bargaining weights are uniquely determined by the derivative of the players' decision weight functions evaluated at the "no risk" point. The subclass of DA utility functions serves as a "numerical" example.

2 The Bargaining Problem

Consider the bargaining problem over one dollar with NEU preferences, i.e. the set X of alternatives is equal to $\{\underline{x} \in \mathbf{R}_+^2 \mid x_1 + x_2 = 1\}$, the NEU preference relation is \succeq_i , $i = 1, 2$, of player i over lotteries and the disagreement outcome is $\underline{0} = (0, 0)$. The NEU definition of the NBS and KSBS are stated in terms of lotteries with at most two prizes. Therefore, only these lotteries are considered. Denote

$p \cdot \underline{x}$ as the lottery which assigns probability p to the partition $\underline{x} \in X$, and the probability $1 - p$ to the disagreement outcome.

Each player's NEU preference relation \succeq_i , $i = 1, 2$, is represented by a DW utility function as analyzed in Tversky and Wakker (1995). This means that the NEU utility function for player i , $i = 1, 2$, over the lottery $p \cdot \underline{x}$ has the form: $w_i(p) u_i(x_i)$, where $u_i(0)$ is normalized to 0. The strictly increasing function $u_i(x)$, $i = 1, 2$, measures the utility over prizes and $w_i(p)$ is the decision weight function, where $w_i(0) = 0$, $w_i(1) = 1$ and $w_i(p)$ is strictly increasing in p . Therefore, the inverse function of $w_i(p)$, $i = 1, 2$, denoted as $w_i^{inv}(q)$, exists, is strictly increasing and twice differentiable. For simplicity, we assume that the functions $u_i(x_i)$, $i = 1, 2$, and $w_i(p)$ are twice continuously differentiable in x_i respectively p and $\ln(u_i(x_i))$ is concave in x_i . We denote the derivatives $\frac{\partial u_i(x_i)}{\partial x_i}$, $i = 1, 2$, and $\frac{\partial w_i(p)}{\partial p}$ as $u'_i(x_i)$ respectively $w'_i(p)$.

The class of DW preferences contains three interesting subclasses of NEU preferences. First of all the subclass of expected utility preferences which correspond to $w_i(p) = p$, $i = 1, 2$. Second, the subclass of DA utility functions as axiomatized in Gul (1991) for which $w_i(p) = \frac{p}{1+(1-p)\beta_i}$, $i = 1, 2$, where the parameter $\beta_i > -1$ measures the disappointment aversion of player i . If $\beta_i > 0$ ($\beta_i < 0$), then $w_i(p)$ is strictly convex (concave). Note that $w_i(p) = p$, $i = 1, 2$, is equivalent with $\beta_i = 0$. Third, the subclass of S-shaped functions $w_i(p)$, $i = 1, 2$, that are characterized in Tversky and Wakker (1995), i.e. $w'_i(0), w'_i(1) \geq 1$, $w''_i(0) \leq 0$, $w''_i(1) \geq 0$ and there exists an $\delta \geq 0$ such that $w'_i(p) \leq \min\{w'_i(0), w'_i(1)\}$ for $p \in (\delta, 1 - \delta)$. These S-shaped functions are consistent with experimental data.

3 The Nash Bargaining Solution

In this section the NBS is fully characterized in theorem 1 as the maximizer of the function $[u_1(x_1)]^{w_2'(1)} [u_2(x_2)]^{w_1'(1)}$. Furthermore, this formula is briefly discussed, which includes a comparison to the asymmetric NBS for expected utility functions and the relation to the interpretation of the symmetry axiom as given in Nash (1953). We also argue that the alternating offer model with DW utilities does not satisfy the stationary preference property. But first we give the new definition of the NBS for NEU bargaining problems in terms of preference relations, as proposed in Rubinstein, Safra and Thomson (1992).

