ON THE RELATION BETWEEN PERFECT EQUILIBRIA IN EXTENSIVE FORM GAMES AND PROPER EQUILIBRIA IN NORMAL FORM GAMES

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ABSTRACT. This paper examines the question of the extent to which it is true that any equilibrium that is quasi-perfect in any extensive form game having a given normal form is necessarily proper. If one fixes not only the equilibrium in question but also a sequence of completely mixed strategies converging to that equilibrium then indeed the notions are equivalent. However the stronger result is not true. An example of a normal form game is given in which there is an equilibrium that is quasi-perfect in any extensive form game having that normal form but not proper.

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1. INTRODUCTION

More than ten years ago van Damme (1984) proved that proper equilibria as defined by Myerson (1978) of a normal form game satisfied a modified version of the perfection requirement of Selten (1975) in any extensive form game having that normal form. At around the same time Kohlberg and Mertens (1986) proved essentially the same result, replacing the notion of perfect equilibrium with that of sequential equilibrium of Kreps and Wilson (1982). Kohlberg and Mertens used this result in showing that one version of their definition of strategic stability was such that stable sets always contained a sequential equilibrium.

The papers on the reformulation of strategic stability by Mertens (1989,1991) also showed that stable sets contain proper equilibria, and hence sequential equilibria. The approach of these papers, while not formally axiomatic, had a certain axiomatic flavour. And the requirement that stable sets contain a proper equilibrium seemed to have some role in motivating the form of the definition of stability. Thus one might be led to ask if nothing, beyond the backward induction property of sequential equilibrium and the admissibility considerations included in van Damme's modification of perfection, was added by requiring that stable sets contain proper equilibria. This leads us to ask if the converse of van Damme's result is true: if in fact an equilibrium that is perfect in any extensive form game having a given normal form is necessarily proper.

It is with this question that this paper is concerned. In Section 3 a normal form game is given in which there is an equilibrium that is perfect (and perfect in the modified sense) in any extensive form game having that normal form. And yet that equilibrium is not proper. Thus the question is answered in the negative. However there is a sense in which the affirmative answer is true. In Section 2 I show that, with a suitable strengthening of the requirement that the equilibrium be a perfect equilibrium of every extensive form game with the given normal form, all such equilibria are proper.

Before starting with the details in the next section I shall make a few comments on terminology. In this paper all games are assumed to be finite and extensive form games will always mean extensive form games with perfect recall. I don't intend anything deep by this restriction to games with perfect recall. The results in the direction already proved by van Damme and by Kohlberg and Mertens are not true for games without perfect recall. Indeed in such games there may be no equilibria in behaviour strategies. The results in the other direction are stronger than if games without perfect recall were permitted.

The notion of an extensive form game having a given normal form will always mean up to the duplication of pure strategies. Here I do mean a little by not being formal about this. I don't think there is any meaningful distinction between games that differ only in that some strategies are duplicated. Further I don't find anything particularly compelling, beyond the verbal convenience, in the standard way of defining and counting strategies in extensive form games. I hope that readers who disagree will have no trouble mentally adding the required phrases in the appropriate places.

2. The Relation Between Perfect and Proper Equilibria

Let us start by briefly laying out the small amount of notation used. Denote by N the set of players in the game and index a typical player by n; S_n is the set of player n's pure strategies and S the set of strategy profiles; σ is a typical mixed strategy profile, b a typical behaviour strategy profile, and similarly σ_n and b_n.

Recall now the definition of perfect equilibrium. The original idea of Selten (1975) was that however close to rational players were they would never be perfectly rational. There would always be some chance that a player would make a mistake. This idea may be implemented by approximating a candidate equilibrium strategy by a nearby completely mixed strategy and requiring that any of the deliberately chosen actions, that is, those given positive probability in the candidate strategy, be optimal, not only against the candidate strategy, but also against the nearby mixed strategy. If we are defining extensive form perfect equilibrium strategy is interpreted to mean behaviour strategy and action to mean an action at some information set.

More formally, a profile of behaviour strategies b is a perfect equilibrium if there is a sequence of completely mixed behaviour strategies $\{b^t\}$ such that at each information set and for each b^t, the behaviour of b at the information set is optimal against b^t, that is, is optimal when behaviour at all other information sets is given by b^t. If the definition is applied instead to the normal form of the game the resulting equilibrium is called a normal form equilibrium.

