VALUE ORIENTED EQUILIBRIA IN REPEATED GAMES OF COMPLETE INFORMATION

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John Payne Bigelow
Department of Economics; 2107 CEBA
Louisiana State University
Baton Rouge, LA 70803-6306
(504) 388 3793
JBigelo@Lsuvm.Sncc.Lsu.Edu

Abstract

Two refinements, called *value oriented equilibria*, of the Nash equilibrium concept are proposed for repeated games of complete information. *Value sufficient equilibria* make each player's strategic response to the another player's previous actions depend only on the value of those actions to the responding player. In *value monotonic equilibria* no player punishes another for taking actions which increase the first player's payoff. The use of value oriented equilibria enables the set of outcomes consistent with equilibrium to be reduced. Outcomes which are *unilaterally inefficient* (i.e., can be Pareto dominated by the unilateral action of one player) are never part of the equilibrium path.

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I. INTRODUCTION

In his 1981 survey paper, Robert Aumann wrote that the aim of the theory of repeated games of complete information is:

> ⋯ to account for phenomena such as cooperation, altruism, revenge, threats (self-destructive or otherwise), etc. - phenomena which may at first seem irrational - in terms of the usual "selfish" utility-maximizing paradigm of game theory and neoclassical economics.[1]

Because of the Folk-Theorem, which provides that every feasible individually rational payoff vector of a stage game can be achieved as the Nash equilibrium of repeated play if the discount factor is close enough to one, the theory has succeeded in meeting this goal. The theory can "account" for a very wide variety of phenomena, because virtually anything can be an equilibrium. Thus, the drawback to the theory's success is that its predictions have no precision. As Aumann put it,

> In a sense, much of the theory of repeated games is an attempt to cut down, in one way or another, the bewildering wealth of equilibrium payoffs provided by [the folk theorem].[2]

With respect to this goal the theory has had little success in the years since Aumann wrote his survey. Rather than "cut down ⋯ the bewildering wealth of equilibrium payoffs" the literature[3] has extended the Folk Theorem to sub-game perfect Nash equilibria in repeated games[4], to finitely repeated games[5], and to overlapping generations games.[6] For purposes of positive economics, this is unfortunate. There are a variety of problems involving long-term relationships between individuals to which it would be natural to apply the theory of repeated games.[7] However, the value of the theory will be limited in these contexts if it makes no precise prediction of the outcome.

The purpose of this paper is to propose some refinements to the Nash equilibrium concept for use in repeated games of complete information. These refinements will eliminate some individually rational outcomes from the set of equilibria, thereby increasing the precision of the theory's predictions. This is accomplished by assuming that in a repeated game, a player's strategic response to the past actions of other players depends only on the value of those actions to the responding player. In other words, if two histories of past action by another player yield (in the context of a particular equilibrium) the same value (i.e. payoffs) to a player, then that player's response to both histories should be

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[1] Aumann (1981), p.9

[2] ibid, p. 16

[3] For a recent survey see Sorin (1992) and for a synthesis see Benoît and Krishna (1996).

[4] See, for example, Fudenberg and Maskin (1986), Abreu (1988), Abreu, Dutta and Smith (1994), and Wen (1994).

[5] See, for example, Benoît and Krishna (1985) and Smith (1995).

[6] See, for example, Kandori (1992) and Smith (1992).

[7] For an application of value oriented equilibria to the Law and Economics of long term relationships see Bowers and Bigelow (1996).

the same. Equilibria in which the players behave this way will be called *value oriented*. The use of value oriented equilibria eliminates, *unilaterally inefficient* outcomes (i.e., those that could be Pareto dominated by the unilateral action of one player) from the set of equilibria.

The paper will emphasize the particular outcomes to be achieved in equilibrium as opposed to the payoffs associated with those outcomes. While it is customary to express the folk theorem in terms of equilibrium payoff vectors, most versions of it could be expressed in terms of outcomes, because most proofs of the folk theorem are constructive. Starting with a feasible individually rational payoff vector, a stage-game outcome which realizes the payoff vector is selected. One then uses appropriate threats to construct repeated-game equilibrium strategies which realize the desired stage-game outcomes in each repetition. Thus, not only does the folk theorem teach that virtually any payoff vector may be achieved in equilibrium, it teaches that virtually any action is consistent with equilibrium.

In some games, eliminating unilaterally inefficient outcomes from the set of equilibria may not reduce the set of equilibrium payoffs at all. If a particular payoff vector can be achieved by both a unilaterally inefficient outcome and by another outcome which is not unilaterally inefficient, then that payoff vector will remain in the set of equilibrium payoffs. Even in that case, however, the results provided here will have served their purpose by enabling the positive economist to better predict outcomes, since one of the outcomes that produces those payoffs is eliminated from the equilibrium set.

The remainder of the paper is organized as follows. Section II contains a formal description of the model and notation to be used. Section III discusses two examples which illustrate unilaterally inefficient outcomes in Nash equilibria of repeated games. The examples also illustrate how a value oriented approach to strategic reaction in a repeated game eliminates these outcomes as possible equilibria. The section also includes formal definitions of two unilateral inefficiency concepts, *innocuous improvement* and *unilateral dominance*, and definitions of two value oriented refinements of the Nash equilibrium, the *value sufficient equilibrium* and the *value monotonic equilibrium*. Section IV relates the unilateral inefficiency concepts to the value oriented equilibrium concepts. Section V contains some concluding remarks.