Definition 1 *The partition $\underline{x}^* \in X$ is a NBS if for every $p \in [0, 1]$ and $\underline{x} \in X$:*

$$[p \cdot \underline{x} \succ_1 1 \cdot \underline{x}^* \Rightarrow p \cdot \underline{x}^* \succeq_2 1 \cdot \underline{x}] \text{ and } [p \cdot \underline{x} \succ_2 1 \cdot \underline{x}^* \Rightarrow p \cdot \underline{x}^* \succeq_1 1 \cdot \underline{x}].$$

In Rubinstein, Safra and Thomson (1992) it is shown that the NBS exists and is unique.

The following theorem states our main result for the NBS.

Theorem 1 *If \underline{x}^* is the NBS, then \underline{x}^* maximizes $[u_1(x_1)]^{w_2'(1)} [u_2(x_2)]^{w_1'(1)}$.*

Proof

Consider $\underline{x} = (x, 1 - x)$, $x \in (x^*, 1]$. Then for all $p \in [0, 1]$ it holds that $p \cdot \underline{x} \prec_2 1 \cdot \underline{x}^*$ and, therefore, for all $p \in [0, 1]$ the condition $[p \cdot \underline{x} \succ_2 1 \cdot \underline{x}^* \Rightarrow p \cdot \underline{x}^* \succeq_1 1 \cdot \underline{x}]$ holds trivially. Next, it holds that

$$p \cdot \underline{x} \succ_1 1 \cdot \underline{x}^* \Leftrightarrow w_1(p) u_1(x) > u_1(x^*) \Leftrightarrow p > w_1^{inv} \left(\frac{u_1(x^*)}{u_1(x)} \right) \quad (1)$$

and

$$p \cdot \underline{x}^* \succeq_2 1 \cdot \underline{x} \Leftrightarrow w_2(p) u_2(1 - x^*) \geq u_2(1 - x) \Leftrightarrow p \geq w_2^{inv} \left(\frac{u_2(1 - x)}{u_2(1 - x^*)} \right). \quad (2)$$

The condition $[p \cdot \underline{x} \succ_1 1 \cdot \underline{x}^* \Rightarrow p \cdot \underline{x}^* \succeq_2 1 \cdot \underline{x}]$ requires that the lower bound on p in (2) should not exceed the lower bound on p in (1). Thus,

$$w_1^{inv} \left(\frac{u_1(x^*)}{u_1(x)} \right) \geq w_2^{inv} \left(\frac{u_2(1-x)}{u_2(1-x^*)} \right).$$

Rewriting yields

$$\frac{u_1(x^*)}{u_1(x)} \geq w_1 \left(w_2^{inv} \left(\frac{u_2(1-x)}{u_2(1-x^*)} \right) \right).$$

This weak inequality holds with an equal sign at $x = x^*$ and both the LHS and the RHS are decreasing in x . In order for this inequality to hold for $x > x^*$ it should hold that the LHS decreases less rapidly than the RHS in x^* , i.e.

$$-\frac{u_1'(x^*)}{u_1(x^*)} \geq -\frac{w_1'(1)}{w_2'(1)} \cdot \frac{u_2'(1-x^*)}{u_2(1-x^*)}. \quad (3)$$

Next, consider $x \in [0, x^*)$. Then similar arguments yield

$$w_1 \left(w_2^{inv} \left(\frac{u_2(1-x^*)}{u_2(1-x)} \right) \right) \geq \frac{u_1(x)}{u_1(x^*)}.$$

This weak inequality holds with an equal sign at $x = x^*$ and both the LHS and the RHS are increasing in x . In order for this inequality to hold for $x < x^*$ it should hold that the LHS increases less rapidly than the RHS in x^* , i.e.

$$\frac{w_1'(1)}{w_2'(1)} \cdot \frac{u_2'(1-x^*)}{u_2(1-x^*)} \leq \frac{u_1'(x^*)}{u_1(x^*)} \quad (4)$$

Combining equations (3) and (4) and rewriting yields

$$w_2'(1) \frac{u_1'(x^*)}{u_1(x^*)} - w_1'(1) \frac{u_2'(1-x^*)}{u_2(1-x^*)} = 0,$$

which is the FOC for maximizing the concave function $w_2'(1) \ln(u_1(x)) + w_1'(1) \ln(u_2(1-x))$.