The definition of perfect equilibrium may be thought of as corresponding to the idea that players really do make mistakes, and that in fact it is not possible to think coherently about games in which there is no possibility of the players making mistakes. On the other hand one might think of the perturbations as instead encompassing the idea that the players should have a little strategic uncertainty, that is, they should not be completely confident as to what the other players are going to do. In such a case a player should not be thought of as being uncertain about his own actions or planned actions. This is (one interpretation of) the idea behind van Damme's (1984) definition of quasi-perfect equilibrium.

That definition is largely the same as the definition of perfect equilibrium. The definitions differ only in that, instead of the limit strategy b being optimal at each information set against behaviour given by b^t at *all* other information sets, it is required that b be optimal at all information sets against behaviour at other information sets given by b for information sets that are owned by the same player who owns the information set in question, and by b^t for other information sets. The assumption of perfect recall guarantees that this requirement of optimality is well defined. That is, the player does not take account of his own "mistakes," except to the extent that they may make one of his information sets reached that otherwise would not be. This change in the definition leads to some attractive properties. Like perfect equilibria, quasi-perfect equilibria are sequential equilibrium strategies. But, unlike perfect equilibria quasi-perfect equilibria are always normal form perfect, and thus admissible. Mertens (1995) argues that quasi-perfect equilibrium seems to be the right combination of admissibility and backward induction.

It is convenient for the statement of the result of this paper to give the definition in an equivalent, but somewhat different, form.

Definition 1. A profile of completely mixed behaviour strategies b is an ε-quasi-perfect equilibrium of the extensive form game Γ if for each player n in N and for each information set of that player every action taken with probability greater than ε is optimal against $(b_n^\varepsilon(b_n), b_{-n})$, where $b_n^\varepsilon(b_n)$ is the behaviour strategy obtained from b_n by replacing probabilities less than ε with zero and rescaling in order to obtain a new behaviour strategy. A profile of behaviour strategies is a quasi-perfect equilibrium if it is the limit as ε goes to zero of a sequence of ε-quasi-perfect equilibria.

A proper equilibrium (Myerson 1978) is defined to be a limit of ε-proper equilibria. An ε-proper equilibrium is a completely mixed strategy profile such that for each player if, given the strategies of the others, one strategy is strictly worse than another the first strategy is played with probability at most ε times the probability with which the second is played. In other words, more costly mistakes are made with lower frequency. van Damme (1984) proved the following result. (Kohlberg and Mertens (1986) proved a similar result, replacing quasi-perfect with sequential.)

Theorem 1 (van Damme, 1984, Kohlberg and Mertens, 1986)**.** *A proper equilibrium of a normal form game is quasi-perfect in any extensive form game having that normal form.*

Van Damme actually states his theorem a little differently referring simply to a pair of games, one an extensive form game and the other the corresponding normal form. (He is also more explicit about the sense in which a quasi-perfect equilibrium, a behaviour strategy, *is* a proper equilibrium, a mixed strategy.) Thus he correctly states that the converse of his theorem is not true. There are such pairs of games and quasi-perfect equilibria of the extensive form that are in no sense equivalent to a proper equilibrium of the normal form. Kohlberg and Mertens state their result in the same form as I have used but refer to sequential equilibria rather than quasi-perfect equilibria. They too correctly state that the converse is not true. For any normal form game one could introduce dummy players, one for each profile of strategies having payoff one at that profile and zero otherwise. In any extensive form having that normal form the set of sequential equilibrium strategies would be the same as the set of equilibrium strategies originally.

However it is not clear from the discussion of van Damme or of Kohlberg and Mertens that the converse of the theorem as I have stated it is not true. Certainly I know of no example in the existing literature that shows it to be false. For example, van Damme (1991) gives the game given in extensive form in Figure 1 and in normal form in Figure 1a to show that a quasi-perfect equilibrium may not be proper. The strategy (BD, R) is quasi-perfect, but not proper. To see that it is not proper one argues as follows. Given that Player II plays R, Player I strictly prefers BD to T and T to BU.