II. The Model

A. The Stage Game

Consider a two person (i = 1, 2) non-cooperative game in normal form. Player i chooses an action x^i from a non-empty compact set X^i and receives a payoff according to the continuous function $u^i{:}X^1 \times X^2 \to \mathbb{R}$. An outcome is a pair of actions $(x^1, x^2) \in X^1 \times X^2$. In the stage game each player's strategy set is the action set X^i. A payoff pair (u^1, u^2) is feasible if there is an outcome $(x^1, x^2) \in X^1 \times X^2$ such that

$$u^1(x^1, x^2) = u^1 \quad \& \quad u^2(x^1, x^2) = u^2.$$

The mini-max payoffs for each player, which are the payoffs that player would achieve if (s)he reacts optimally to the worst possible choice of strategy by the other player, are

$$\underline{u}^1 \equiv \underset{x^2 \in X^2}{\text{Min}} \; \underset{x^1 \in X^1}{\text{Max}} \; u^1(x^1, x^2) \quad \& \quad \underline{u}^2 \equiv \underset{x^1 \in X^1}{\text{Min}} \; \underset{x^2 \in X^2}{\text{Max}} \; u^2(x^1, x^2).$$

The mini-max strategies, which are the strategies each player could play to ensure that the other player does no better than his/her mini-max payoff, are the maximally effective threat against a rational opponent. They are denoted by

$$\underline{x}^2 \equiv \underset{x^2 \in X^2}{\text{ArgMin}} \; \underset{x^1 \in X^1}{\text{Max}} \; u^1(x^1, x^2) \quad \& \quad \underline{x}^1 \equiv \underset{x^1 \in X^1}{\text{ArgMin}} \; \underset{x^2 \in X^2}{\text{Max}} \; u^2(x^1, x^2).$$

Because a player can guarantee him(her)self at least the mini-max payoff, payoff pairs, (u^1, u^2) for which $u^1 > \underline{u}^1$ and $u^2 > \underline{u}^2$ are called individually rational. Likewise an outcome $(x^1, x^2) \in X^1 \times X^2$ is individually rational if

$$u^1(x^1, x^2) > \underline{u}^1 \quad \& \quad u^2(x^1, x^2) > \underline{u}^2.$$

B. THE REPEATED GAME

In a repeated game the interaction described by the stage game takes place once for each $t = 0, 1, 2, \cdots$. At any repetition, $t > 0$, the history of the game before t is[8]

$$h_t = \left((x_0^1, x_0^2), (x_1^1, x_1^2), (x_2^2, x_2^2), \cdots (x_{t-1}^1, x_{t-1}^2) \right) \in H_t \equiv \prod_{s=0}^{t-1} \left(X^1 \times X^2\right)$$

where x_t^i is the action that was taken by player i at repetition t. At repetition t the players choose response functions, $r_t^i{:}H_t \to X^i$ $(i = 1,2)$, which make their actions at t functions of the history. A strategy for i is a sequence,

$$\sigma^i \equiv \left(r_0^i{:}H_0 \to X^i,\ r_1^i{:}H_1 \to X^i,\ r_2^i{:}H_2 \to X^i,\ \cdots,\ r_t^i{:}H_t \to X^i,\ \cdots \right)$$

of response functions. Let S^i denote the set of all such strategies.

Once both parties have chosen strategies $(\sigma^1, \sigma^2) \in S^1 \times S^2$, the outcomes and history of the game are determined by the outcome functions $\xi_t{:}S^1 \times S^2 \to X^1 \times X^2$ and history functions $\theta_t{:}S^1 \times S^2 \to H_t$ which are defined inductively by

$$\theta_0(\sigma^1, \sigma^2) \equiv h_0 \qquad \xi_0(\sigma^1, \sigma^2) \equiv \left(r_0^1(h_0), r_0^2(h_0)\right)$$

$$\theta_t(\sigma^1, \sigma^2) \equiv \left(\theta_{t-1}(\sigma^1, \sigma^2), \xi_{t-1}(\sigma^1, \sigma^2)\right) \qquad \xi_t(\sigma^1, \sigma^2) \equiv \left(r_t^1\left(\theta_t(\sigma^1, \sigma^2)\right), r_t^2\left(\theta_t(\sigma^1, \sigma^2)\right)\right)$$

The action of one player will be denoted by

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[8] In order to have a consistent notation, let h_0 denote the null history at $t = 0$ and let $H_0 \equiv \{ h_0 \}$.

$$\xi_t^1(\sigma^1,\ \sigma^2) \;=\; r_t^1\left(\theta_t(\sigma^1,\ \sigma^2)\right) \quad \text{or} \quad \xi_t^2(\sigma^1,\ \sigma^2) \;=\; r_t^2\left(\theta_t(\sigma^1,\ \sigma^2)\right).$$

Letting $\delta \in (0, 1)$ denote the common discount factor, the payoff to player i from (σ^1, σ^2) is

$$W^i(\sigma^1,\ \sigma^2) \;=\; \sum_{t=0}^{\infty} (1-\delta)\delta^t u^i\left(\xi_t(\sigma^1,\ \sigma^2)\right).$$

A Nash equilibrium is a pair (σ^1*, σ^2*) such that

$$W^1(\sigma^1*,\ \sigma^2*) \;\geq\; W^1(\sigma^{1'},\ \sigma^2*) \quad \forall\ \sigma^{1'} \in S^1 \qquad \&$$
$$W^2(\sigma^1*,\ \sigma^2*) \;\geq\; W^1(\sigma^1*,\ \sigma^{2'}) \quad \forall\ \sigma^{2'} \in S^2.$$

