ON THE FINITENESS OF STABLE SETS

JOHN HILLAS, DRIES VERMEULEN, AND MATHIJS JANSEN

Abstract. For two person games, stable sets in the sense of Kohlberg and Mertens and quasi-stable sets in the sense of Hillas are finite. In this paper we present an example to show that these sets are not necessarily finite in games with more than two players.

Contents

Date: June, 1996.

1. Introduction

In their seminal paper, Kohlberg and Mertens (1986) defined a number of solutions of noncooperative games in which the solution of a game is a collection of closed subsets of equilibria. Formally this means that a solution is a map assigning to a game a collection of closed subsets of its strategy space. As well as defining a number of particular solutions Kohlberg and Mertens gave a list of properties such solutions should satisfy. Because these definitions were all based on the idea that a "solution set" must be stable against certain perturbations of the game, they called the solutions sets stable sets.

To differentiate it from other definitions that have appeared in the literature we shall call KM-stable sets the solution that they called "stable sets." None of the solutions defined by Kohlberg and Mertens actually satisfies all of the properties they proposed. A number of those properties already imply that stable sets must in some games be infinite. In many games it is clear that sets satisfying the requirements cannot be singletons. Thus if the solution were to be a connected set (one of the requirements) it must be infinite. Two other requirements are that the stable set should contain a sequential equilibrium of the game and that the solution depend only on the reduced normal form of the game and not on other "irrelevant" details of the game. Kohlberg and Mertens give an example in which, as the game is changed in a manner that leaves the reduced normal form unchanged, the unique sequential equilibrium of the game traces out an interval of equilibria. Thus any solution satisfying the requirements of Kohlberg and Mertens must necessarily be infinite valued in this game. Nevertheless Jansen, Jurg, and Borm (1994) showed that for a two person game all KM-stable sets are finite.

Mertens (1989) and Hillas (1990) defined solutions that do satisfy all of the properties of Kohlberg and Mertens. Hillas also introduced an intermediate concept, not satisfying the requirements, that he called quasi-stable sets. Vermeulen, Potters, and Jansen (1996) proved that for two person games these sets are also finite.

In this paper we will show that these results cannot be extended to normal form games with more than two players. We will present a three-player normal form game whose unique KM-stable set includes a line segment (so it is clearly not a finite set). Since each quasi-stable set contains a KM-stable set, this example is also a counterexample for the finiteness of quasi-stable sets for games with more than two players.

2. Preliminaries

In this section we give a brief and informal description of the concepts with which we are concerned. More formal and precise treatments are to be found in Kohlberg and Mertens (1986), and in the other references. We assume that the reader is familiar with the basic notions of a game and of a strategic equilibrium.

A KM-perturbation (or a Selten-perturbation) of a game associates with each pure strategy of each player a positive number representing the minimum probability with which that strategy may be played. A perturbation is small if each of these minimum probabilities is small. The game in which each player is required to play each strategy with that minimum probability is called the perturbed game. A closed set S of strategy profiles is KM-stable if it is minimal with respect to the following property: every neighborhood of S contains at least one equilibrium of any sufficiently small perturbation of the game.

A strategy profile is a perfect equilibrium (Selten 1975) if there is a sequence of KM-perturbations converging to zero and a sequence of equilibria of the perturbed games that converge to the perfect equilibrium.

3. The Counterexample

Consider the three player game of Figure 1. Here Player I chooses rows, Player II columns, and Player III matrices. We shall show that any KM-stable set of this game contains a line segment of equilibria. Let $\eta = \big((\gamma_1, \gamma_2), (\delta_1, \delta_2), (\varepsilon_1, \varepsilon_2, \varepsilon_3)\big)$ be a KM-perturbation for this game and let $\big((x_1, x_2), (y_1, y_2), (z_1, z_2, z_3)\big)$ be an equilibrium of the perturbed game. Because the pure strategies C and E of Player III are strictly dominated by W, Player III will play those two strategies with minimum weight in any equilibrium, that is, $(z_1, z_2, z_3) = (1 - \varepsilon_2 - \varepsilon_3, \varepsilon_2, \varepsilon_3)$. And so $\big((x_1, x_2), (y_1, y_2)\big)$ must be an equilibrium of the $\big((\gamma_1, \gamma_2), (\delta_1, \delta_2)\big)$-perturbation of the game of Figure 2.

	L	R	L	R	L	R
T	$1,0,1$	$0,0,1$	$1,0,0$	$0,0,0$	$1,0,0$	$0,1,0$
B	$0,0,1$	$1,0,1$	$0,1,0$	$1,0,0$	$0,0,0$	$1,0,0$
	W		C		E	

Figure 1

	L	R
T	$1,0$	$0,\varepsilon_3$
B	$0,\varepsilon_2$	$1,0$

Figure 2

For $\gamma_1, \gamma_2 < \frac{1}{2}$, the perturbations may be divided into the following five mutually exclusive and completely exhaustive sets.

$$\begin{aligned} S_2 &= \{\eta \mid \varepsilon_2 < \gamma_1(\varepsilon_2 + \varepsilon_3)\} \\ T_2 &= \{\eta \mid \varepsilon_2 = \gamma_1(\varepsilon_2 + \varepsilon_3)\} \\ S_{23} &= \{\eta \mid \varepsilon_2 > \gamma_1(\varepsilon_2 + \varepsilon_3) \text{ and } \varepsilon_3 > \gamma_2(\varepsilon_2 + \varepsilon_3)\} \\ T_3 &= \{\eta \mid \varepsilon_3 = \gamma_2(\varepsilon_2 + \varepsilon_3)\} \\ S_3 &= \{\eta \mid \varepsilon_3 < \gamma_2(\varepsilon_2 + \varepsilon_3)\} \end{aligned}$$

On these five sets we now calculate the equilibria of the perturbed games.