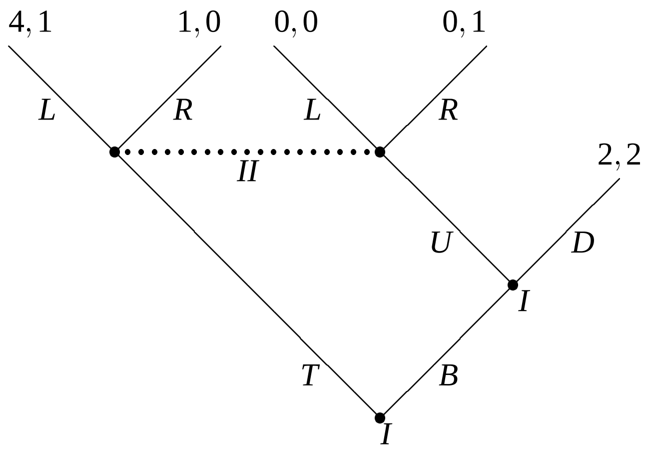

Figure 1

	L	*R*
T	4, 1	1, 0
BU	0, 0	0, 1
BD	2, 2	2, 2

Figure 1a

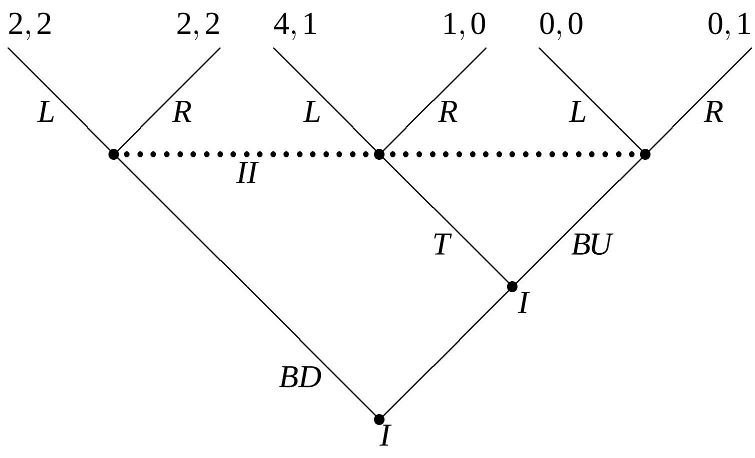

Figure 2

Thus in an ε-proper equilibrium BU is played with at most ε times the probability of T. Given this, in an ε-proper equilibrium Player II strictly prefers L to R, contradicting the fact that R is Player II's (proper) equilibrium strategy. Nevertheless in the game of Figure 2, which has the same normal form, up to duplication of strategies, that strategy is not quasi-perfect. At Player I's second decision node the action T is strictly preferred to BU. Thus in a quasi-perfect equilibrium T will have higher probability than BU and so Player II strictly prefers L to R. And so (BD, R) is not a quasi-perfect equilibrium.

In fact, the same can be done in all the games of which I am aware in which the questions of properness or perfection have been addressed. Thus one might conjecture that for any normal form game, any strategy profile that is quasi-perfect in any extensive form game having that normal form is a proper equilibrium of the normal form game. However, as the example of the following section shows, this is not so. A fairly weak version of this conjecture is true.

Theorem 2. *An equilibrium σ of a normal form game G is proper if and only if there exists a sequence of completely mixed strategies $\{\sigma^t\}$ with limit σ such that for any extensive form game Γ having the normal form G, for any sufficiently small $\varepsilon > 0$, for sufficiently large t, some behaviour strategy corresponding to σ^t is an ε-quasi-perfect equilibrium of Γ.*

A slightly weaker version of this theorem is proved in Mailath, Samuelson, and Swinkels (1995), using quite different methods.

Proof. The only if direction is implied by van Damme's proof. (Though not quite by the result as he states it, since that leaves open the possibility that different extensive forms may require different supporting sequences.)

The other direction is quite straightforward. Suppose that there does exist such a sequence $\{\sigma^t\}$. Since the σ^t are assumed to be completely mixed they define, for each player and for each nonempty subset of that player's strategies a conditional probability distribution on that subset. We first extract a subsequence of the σ^t for which all of these conditional probabilities converge. In fact, in order to avoid introducing new notation we shall assume, without loss of generality, that we start with such a sequence. Thus the sequence $\{\sigma^t\}$ partitions the strategy space S_n of each player into the sets $S_n^0, S_n^1, \ldots, S_n^J$ where S_n^j is the set of those strategies that receive positive probability conditional on one of the strategies in $S_n \backslash (\cup_{i<j} S_n^i)$ being played.