III. Unilateral Inefficiency and Value Orientation

Consider the two person game shown in Example 1. This game is a variation on the Prisoner's Dilemma game. Player 1 has a cooperative strategy (x_1^1) and a non-cooperative strategy (x_2^1), which dominates the cooperative. Player 2 has a dominant non-cooperative strategy (x_3^2) and two cooperative strategies (x_1^2 and x_2^2). Player 1 is indifferent between 2's two cooperative strategies, but 2 prefers x_1^2 to x_2^2. One could think of x_1^2 as "cooperate at low cost" and x_2^2 as "cooperate at high cost." In this game, the individually rational pairs are (x_1^1, x_1^2) and (x_1^1, x_2^2). Thus, according to the folk theorem, if the discount factor is close enough to one, repeated play of each of these pairs is the outcome of an equilibrium of the repeated game.[9]

Example 1

		Player 2		
		x_1^2	x_2^2	x_3^2
Player 1	x_1^1	$u^2 = 5$ $u^1 = 5$	$u^2 = 4$ $u^1 = 5$	$u^2 = 6$ $u^1 = 1$
	x_2^1	$u^2 = 1$ $u^1 = 6$	$u^2 = 0$ $u^1 = 6$	$u^2 = 2$ $u^1 = 2$

In the equilibrium that sustains repeated play of the second pair, (x_1^1, x_2^2), player 2 would like to deviate because x_2^2 is dominated by both of 2's other actions. Player 2 is dissuaded from doing so by player 1's strategy which calls for player 1 to punish any deviation from x_2^2 by switching to x_2^1 for at least long enough to deny 2 any net gains from switching away from x_2^2. Where a deviation from x_2^2 to x_3^2 is concerned, it is easy to understand why player 1 would be motivated to punish 2's deviation. Switching from x_2^2 to x_3^2 is costly to 1. However, the deviation from x_2^2 to x_1^2 is completely innocuous as far as player 1 is concerned. Yet, such is the logic of the repeated game equilibrium that player 1 punishes player 2 for choosing x_1^2 instead of x_2^2.

The innocuous improvement, of which the relationship between (x_1^1, x_1^2) and (x_1^1, x_2^2) is an example, may be defined formally.

[9] Repeated play of (x_2^1, x_3^2) is also an equilibrium, although not on the basis of the folk theorem.

Definition: *Player 1 [or 2] can make an* **innocuous improvement** *to* $(\tilde{x}^1, x^2) \in X^1 \times X^2$ *[or* $(x^1, \tilde{x}^2) \in X^1 \times X^2$*] if there is some* $\hat{x}^1 \in X^1$ *[or* $\hat{x}^2 \in X^2$ *] such that*

$$u^1(\hat{x}^1, x^2) > u^1(\tilde{x}^1, x^2) \quad \& \quad u^2(\hat{x}^1, x^2) = u^2(\tilde{x}^1, x^2).$$

$$\left[\text{ or } \quad u^2(x^1, \hat{x}^2) > u^2(x^1, \tilde{x}^2) \quad \& \quad u^1(x^1, \hat{x}^2) = u^1(x^1, \tilde{x}^2). \right]$$

A pair like (x^1_1, x^2_2) in Example 1 which can be innocuously improved upon is an unpersuasive candidate for part of an equilibrium path since one player will have an incentive to deviate from it, and the other player should have no reason for trying to alter that incentive.

Repeated play of the innocuously improvable outcome would not be an equilibrium if 1's strategy depended only on the payoffs 1 had received in the past. In that case when 2 deviated from x^2_2 to x^2_1, player 1 would continue to receive the same payoff, and so would not punish player 2's deviation. Player 2 would receive a higher payoff, so repeated play of (x^1_1, x^2_2) would not be an equilibrium. Indeed, (x^1_1, x^2_2) could not be part of any equilibrium path.

To aid in formalizing this reasoning, define the value to one player of another player's actions as follows:

Definition: *The* **value** *to player 1 [2] of player 2's [1's] choice of* x^2_t $[x^1_t]$ *when player 1 [2] expects both parties to play* (σ^1, σ^2) *is*

$$\begin{aligned} v^1_t\left(x^2_t \mid \sigma^1, \sigma^2\right) &\equiv u^1\left(\xi^1_t(\sigma^1, \sigma^2), x^2_t\right) \\ \left[\quad v^2_t\left(x^1_t \mid \sigma^1, \sigma^2\right) \right. &\equiv \left. u^2\left(x^1_t, \xi^2_t(\sigma^1, \sigma^2)\right). \quad \right] \end{aligned} \tag{1}$$

Note that the value to one player of the other's choice depends on the strategies (σ^1, σ^2) which the first player expects both players to be using.

To say that each player's strategic reactions depend only on the value of the other player's actions is, in effect, to say that a player will react the same way to two different histories if they have the same value. In other words, for strategic purposes the values of a player's past actions are a sufficient statistic for the actions themselves. This is formalized in the following definition:

Definition: *A pair of strategies,* (σ^1, σ^2) *is* **value sufficient**[1] *if* $r^1_t(\hat{h}_t) = r^1_t(\tilde{h}_t)$ *for every* $\hat{h}_t$ *and* $\tilde{h}_t$ *with* $v^1_\tau\left(\hat{x}^2_\tau \mid \sigma^1, \sigma^2\right) = v^1_\tau\left(\tilde{x}^2_\tau \mid \sigma^1, \sigma^2\right) \ \forall\, \tau < t,$ *and* $r^2_t(\hat{h}_t) = r^2_t(\tilde{h}_t)$ *for every* $\hat{h}_t$ *and* $\tilde{h}_t$ *with* $v^2_\tau\left(\hat{x}^1_\tau \mid \sigma^1, \sigma^2\right) = v^2_\tau\left(\tilde{x}^1_\tau \mid \sigma^1, \sigma^2\right) \ \forall\, \tau < t.$ *A pair of strategies,* $(\sigma^{1*}, \sigma^{2*})$*, is a* **value sufficient equilibrium** *if it is a Nash equilibrium and is value sufficient.*

As we shall see in section IV, a pair like (x_1^1, x_2^2) which can be innocuously improved upon is never part of the equilibrium path for a value sufficient equilibrium, and repeated play of any pair like (x_1^1, x_1^2), which cannot be innocuously improved upon, is a value sufficient equilibrium for discount factors close enough to 1.