Finally, the stated result follows after taking the monotonic transformation $f(y) =$

e^y .

The derived formula is strikingly similar to the standard asymmetric NBS for expected utility preferences (e.g. Roth, 1979). Recall that for some given $\alpha \in (0, 1)$ this asymmetric NBS maximizes the function $[v_1(x)]^\alpha [v_2(1-x)]^{1-\alpha}$, where the function v_i , $i = 1, 2$, corresponds to expected utility preferences. For $\alpha = \frac{w_2'(1)}{w_1'(1)+w_2'(1)}$ and $v_i(x_i) = u_i(x_i)$, $i = 1, 2$, it follows that after a monotonic transformation the new formula for the NBS with DW utilities coincides with the formula for the asymmetric NBS. Moreover, the decision weight functions w_1 and w_2 that are part of the DW utility functions uniquely determine the weight α . For $w_1'(1) = w_2'(1)$ we obtain the “symmetric” NBS. Furthermore, an increase in $w_i'(1)$, $i = 1, 2$, weakens player i 's bargaining position. For S-shaped functions $w_i(p)$, $i = 1, 2$, a sufficient condition for this to happen is an increase in player i 's “subadditivity” (e.g. Tversky and Wakker, 1995).

The decision weight functions $w_1(p)$ and $w_2(p)$ uniquely determine the players' weights and only the marginal attitudes around $p = 1$ matter, i.e. only $w_1'(1)$ and $w_2'(1)$ matter. Note that even if $w_1(p) \neq w_2(p)$ for all $p \in (0, 1)$ it is still possible that $w_1'(1) = w_2'(1)$, i.e. $\alpha = \frac{1}{2}$. The role of marginal attitudes against risk can be explained as follows. In Rubinstein, Safra and Thomson (1992) it is argued that the NBS should be robust in the sense that every objection made by one of the players should be credibly counter objected by the other player. In particular this should hold for an objection in which one player wants an incremental higher share. For such an incremental extra share this player can only afford lotteries with a low risk of breakdown, i.e. $p \approx 1$. Otherwise, the costs of losing exceed the incremental gain. Similarly, for the other player, who can only afford credible

counter objections with a low risk of breakdown. Thus for both players there risk attitudes around $p = 1$ matter and this is reflected in the formula.

In Nash (1953) it is argued that the symmetry axiom means that the only differences between the players can be differences in preferences. The formula of theorem 1 nicely reflects this point of view. If both players have equal attitudes towards risk around $p = 1$, i.e. $w'_1(1) = w'_2(1)$, then only the asymmetries in the functions u_i , $i = 1, 2$, can explain asymmetries in the players' shares. This case includes the standard expected utilities case, i.e. $w'_1(1) = w'_2(1) = 1$. Similarly, if the functions u_i , $i = 1, 2$, are identical, then only the differences in risk attitudes $w'_1(1)$ and $w'_2(1)$ matter.

A final remark concerns the alternating offer model with DW preferences. This model does not have the property of stationary preferences. Let (\underline{z}, τ) denote the outcome in which the players agree upon $\underline{z} \in X$ at time τ and let $\delta \in (0, 1)$ be the probability of a next bargaining round, i.e. $1 - \delta$ is the probability of breakdown. Then (\underline{z}, τ) induces the lottery $\delta^\tau \cdot \underline{z}$ and stationary preferences require

$$w_i(\delta^t) u_i(x_i) > w_i(\delta^{t+s}) u_i(y_i) \Leftrightarrow u_i(x_i) > w_i(\delta^s) u_i(y_i).$$

Hence, stationary DW preferences require $w_i(\delta^{t+s}) = w_i(\delta^t) w_i(\delta^s)$ for all t and s . Clearly, this latter relation does not hold in general and DW preferences fail the property of stationarity. The consequences are that the alternating offer model with DW preferences is a non-stationary bargaining game and for general functions $w_i(p)$ and $u_i(x_i)$ one can only state recursive relations between proposals of subsequent bargaining rounds that cannot be solved. This simple observation casts serious doubts whether it would be possible to extend the relation between the unique SPE in the alternating offer model and the NBS known for the standard expected utility case.

