

A reformulation of Aumann-Shapley random order values of non-atomic games using invariant measures ¹

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Abstract

In this paper the random order approach to values of non-atomic games is reformulated by generating random orders from a fixed subgroup of automorphisms, Θ that admits an invariant probability measurable group structure. The resulting Θ -symmetric random order value operator is unique and satisfies all the axioms of a Θ -symmetric axiomatic value operator. It is shown that for the uncountably large invariant probability measurable group $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$ of Lebesgue measure preserving automorphisms constructed in Raut [1996], $\check{\Theta}$ -symmetric random order value exists for most games in BV and it coincides with the fully symmetric Aumann-Shapley axiomatic value on pNA. Thus by restricting the set of admissible orders suitably the paper provides a possibility result to the Aumann-Shapley Impossibility Principle for the random order approach to values of non-atomic games.

¹Draft prepared for presentation at the conference on "New Directions in the Theory of Markets and Games in honour of Bob Aumann", Toronto, Canada, October 19-23, 1995. I got the idea of using Invariant measures to characterize random order values of games with a continuum of players (Raut[1981]) when I was a junior research fellow at the Indian Statistical Institute. I benefited from delightful introduction to game theory by Lloyd Shapley, to measure theory by K.R. Parthasarathy, and research advising by T. Parthasarathy during my stay there; comments from Robert Aumann, J.F. Mertens, Abraham Neyman and Joel Sobel were very useful in writing this revised draft. My gratitude goes to all.

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1. Introduction

Given a fixed set of players (known as grand coalition) and an algebra of its subsets as the possible coalitions, a cooperative game is a set function that assigns to each coalition its worth which the players can earn cooperatively. A basic problem in cooperative game theory is to find rules for dividing the worth of the grand coalition among its members such that the solution has certain "nice properties". More precisely, the problem is to find a mapping or an operator from the space of all set functions to the space of additive set functions such that the mapping possesses certain properties. Shapley value is one such solution concept and its wide commendation is due to its fairness properties which are the axioms that Shapley [1953] postulated to characterize it. Utilizing the linear vector space structure of the space of games, he proved the existence and uniqueness of the operator satisfying these axioms on the whole space of games and provided a formula to compute the value of individual games. The solution thus obtained is known as *axiomatic value*. Shapley also postulated an alternative fairness principle for the solution of the above basic problem of cooperative game theory, which later came to be known as *random order value*. In this approach, a player is given his expected marginal contribution in a random ordering of players, each ordering being equally likely among all possible orderings of the players. He showed that the formulas for value from both approaches coincide.

Aumann and Shapley [1974] extended the concept of axiomatic value to games with a continuum of players and proved the existence and uniqueness of axiomatic value operator on the spaces of games such as pNA and vNA (definitions of unknown terms in the introduction can be found in subsequent sections) and provided a "diagonal formula" for games in pNA. In an attempt to extend the random order value to continuum case, Aumann and Shapley considered the set Ω of all orderings of players satisfying some measurability condition and derived an impossibility principle: *There does not exist a measure structure on Ω with respect to which a random order value could be assigned to games in pNA.*

There have been several developments in the axiomatic value over the past several years, of which I point out briefly the ones relevant to our issues. One most widely studied issue has been to find larger spaces of games on which an axiomatic value, possibly a unique one, exists. pNA is economically the most important space which contains smooth market games and fair cost allocation schemes. However, non-smooth games that arise from markets with strong complementarity, do not belong to above spaces, nor even to ASYMP which is the largest space on which value was shown to exist by approximating a continuum game with a sequence of finite games and then using some limiting argument on the sequence of Shapley values of the approximating finite games, see Kannai [1966] and Aumann and Shapley [1974]. Mertens [1988] extended the diagonal formula to a very large space, known as *Mertens space* which include these non-smooth games and also all above spaces; using this formula he proved the existence of axiomatic value on Mertens space.

The most important fairness properties of Shapley value are derived from the symmetry axiom which was originally specified with respect to the whole group \mathcal{G} of automorphisms of the players set. More generally, we view each axiomatic value with symmetry restricted to a subgroup $\Theta \subset \mathcal{G}$ to represent a particular fairness principle. We shall refer to such a value as *Θ -symmetric axiomatic value* and the mapping that assigns to each game its axiomatic value as a *Θ -symmetric axiomatic value operator*. Some economic applications may dictate the use of Θ -symmetric value, Θ being a particular proper subgroup of \mathcal{G} . Monderer [1986, 1989] argued that the non-atomic games that arise from smooth market economies have certain characteristics in which symmetry group could be restricted to the group $\mathcal{G}(\mu)$ of μ -measure preserving automorphisms, where μ is a non-atomic probability measure on the unit interval determined by the data of the economy, and the domain of the value operator can be restricted to the space pNA(μ), which is the linear subspace of pNA generated by powers of this single μ . He showed that any $\mathcal{G}(\mu)$ -symmetric axiomatic value operator on pNA(μ) is also symmetric with respect to the full group \mathcal{G} .