Example 2

		Player 2		
		x_1^2	x_2^2	x_3^2
Player 1	x_1^1	$u^2 = 5$, $u^1 = 5$	$u^2 = 4$, $u^1 = 4$	$u^2 = 6$, $u^1 = 1$
	x_2^1	$u^2 = 1$, $u^1 = 6$	$u^2 = 0$, $u^1 = 5$	$u^2 = 2$, $u^1 = 2$

Example 2 is similar to Example 1 except that both players agree in preferring x_1^2 to x_2^2. One could think of x_2^2 as "cooperate inefficiently" and x_1^2 as "cooperate efficiently," where inefficient cooperation both increases costs to player 2 and reduces benefits to player 1. The same two outcomes, (x_1^1, x_1^2) and (x_1^1, x_2^2), are individually rational and so repeated play of each of them is the outcome of an equilibrium of the repeated game.[10]

The two pairs, (x_1^1, x_1^2), (x_1^1, x_2^2) provide an example of unilateral dominance. The first Pareto dominates the second and the Pareto improvement can be achieved by the action of only one player. A general definition of unilateral dominance is:

Definition: *Player 1 [or 2] can* **unilaterally dominate** $(\tilde{x}^1, x^2) \in X^1 \times X^2$ *[or* $(x^1, \tilde{x}^2) \in X^1 \times X^2$*] if there is some* $\hat{x}^1 \in X^1$ *[or* $\hat{x}^2 \in X^2$ *] such that*

$$u^1(\hat{x}^1, x^2) > u^1(\tilde{x}^1, x^2) \quad \& \quad u^2(\hat{x}^1, x^2) \geq u^2(\tilde{x}^1, x^2).$$

$$\left[\text{ or } \quad u^2(x^1, \hat{x}^2) > u^2(x^1, \tilde{x}^2) \quad \& \quad u^1(x^1, \hat{x}^2) \geq u^1(x^1, \tilde{x}^2). \right]$$

The second equilibrium in Example 2 is sustained by player 1's threat to punish player 2 for any deviation from x_2^2, including a deviation to x_1^2, which would make both player 1 and player 2 better off. This anomalous equilibrium will not be eliminated by resort to a value sufficient equilibrium because switching from x_2^2 to x_1^2 changes u^1. Even if 1's response depends only on his payoffs, he could play a strategy which punishes such a deviation, although it is very hard to understand why he would want to.[11] Player 2's deviation has increased his payoff. He ought to want to encourage such

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[10] As before, repeated play of (x_2^1, x_3^2) is also the outcome of an equilibrium of the repeated game.

[11] The issue here is not one of credibility. By theorem 1 in Fudenberg and Maskin (1986), repeated play of the outcome (x_1^1, x_2^2) is the outcome of a *sub-game perfect* Nash equilibrium of the repeated game if the players' discount factor is sufficiently close to 1. That is, player 1's threat to punish (continued...)

action. If player 1 followed a strategy in which any deviation that increases his payoff will be met by a response that does not diminish the payoff of player 2, then player 2 would have an incentive to switch from x_2^2 to x_1^2, so (x_1^1, x_2^2) could not be part of an equilibrium path. Such a strategy is defined below.

Definition: *A pair of strategies,* $(\sigma^1, {}^2\sigma)$ *is* **value monotonic** *if it is value sufficient, if* $v_\tau^1\left(x_\tau^2 \mid \sigma^1, \sigma^2\right) \geq u^1\left(\xi_\tau(\sigma^1, \sigma^2)\right) \ \forall\ \tau < t$ *implies* $v_t^2\left(r_t^1(h_t) \mid \sigma^1, \sigma^2\right) \geq u^2\left(\xi_t(\sigma^1, \sigma^2)\right)$, *and if* $v_\tau^2\left(x_\tau^1 \mid \sigma^1, \sigma^2\right) \geq u^2\left(\xi_\tau(\sigma^1, \sigma^2)\right) \ \forall\ \tau < t$ *implies* $v_t^1\left(r_t^2(h_t) \mid \sigma^1, \sigma^2\right) \geq u^1\left(\xi_t(\sigma^1, \sigma^2)\right)$.

As we shall see in section IV, a pair like (x_1^1, x_2^2) which can be unilaterally dominated is never part of the equilibrium path for a value monotonic equilibrium, and repeated play of any pair like (x_1^1, x_1^2) which cannot be unilaterally dominated is a value monotonic equilibrium for discount factors close enough to 1.

IV. RESULTS

This first part of this section contains results relating value sufficient equilibria to innocuous improvements. A version of the folk theorem is obtained that applies *only* to outcomes which cannot be innocuously improved. Any such pair is, for a discount factor close enough to 1, a value sufficient equilibrium. Secondly, the equilibrium path of a value-sufficient equilibrium will never include an outcome which can be innocuously improved. The second part of this section contains parallel results for value monotonic equilibria and outcomes which can be unilaterally dominated.