The following example discusses the NBS for the special case of DA utility functions.

Example 1

For DA utility functions we have $w'_i(1) = 1 + \beta_i > 0$, $i = 1, 2$. Thus, the NBS maximizes $u_1(x_1)^{1+\beta_2}u_2(x_2)^{1+\beta_1}$ and equal weights is equivalent to $\beta_1 = \beta_2$. In Gul (1991) it is shown that *risk aversion* is equivalent to $\beta_1, \beta_2 > 0$ and u_i , $i = 1, 2$, concave. Risk aversion weakens a player's bargaining position, a result that is already shown in Rubinstein, Safra and Thomson (1992). For DA utility functions an alternative derivation of this result can be given by replicating the arguments in Osborne and Rubinstein (1990). There are two sources that can lead to an increased risk aversion for player i , $i = 1, 2$, namely either an increase in β_i or a concave transformation of the utility function u_i (or both). The negative effect for player i , $i = 1, 2$, of an increase in β_i is nicely illustrated in the new formula where the weight given to player j , $j = 1, 2$ and $j \neq i$, will increase.

4 The Kalai-Smorodinsky Bargaining Solution

In this section an elegant formula is derived that fully characterizes the KSBS. But first the definition of the KSBS for NEU bargaining problems, as proposed in Rubinstein, Safra and Thomson (1992), is given.

Definition 2 *The partition $\underline{y}^* \in X$ is the KSBS if there exists a $p \in [0, 1]$ such that*

$$p \cdot \underline{e}_1 \sim_1 1 \cdot \underline{y}^* \text{ and } p \cdot \underline{e}_2 \sim_2 1 \cdot \underline{y}^*,$$

where \underline{e}_i , $i = 1, 2$, denotes the i -th unit vector in \mathbf{R}^2 .

The following theorem gives the full characterization of the KSBS.

Theorem 2 *If $\underline{y}^* = (y^*, 1 - y^*)$ is the KSBS, then y^* solves*

$$\frac{u_1(y^*)}{u_1(1)} = w_1 \left(w_2^{inv} \left(\frac{u_2(1 - y^*)}{u_2(1)} \right) \right).$$

Proof

Making use of DW utility functions yields the two-equation system

$$w_1(p^*) u_1(1) = u_1(y^*) \text{ and } w_2(p^*) u_2(1) = u_2(1 - y^*),$$

where p and y^* are the two unknowns. Rewriting yields

$$p^* = w_1^{inv} \left(\frac{u_1(y^*)}{u_1(1)} \right) \text{ and } p^* = w_2^{inv} \left(\frac{u_2(1 - y^*)}{u_2(1)} \right).$$

Equating both expressions and rewriting yields the stated expression for the KSBS.

In Rubinstein, Safra and Thomson (1992) existence and uniqueness of the KSBS has been proved. The derived expression in the latter theorem can be used to construct a simple alternative proof of this result based upon the mean value theorem and the increasing utility in a player's own share.

For $w_1(p) = w_2(p)$ it simply follows that $\frac{u_1(1)}{u_1(y^*)} = \frac{u_2(1)}{u_2(1 - y^*)}$, which includes the special case of expected utility, i.e. $w_1(p) = w_2(p) = p$. This once more illustrates the interpretation of the symmetry axiom as given in Nash (1953) already discussed extensively for the NBS. For $w_1(p) \neq w_2(p)$ the new formula is a straightforward and simple extension of the well known standard formula for expected utility. The concluding example states the KSBS for the special case of DA utility functions.

Example 2

The KSBS $(y^*, 1 - y^*)$ for bargaining problems in which the players have DA utility functions solves

$$(1 + \beta_2) \frac{u_1(1)}{u_1(y^*)} = (1 + \beta_1) \frac{u_2(1)}{u_2(1 - y^*)} + (\beta_2 - \beta_1).$$

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