S_2: $x_2\varepsilon_2 \leq (1 - \gamma_1)\varepsilon_2 < \gamma_1\varepsilon_3 \leq x_1\varepsilon_3$. Thus Player II strictly prefers R to L, and so $(y_1, y_2) = (\delta_1, 1 - \delta_1)$. Then Player I strictly prefers B to T, and so $(x_1, x_2) = (\gamma_1, 1 - \gamma_1)$.

T_2: If $x_1 > \gamma_1$ then, as in the previous case (changing only the location of the strict inequality), $x_1 = \gamma_1$, a contradiction. Thus $(x_1, x_2) = (\gamma_1, 1-\gamma_1)$, and so B must be at least as good as T for Player I. Thus (y_1, y_2) is an element of the set $\{(y, 1-y) \mid \delta_1 \leq y \leq \frac{1}{2}\}$.

S_{23}: There is a unique equilibrium. In it $(x_1, x_2) = (\varepsilon_2/(\varepsilon_2+\varepsilon_3), \varepsilon_3/(\varepsilon_2+\varepsilon_3))$ and $(y_1, y_2) = (\frac{1}{2}, \frac{1}{2})$.

T_3: Similarly to T_2, $(x_1, x_2) = (1-\gamma_2, \gamma_2)$ and (y_1, y_2) is an element of the set $\{(y, 1-y) \mid \frac{1}{2} \leq y \leq 1-\delta_2\}$.

S_3: Similarly to S_2, $(x_1, x_2) = (1-\gamma_2, \gamma_2)$ and $(y_1, y_2) = (1-\delta_2, \delta_2)$.

The behavior of the equilibrium correspondence on S_{23} means that any KM-stable set must include $\{((s, 1-s), (\frac{1}{2}, \frac{1}{2}), (1,0,0)) \mid 0 \leq s \leq 1\}$; on S_2 that it must include $((0,1),(0,1),(1,0,0))$; and on S_3 that it must include $((1,0),(1,0),(1,0,0))$. Moreover any set that contains all of these points *is* a KM-stable set, and so, since KM-stability includes a minimality requirement, this is the unique KM-stable set. Since any quasi-stable set contains a KM-stable set, any quasi-stable set of this game must also be infinite.

For completeness we note that the set of perfect equilibria also includes the sets

$$\{((0,1),(t,1-t),(1,0,0)) \mid 0 \leq t \leq \tfrac{1}{2}\}$$

and

$$\{((1,0),(t,1-t),(1,0,0)) \mid \tfrac{1}{2} \leq t \leq 1\}.$$

Since any stable set in the sense of Mertens (1989) or of Hillas (1990) is a connected set of perfect equilibria containing a KM-stable set, in this game the unique stable set in either of these senses coincides with the set of all perfect equilibria.

References

John Hillas (1990): "On the Definition of the Strategic Stability of Equilibria," *Econometrica*, 58, 1365–1390.

Mathijs Jansen, Peter Jurg, and Peter Borm (1994): "On Strictly Perfect Sets," *Games and Economic Behavior*, 6, 400–415.

Elon Kohlberg and Jean-François Mertens (1986): "On the Strategic Stability of Equilibria," *Econometrica*, 54, 1003–1038.

Jean-François Mertens (1989): "Stable Equilibria—A Reformulation, Part I: Definition and Basic Properties," *Mathematics of Operations Research*, 14, 575–624.

Jean-François Mertens (1991): "Stable Equilibria—A Reformulation, Part II: Discussion of the Definition and Further Results," *Mathematics of Operations Research*, 16, 694–753.

John Nash (1950): "Equilibrium Points in N-Person Games," *Proceedings of the National Academy of Sciences*, 36, 48–49.

Reinhard Selten (1975): "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4, 25–55.

Dries Vermeulen, Jos Potters, and Mathijs Jansen (1996): "On Quasi-Stable Sets," *International Journal of Game Theory*, 25, 34–49.

Institute for Decision Sciences and Department of Economics, SUNY at Stony Brook, Stony Brook, NY 11794-4384 USA
E-mail address: `John.Hillas@sunysb.edu`

Department of Quantitative Economics, University of Limburg, P.O. Box 616, 6200 MD Maastricht, The Netherlands
E-mail address: `a.vermeulen@ke.rulimburg.nl`

Department of Quantitative Economics, University of Limburg, P.O. Box 616, 6200 MD Maastricht, The Netherlands
E-mail address: `m.jansen@ke.rulimburg.nl`