Theorem 1: *If* $(x_1^*, x_2^*) \in X^1 \times X^2$ *is individually rational and cannot be innocuously improved upon, then for* δ *close enough to 1 there is a value-sufficient equilibrium* $(\sigma^{1*}, \sigma^{2*})$ *for which* $\xi_t(\sigma^{1*}, \sigma^{2*}) = (x^{1*}, x^{2*})$ *for all* $t = 0, 1, 2, \cdots$

Proof: The proof is essentially the same as proofs of the folk theorem[12], except that care must be taken to construct strategies that are value-sufficient.

1) Construct $(\sigma^{1*}, \sigma^{2*})$. Let $r_0^1{}^*(h_0) \equiv x^{1*}$, and let

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[11](...continued)
player 2 for the mutually beneficial switch to x_1^2 is credible if, as in Fudenberg and Maskin's equilibrium, player 1 expects to be punished by player 2 for not punishing player 2. Player 2's threat to punish player 1 for not punishing player 2 is credible if player 2 expects to be punished for not punishing player 1 for not punishing player 2. And so on.

[12] See, for example, Aumann (1981).

$$r_t^1*(h_t) \equiv \begin{cases} x^1* & | \quad u^1(x^1*,\ x_0^2) = u^1(x^1*,\ x_1^2) = \cdots = u^1(x^1*,\ x_{t-1}^2) = u^1* \\ \\ \underline{x}^1 & | \quad \text{otherwise} \end{cases}$$

Also let $r_0^2*(h_0) \equiv x^2*$, and let

$$*(h_t) \equiv \begin{cases} x^2* & | \quad u^1(x_0^1,\ x^2*) = u^1(x_1^1,\ x^2*) = \cdots = u^1(x_{t-1}^1,\ x^2*) = u \\ \\ \underline{x}^2 & | \quad \text{otherwise} \end{cases} \tag{2}$$

where $u^1* = u^1(x^1*, x^2*)$ and $u^2* = u^2(x^1*, x^2*)$.

2) By construction $\xi_t(\sigma^1*, \sigma^2*) = (x^1*, x^2*)$ for all t.

3) Because of 2), $v_t^1(x^2 \mid \sigma^1*, \sigma^2*) = u^1(x^1*, x^2)$ and $v_t^2(x^1 \mid \sigma^1*, \sigma^2*) = u^2(x^1, x^2*)$ for all t, so (σ^1*, σ^2*) is value sufficient.

4) Suppose player 1 considers some deviation $\hat{\sigma}^1 \in S^1$.

4a) If

$$u^2\Big(\xi_t^1\big(\hat{\sigma}^1,\ \sigma^2*\big),\ x^2*\Big) = u^2* \tag{3}$$

for all t, then $\xi_t^2(\hat{\sigma}^1, \sigma^2*) = x^2*$ for all t and because (x^1*, x^2*) cannot be innocuously improved upon

$$u^1\Big(\xi_t(\hat{\sigma}^1,\ \sigma^2*)\Big) \leq u^1(x^1*,\ x^2*) \leq u^1\Big(\xi_t(\sigma^1*,\ \sigma^2*)\Big). \tag{4}$$

so $W^1(\hat{\sigma}^1, \sigma^2*) \leq W^1(\sigma^1*, \sigma^2*)$ and there is no gain to player 1 from deviating.

4b) Therefore, suppose T (finite) is the first t for which (3) does not hold. By (2), $\xi_t^2(\hat{\sigma}^1, \sigma^2*) = x^2*$ for all $t \leq T$. Since (x^1*, x^2*) cannot be innocuously improved upon, (4) holds for all $t < T$.

4c) Since X^1 is compact and u^1 is continuous

$$u^1\Big(\xi_T(\hat{\sigma}^1,\ \sigma^2*)\Big) \leq M^1 \equiv \operatorname*{Max}_{x^1 \in X^1} u^1(x^1,\ x^2*) \tag{5}$$

4d) (2) implies $\xi_t(\hat{\sigma}^1, \sigma^2*) = \underline{x}^2$ for all $t > T$. Therefore,

$$u^1\Big(\xi_t(\hat{\sigma}^1,\ \sigma^2*)\Big) = u^1\Big(\xi_t^1(\hat{\sigma}^1,\ \sigma^2*),\ \underline{x}^2\Big) \leq \underline{u}^1. \tag{6}$$

5) Adding together (4), (5), and (6),

$$W^1(\hat{\sigma}^1,\ \sigma^2*) \leq (1 - \delta^T)u^1* + (1 - \delta)\delta^T M^1 + \delta^{T+1}\underline{u}^1.$$

Since $W^1(\sigma^1*, \sigma^2*) = u^1*$,

$$W^1(\sigma^1*,\ \sigma^2*) - W^1(\hat{\sigma}^1,\ \sigma^2*) \geq \delta^T\Big((1 - \delta)(u^1* - M^1) + \delta(u^1* - \underline{u}^1)\Big)$$

Since (x^1*, x^2*) is individually rational, $u^1* > \underline{u}^1$ so this difference is positive for δ close enough to 1.

6) By the same reasoning if δ is close enough to 1 player 2 will not be able to gain by deviating from $(\sigma^{1*}, \sigma^{2*})$, so $(\sigma^{1*}, \sigma^{2*})$ is an equilibrium for δ close enough to 1. ■

The next result establishes that outcomes which can be unilaterally improved upon are not part of the equilibrium path for any value-sufficient equilibrium.