Now consider the following extensive form. The players move in order of their names, but without observing anything about the choices of those who chose earlier. (That is, they move essentially simultaneously.) Each player n first decides whether to play one of the strategies in S_n^0 or not. If he decides to play one of the strategies in S_n^0, then he chooses one of those strategies; if he decides not to, then he decides whether to play one of the strategies in S_n^1, and so on.

Now the construction of the S_n^j's implies that the only behaviour strategy in such a game consistent with (the limiting behaviour of) $\{\sigma^t\}$ has each player choosing at each opportunity to play one of the strategies in S_n^j rather than continuing with the process, and then choosing among those strategies according to the (strictly positive) limiting conditional (on S_n^j) probability distribution. More formally, for any $\varepsilon > 0$ there is T such that for all $t > T$ any behaviour strategy equivalent to σ^t is such that: at an information set at which player n chooses between S_n^j and not S_n^j, he chooses S_n^j with probability at least $1-\varepsilon$; and, at an information set at which he chooses among the strategies in S_n^j, his (mixed) choice is within ε of the (strictly positive) limiting conditional distribution. Let ε be small enough that each such choice is taken with probability at least $\sqrt{\varepsilon}$.

Now suppose that for such a t and ε there is a behaviour strategy b^t equivalent to σ^t with b^t an ε-quasi-perfect equilibrium. For such a strategy to be an ε-quasi-perfect equilibrium it must be that, for all n and for all j, player n is indifferent between any two strategies in S_n^j, and weakly prefers any strategy in S_n^j to any strategy in $S_n^{j'}$ for any $j' > j$. Also any strategy in $S_n^{j'}$ is played with probability at most $\sqrt{\varepsilon}/(1-\varepsilon)$ times the probability of any strategy in S_n^j for any $j' > j$. Thus σ^t is a $\sqrt{\varepsilon}/(1-\varepsilon)$-proper equilibrium, and so $\{\sigma^t\}$ supports σ as a proper equilibrium. □

In the new direction of this result (the if direction) quasi-perfect can be replaced by perfect. Since the other direction is not true with this change, this result is somewhat less interesting. In fact, for some games it is a statement about the empty set. Nevertheless, since the notion of perfect equilibrium is more well known, and the proof given above requires some significant modification we give the result as a separate theorem.

Theorem 3. *An equilibrium σ of a normal form game G is proper if there exists a sequence of completely mixed strategies $\{\sigma^t\}$ with limit σ such that for any extensive form game Γ having the normal form G, for any sufficiently small $\varepsilon > 0$, for sufficiently large t, some behaviour strategy corresponding to σ^t is an ε-perfect equilibrium of Γ.*

Sketch of Proof. The proof is similar to the proof of Theorem 2. However the extensive form game constructed in that proof will not work in this case. (This is so in even the simplest games in which perfect and quasi-perfect equilibria differ.) We can however use a very similar construction;

we simply need a somewhat finer partition on the strategy space of each player.

Each strategy profile σ^t in the sequence induces, for each player, a weak preference ordering on his strategies. There are only a finite number of such orderings, and so at least one must occur infinitely often in the sequence. We now take a subsequence in which the order is constant for all t, and again take a subsequence for which all the relevant conditional probabilities converge. We first partition each player's pure strategies into a sequence of subsets such that one strategy is strictly preferred to another if and only if it is in an earlier subset. We now refine this partition in the same way that the partition was defined in the proof of Theorem 2. If in the limit one strategy is played with infinitely more probability than another then it is in an earlier element of the partition. We now construct the extensive form game as in Theorem 2.

For this game, for t sufficiently large, for a behaviour strategy b^t equivalent to σ^t from the sequence, and with b^t an ε-perfect equilibrium for each player n the following hold.

1. If one pure strategy is strictly preferred to another then the preferred strategy occurs earlier in the tree.
2. When choosing among the strategies in S_n^j all strategies are chosen with probability at least $\sqrt{\varepsilon}$.
3. When choosing between whether to play one of the strategies in S_n^j or not to, S_n^j is chosen with probability at least $1-\varepsilon$. This is so either because the strategies in S_n^j are strictly preferred to those occurring later and b^t is a ε-perfect equilibrium, or because they are played with much greater probability in the limit, and t is sufficiently large.