Theorem 2: *If* $(\sigma^{1*}, \sigma^{2*})$ *is a value sufficient equilibrium, then* $\xi_t(\sigma^{1*}, \sigma^{2*})$ *cannot be innocuously improved upon for any t.*

Proof by Contradiction: 1) Suppose $\xi_T(\sigma^{1*}, \sigma^{2*})$ can be innocuously improved upon by $(\hat{x}^1, \xi_T^2(\sigma^{1*}, \sigma^{2*}))$. Have player 1 adopt a new strategy, $\hat{\sigma}^1$, where $\hat{r}_t^1(h_t) = \xi_t^1(\sigma^{1*}, \sigma^{2*})$ for all $h_t \in H_t$ for all $t \neq T$, and $\hat{r}_T^1(h_T) = \hat{x}^1$ for all $h_T \in H_T$.

2) Consider the outcomes and histories generated by $(\hat{\sigma}^1, \sigma^{2*})$.

2a) $\xi_t(\hat{\sigma}^1, \sigma^{2*}) = \xi_t(\sigma^{1*}, \sigma^{2*})$ for all $t < T$, so

$$u^1\Big(\xi_t(\hat{\sigma}^1,\ \sigma^2*)\Big) \quad = \quad u^1\Big(\xi_t(\sigma^1*,\ \sigma^2*)\Big), \tag{7}$$

$$v_t^2\Big(\xi_t^1(\hat{\sigma}^1,\ \sigma^2*)\ \Big|\ \sigma^1*,\ \sigma^2*\Big) \quad = \quad u^2\Big(\xi_t(\sigma^1*,\ \sigma^2*)\Big), \tag{8}$$

and

$$\theta_{t+1}(\hat{\sigma}^1,\ \sigma^2*) \quad = \quad \theta_{t+1}(\sigma^1*,\ \sigma^2*). \tag{9}$$

2b) At $t = T$, $\xi_T(\hat{\sigma}^1, \sigma^{2*}) = (\ \hat{x}^1, \xi_T^2(\sigma^{1*}, \sigma^{2*})\ )$ so

$$u^1\Big(\xi_T(\hat{\sigma}^1,\ \sigma^2*)\Big) \quad = \quad u^1\Big(\hat{x}^1,\ \xi_T^2(\sigma^1*,\ \sigma^2*)\Big) \quad > \quad u^1\Big(\xi_T(\sigma^1*,\ \sigma^2*)\Big), \tag{10}$$

and

$${}_T^2\Big(\xi_T^1(\hat{\sigma}^1,\ \sigma^2*)\ \Big|\ \sigma^1*,\ \sigma^2*\Big) \quad = \quad u^2\Big(\hat{x}^1,\ \xi_T^2(\sigma^1*,\ \sigma^2*)\Big) \quad = \quad u^2\Big(\xi_T(\sigma^1*,\ \sigma^2*)\Big), \tag{11}$$

2c) At $t > T$ proceed inductively. If

$$v_\tau^2\Big(\xi_\tau^1(\hat{\sigma}^1,\ \sigma^2*)\ \Big|\ \sigma^1*,\ \sigma^2*\Big) \quad = \quad u^2\Big(\xi_\tau(\sigma^1*,\ \sigma^2*)\Big)$$

for all $\tau < t$, then since $(\sigma^{1*}, \sigma^{2*})$ is value sufficient

$$\xi_t^2(\hat{\sigma}^1,\ \sigma^2*) \quad = \quad r_t^2*\Big(\theta_t(\hat{\sigma}^1,\ \sigma^2*)\Big) \quad = \quad r_t^2*\Big(\theta_t(\sigma^1*,\ \sigma^2*)\Big) \quad = \quad \xi_t^2(\sigma^1*,\ \sigma^2*).$$

By definition $\xi_t^1(\hat{\sigma}^1, \sigma^{2*}) = \xi_t^1(\sigma^{1*}, \sigma^{2*})$. Therefore

$$u^1\Big(\xi_t(\hat{\sigma}^1,\ \sigma^2*)\Big) \quad = \quad u^1\Big(\xi_t(\sigma^1*,\ \sigma^2*)\Big) \tag{12}$$

$$v_t^2\Big(\xi_t^1(\hat\sigma^1,\ \sigma^2*)\ \mid\ \sigma^1*,\ \sigma^2*\Big) \quad = \quad u^2\Big(\xi_t(\sigma^1*,\ \sigma^2*)\Big). \tag{13}$$

3) By (7), (10), and (12) $W^1(\hat\sigma^1, \sigma^2*) > W^1(\sigma^1*, \sigma^2*)$ so (σ^1*, σ^2*) cannot be an equilibrium. ■

Theorem 3, which provides that repeated play of any outcome that cannot be unilaterally dominated is the outcome of a value monotonic equilibrium, is analogous to theorem 1.

Theorem 3: *If* $(x_1*, x_2*) \in X^1 \times X^2$ *is individually rational and cannot be unilaterally dominated, then for* δ *close enough to 1 there is a value-monotonic equilibrium* (σ^1*, σ^2*) *for which* $\xi_t(\sigma^1*, \sigma^2*) = (x^1*, x^2*)$ *for all* $t = 0, 1, 2, \cdots$

Proof: The proof is very similar to the proof of theorem 1. 1) Construct (σ^1*, σ^2*). Let $r_0^1*(h_0) \equiv x^1*$, and let

$$r_t^1*(h_t) \quad \equiv \quad \begin{cases} x^1* & \mid \quad u^1(x^1*,\ x_\tau^2)\ \geq\ u^1(x^1*,\ x^2*) \quad \forall\ \tau\ <\ t \\ \\ \underline{x}^1 & \mid \quad \text{otherwise} \end{cases}$$

Also let $r_0^2*(h_0) \equiv x^2*$, and let

$$r_t^2*(h_t) \quad \equiv \quad \begin{cases} x^2* & \mid \quad u^2(x_\tau^1,\ x^2*)\ \geq\ u^2(x^1*,\ x^2*) \quad \forall\ \tau\ <\ t \\ \\ \underline{x}^2 & \mid \quad \text{otherwise} \end{cases} \tag{14}$$

where $u^1* = u^1(x^1*, x^2*)$ and $u^2* = u^2(x^1*, x^2*)$.

2) By construction $\xi_t(\sigma^1*, \sigma^2*) = (x^1*, x^2*)$ for all t.

3) By 2), $v_t^1(x^2 \mid \sigma^1*, \sigma^2*) = u^1(x^1*, x^2)$ and $v_t^2(x^1 \mid \sigma^1*, \sigma^2*) = u^2(x^1, x^2*)$ for all t so (σ^1*, σ^2*) is value monotonic.

4) Suppose player 1 considers some deviation $\hat\sigma^1 \in S^1$.

4a) If

$$v^2\Big(\xi_t^1\big(\hat\sigma^1,\ \sigma^2*\big)\ \mid\ \sigma^1*,\sigma^2*\Big) \quad \geq \quad u^2* \tag{15}$$

for all t, then $\xi_t^2(\hat\sigma^1, \sigma^2*) = x^2*$ for all t and because (x^1*, x^2*) cannot be unilaterally dominated

$$u^1\Big(\xi_t(\hat\sigma^1,\ \sigma^2*)\Big) \quad \leq \quad u^1(x^1*,\ x^2*) \quad \leq \quad u^1\Big(\xi_t(\sigma^1*,\ \sigma^2*)\Big). \tag{16}$$

so $W^1(\hat\sigma^1, \sigma^2*) \leq W^1(\sigma^1*, \sigma^2*)$ and there is no gain to player 1 from deviating.

4b) Therefore, suppose T (finite) is the first t for which (15) does not hold. By (14), $\xi_t^2(\hat\sigma^1, \sigma^2*) = x^2*$ for all $t \leq T$. Since (x^1*, x^2*) cannot be unilaterally dominated, (16) holds for all $t < T$.

4c) Since X^1 is compact and u^1 is continuous

$$u^1\left(\xi_T(\hat{\sigma}^1,\ \sigma^2*)\right) \quad \leq \quad M^1 \quad \equiv \quad \max_{x^1 \in X^1} u^1(x^1,\ x^2*) \tag{17}$$

4d) (14) implies $\xi_t(\hat{\sigma}^1, \sigma^2*) = \underline{x}^2$ for all $t > T$. Therefore,

$$u^1\left(\xi_t(\hat{\sigma}^1,\ \sigma^2*)\right) \quad = \quad u^1\left(\xi_t^1(\hat{\sigma}^1,\ \sigma^2*),\ \underline{x}^2\right) \quad \leq \quad \underline{u}^1. \tag{18}$$

5) Adding together (16), (17), and (18),

$$W^1(\hat{\sigma}^1,\ \sigma^2*) \quad \leq \quad (1 - \delta^T)u^1* \ + \ (1 - \delta)\delta^T M^1 \ + \ \delta^{T+1}\underline{u}^1.$$

Since $W^1(\sigma^1*, \sigma^2*) = u^1*$,

$$W^1(\sigma^1*,\ \sigma^2*) \ - \ W^1(\hat{\sigma}^1,\ \sigma^2*) \quad \geq \quad \delta^T\left((1 - \delta)(u^1* - M^1) \ + \ \delta(u^1* \ - \ \underline{u}^1)\right).$$

Since (x^1*, x^2*) is individually rational, $u^1* > \underline{u}^1$ so this difference is positive for δ close enough to 1.

6) By the same reasoning, if δ is close enough to 1, player 2 will not be able to gain by deviating from (σ^1*, σ^2*), so (σ^1*, σ^2*) is an equilibrium for δ close enough to 1. ■

Finally, theorem 4, which is analogous to theorem 2, establishes that an outcome which can be unilaterally dominated is never part of a value monotonic equilibrium path.

Theorem 4: *If* (σ^1*, σ^2*) *is a value monotonic equilibrium, then* $\xi_t(\sigma^1*, \sigma^2*)$ *cannot be unilaterally dominated for any t.*

Proof by Contradiction: 1) Suppose $\xi_T(\sigma^1*, \sigma^2*)$ can be unilaterally dominated by $(\hat{x}^1, \xi_T^2(\sigma^1*, \sigma^2*))$. Have player 1 adopt a new strategy, $\hat{\sigma}^1$, where $\hat{r}_t^1(h_t) = \xi_t^1(\sigma^1*, \sigma^2*)$ for all $h_t \in H_t$ for all $t \neq T$, and $\hat{r}_T^1(h_T) = \hat{x}^1$ for all $h_T \in H_T$.

2) Consider the outcomes and histories generated by $(\hat{\sigma}^1, \sigma^2*)$.