Now the same argument as in the proof of Theorem 2 shows that $\{\sigma^t\}$ supports σ as a proper equilibrium. □

3. A Counter Example

In this section I analyse a game that shows that, as I said in Section 2, it is not true that an equilibrium that is quasi-perfect in any extensive form game with a given normal form is necessarily proper in that normal form. Nor is an equilibrium that is perfect in any extensive form game with a given normal form necessarily proper in that normal form. Thus the results of Section 2 cannot be very much strengthened, and certainly cannot be strengthened in the most immediately natural way.

Consider the game given in Figure 3. The equilibrium (A,V) is not proper. To see this we argue as follows. Given that Player I plays A, Player II strictly prefers W to Y and X to Y. Thus in any ε-proper equilibrium Y is played with at most ε times the probability of W, and also at most ε times the probability of X. The fact that Y is less likely than W implies that Player I strictly prefers B to C, while the fact that Y is less likely than X implies that Player I strictly prefers B to D. Thus in an ε-proper equilibrium C and D are both played with at most ε times the probability of B. This in turn implies that Player II strictly prefers Z to V, and so there can be no ε-proper equilibrium in which V is played with probability close to 1. Thus (A,V) is not proper.

	V	W	X	Y	Z
A	1,1	3,1	3,1	0,0	1,1
B	1,1	2,0	2,0	0,0	1,2
C	1,1	1,0	2,1	1,0	1,0
D	1,1	2,1	1,0	1,0	1,0

Figure 3

Nevertheless, in any extensive form game (with perfect recall) having this normal form there are perfect—and quasi-perfect—equilibria equivalent to (A,V). The proof of this fact is a little tedious, but the idea is quite simple. It is that there is no way of arranging Player II's decisions in an extensive form so that the requirements for a quasi-perfect equilibrium will imply both that W is much more likely than Y and that X is much more likely than Y. Recall that this conjunction was needed in the argument that (A,V) is not proper.

Let us suppose that we have an extensive form game having the normal form of Figure 3. We wish to show that there is a behaviour strategy equivalent to (A,V) that is quasi-perfect. To simplify things a bit let us first transform the game by expanding information sets so that neither player learns anything about what the other does. While this might change, for example, the set of sequential equilibria, it will not change the set of quasi-perfect equilibria—or, for that matter, the set of perfect equilibria. (The proof of this is straightforward, and is left as an exercise for the reader.)

Now what we have is essentially a one person extensive form for each player. And by the assumption of perfect recall these one person problems will have perfect information. Consider Player II's problem. For the moment remove all the branches leading to only V or Z. In this reduced problem consider one of the nodes at which Player II has a nontrivial decision and has, for the last time, the option to play Y. There are three possible cases: W is available, but not X; X is available, but not W; and both W and X are available. I shall make the argument for the first case; the second case is symmetric; and the third even easier.

We now construct the behaviour strategy in the following way. On the path to this node each branch is taken with probability at least ε. Action Y at any node following this one, is taken with probability ε. At this node, if W is "optimal" then Y is taken with probability ε; if not, then W is taken with probability ε and Y with probability ε^2. Every other "suboptimal" action is taken with probability ε^K, where K is some very large number, larger than the total number of nodes in the tree, say. Here"optimal" and "suboptimal" mean with respect to the order $V \succ W \succ X \succ Z \succ Y$. For this order optimal actions are always unique so, since we have assigned probability to all suboptimal actions we have uniquely defined a behaviour strategy.

Now for the defined strategy we have the following: W is played with probability of order ε^{K_W}; Y is played with probability of order ε^{K_Y}; and X is played with probability of order ε^{K_X}, where $K_W < K_Y < K < K_X$. (Also, though we shall not need these facts since they do not affect Player I's preferences, V is played with probability of order 1 and Z is played with probability of order ε^{K_Z}, where $K < K_Z$.) Now this strategy for Player II can be seen to induce the following preferences for Player I: $A \succ D \succ B \succ C$. So, by the same argument as that given by van Damme for the proof of Theorem 1, there is a behaviour strategy satisfying the conditions for a quasi-perfect equilibrium equivalent to the mixed strategy $(1-\varepsilon-\varepsilon^2-\varepsilon^3, \varepsilon^2, \varepsilon^3, \varepsilon)$. And this strategy induces the preferences that we hypothesized for Player II. Thus this pair of strategies is an ε-quasi-perfect equilibrium, as required.

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