2a) $\xi_t(\hat{\sigma}^1, \sigma^2*) = \xi_t(\sigma^1*, \sigma^2*)$ for all $t < T$, so

$$u^1\left(\xi_t(\hat{\sigma}^1,\ \sigma^2*)\right) \quad = \quad u^1\left(\xi_t(\sigma^1*,\ \sigma^2*)\right), \tag{19}$$

$$v_t^2\left(\xi_t^1(\hat{\sigma}^1,\ \sigma^2*) \mid \sigma^1*,\ \sigma^2*\right) \quad = \quad u^2\left(\xi_t(\sigma^1*,\ \sigma^2*)\right), \tag{20}$$

and

$$\theta_{t+1}(\hat{\sigma}^1,\ \sigma^2*) \quad = \quad \theta_{t+1}(\sigma^1*,\ \sigma^2*). \tag{21}$$

2b) At $t = T$ the outcome is $\xi_T(\hat{\sigma}^1, \sigma^2*) = (\hat{x}^1, \xi_T^2(\sigma^1*, \sigma^2*))$ so

$$u^1\left(\xi_T(\hat{\sigma}^1,\ \sigma^2*)\right) \quad = \quad u^1\left(\hat{x}^1,\ \xi_T^2(\sigma^1*,\ \sigma^2*)\right) \quad > \quad u^1\left(\xi_T(\sigma^1*,\ \sigma^2*)\right), \tag{22}$$

and

$$v_T^2\left(\xi_T^1(\hat{\sigma}^1,\ \sigma^2*)\ \middle|\ \sigma^1*,\ \sigma^2*\right) \;=\; u^2\left(\hat{x}^1,\ \xi_T^2(\sigma^1*,\ \sigma^2*)\right) \;\geq\; u^2\left(\xi_T(\sigma^1*,\ \sigma^2*)\right) \tag{23}$$

2c) At $t > T$ proceed inductively. If

$$v_\tau^2\left(\xi_\tau^1(\hat{\sigma}^1,\ \sigma^2*)\ \middle|\ \sigma^1*,\ \sigma^2*\right) \;\geq\; u^2\left(\xi_\tau(\sigma^1*,\ \sigma^2*)\right)$$

for all $\tau < t$, then since (σ^1*, σ^2*) is value monotonic,

$$v_t^1\left(r_t^2\left(\theta_t(\hat{\sigma}^1,\ \sigma^2*)\right)\ \middle|\ \sigma^1*,\ \sigma^2*\right) \;\geq\; u^1\left(\xi_t(\sigma^1*,\ \sigma^2*)\right).$$

By definition $\xi_t^1(\hat{\sigma}^1, \sigma^2*) = \xi_t^1(\sigma^1*, \sigma^2*)$. Therefore

$$u^1\left(\xi_t(\hat{\sigma}^1,\ \sigma^2*)\right) \;=\; v_t^1\left(r_t^2\left(\theta_t(\hat{\sigma}^1,\ \sigma^2*)\right)\ \middle|\ \sigma^1*,\ \sigma^2*\right) \;\geq\; u^1\left(\xi_t(\sigma^1*,\ \sigma^2*)\right) \tag{24}$$

and

$$v_t^2\left(\xi_t^1(\hat{\sigma}^1,\ \sigma^2*)\ \middle|\ \sigma^1*,\ \sigma^2*\right) \;=\; u^2\left(\xi_t(\sigma^1*,\ \sigma^2*)\right).$$

3) By (19), (22), and (24), $W^1(\hat{\sigma}^1, \sigma^2*) > W^1(\sigma^1*, \sigma^2*)$ so (σ^1*, σ^2*) cannot be an equilibrium. ■

V. Conclusion

An interpretation of the value oriented approach to repeated games may be found in Aumann's assessment of the significance of the Folk theorem, which is that

> ⋯ it relates cooperative behavior in the [stage game] to non-cooperative behavior in [the repeated game]. This is the fundamental message of the theory of repeated games of complete information; that cooperation may be explained by the fact that the "games people play" ⋯ are not one-time affairs, but are repeated over and over. In game-theoretic terms, an outcome is cooperative if it requires an outside enforcement mechanism to make it "stick." Equilibrium points are self-enforcing; once an equilibrium point is agreed upon, it is not worthwhile for any player to deviate from it.[13]

Equilibrium points are self enforcing in the repeated game because each player knows that any deviation from the cooperative pattern dictated by the equilibrium will result in punishment. Thus equilibrium in a repeated game may be understood as a means of sustaining cooperative behavior by punishing deviations from cooperation.

With this interpretation in mind, the question may reasonably be asked, "What constitutes a `deviation from cooperative behavior?'" If the answer given to this question is, "any deviation from a prescribed pattern of behavior," then equilibria like the conventional Nash equilibria illustrated in Example 1 and Example 2 are obtained. It seems more likely, however, that one player will only view another player's behavior as deviating from a cooperative standard if the

[13] ibid p. 13

deviation is costly. In that case, only value oriented equilibria are obtained.

An interesting topic for further research would be to study sub-game perfect value oriented equilibria, i.e. pairs of strategies (σ^{1*}, σ^{2*}) which are value oriented equilibria in each sub-game of the repeated game. On the one hand, the set of such equilibria will not be empty since repeated unconditional play of a Nash equilibrium from the stage game will be such an equilibrium.[14] On the other hand, the set of such equilibria may well be considerably smaller than the set of all sub-game perfect Nash equilibria. Consider the type of punishment strategies used by Fudenberg and Maskin (1986) to obtain a Folk Theorem for sub-game perfect Nash equilibria in a two person game. Each player is dissuaded from making an initial deviation by the prospect of being punished for doing so. The punishments, which are costly to the punisher, are made credible by the fact that both agents know that failure to take a costly punishment action will result in the putative punisher being punished by *the player who made the initial deviation.* Now suppose the equilibrium is a sub-game perfect value monotonic equilibrium. If the putative punisher fails to punish, that failure will increase the value of his action to the other player. In a value monotonic equilibrium the other player will not punish such an action. The effect of this is that the initial punishment is not credible, so the initial deviation may be profitable. Thus, outcomes which can be achieved as sub-game perfect Nash equilibria may not be achievable as sub-game perfect value monotonic equilibria.

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