Aumann and Shapley [1974] proved that there does not exist \mathcal{G} -symmetric axiomatic value operator on all of BV. Thus to have a value on all of BV, the symmetry axiom must be restricted to a proper subgroup. Ruckle [1982] has shown that when the symmetry is restricted to any "locally finite" subgroup of automorphisms, there exists a value operator on all of BV. This result is further refined by Monderer and Ruckle [1990].

It is important to note that the main fairness property of the random order value arises from the fact that each player has an equal chance of forming a coalition with a set of players of any size and any names and random order value gives every player its average marginal contribution out of all such coalitions. In the finite player case, the group of automorphisms of the players set and the set of orderings of players generated by them are identical, and the unweighted averaging of the marginal contributions of a player over all orderings generated by a given group of automorphisms basically symmetrizes the random order value for the group of automorphisms; in Raut [1993] I have shown that for the average marginal contribution to be symmetric for all games and all players it is necessary and sufficient that the randomness of the set of orders that are induced by the symmetry group assigns equal likelihood to each order, i.e., a random order has the uniform distribution. I use these insights from finite games to reformulate a Θ -symmetric random value for continuum case.

We begin with a fixed group of automorphisms $\Theta \subset \mathcal{G}$ with an invariant probability measurable group structure in it. We take the set of orders Ω as those which are determined by Θ and induce a uniform distribution on Ω from the action of the invariant probability measurable group Θ . We first show that the expected value of the marginal contribution function on Ω is unique (i.e., independent of any particular invariant measurable group structure on Θ) and it satisfies all the axioms of Θ -symmetric value; we name such a mathematical expectation operator as *Θ -symmetric random order value operator* on any subspace of games on which it is well defined. In particular then the \mathcal{G} -symmetric random order value operator is also a \mathcal{G} -symmetric axiomatic value operator. But, we shall see that the set of orders induced by \mathcal{G} is a proper subset of the set of Aumann-Shapley measurable orders. Could we not then hope to have a possibility result for the Aumann-Shapley Impossibility Principle with respect to the \mathcal{G} -symmetric random order value operator? We do not have an answer to this question yet. However, with the help of even more restricted set of orders induced from a suitably constructed uncountably large invariant probability measurable group Θ of Lebesgue measure preserving (l.m.p.) automorphisms, we have proved a possibility theorem to the Aumann-Shapley Impossibility Principle. More specifically, we show that there exists a unique Θ -symmetric random order value operator on a large space of games (we name it as NBV) which contains most of the games in BV and some of the non-smooth games that did not belong even to ASYMP; we derive a

diagonal formula for the Θ -symmetric random order value which coincides with the diagonal value formula for the \mathcal{G} -symmetric Aumann and Shapley axiomatic value for games in pNA. Thus, when the set of admissible orders Ω are suitably restricted we have a measure structure on Ω with respect to which the random order value exists for all games in pNA and it coincides with the \mathcal{G} -symmetric axiomatic value of Aumann and Shapley.

In section 2 we lay out the basic framework for our random order approach. In section 3 we discuss the relevant issues, state our main results and further remarks for future research. In section 4, we discuss the issues concerning choice of a suitable symmetry group, and sketch the construction of invariant probability measurable group Θ which was studied in more detail in Raut [1996]. Finally, section 5 gets into the details of proofs of the main results stated earlier and states and proves all other subsidiary results.

2. The basic framework

We adopt the convention of using a subscripted notation \mathcal{B}_X to denote a Borel σ -algebra of a topological space X (i.e., the σ -algebra generated by the class of open sets of X) and to denote any general σ -algebra, we do not use a subscript. Let $I = [0, 1] \subset \mathfrak{R}$ be the set of players. Let \mathcal{B}_I be the Borel σ -algebra of I . The elements of \mathcal{B}_I are the set of admissible coalitions. A *game* is a set function $V : \mathcal{B}_I \rightarrow \mathfrak{R}$ such that $V(\emptyset) = 0$. Let \mathcal{G}_I be the set of all games. Let FA be the set of finitely additive set functions on (I, \mathcal{B}_I) . A *measure* is a countably additive set function. One can check easily that \mathcal{G}_I and FA are linear vector spaces. A game V is *monotonic* if $V(S) < V(T)$ for any $S, T \in \mathcal{B}_I$, $S \subset T$. A *Borel automorphism* is a measurable map $\theta : (I, \mathcal{B}_I) \rightarrow (I, \mathcal{B}_I)$ such that it is one-one, onto and θ^{-1} is also measurable. Let \mathcal{G} be the set of all Borel automorphisms on (I, \mathcal{B}_I) . One can check that \mathcal{G} is a non-commutative (also known as non-abelian) group with composition of functions as group multiplication operation and identity function as the group identity.

For each $\theta \in \Theta$, define the linear operator $\theta^* : \mathcal{G}_I \rightarrow \mathcal{G}_I$ by

$$(\theta^*V)(S) = V(\theta^{-1}(S)), \quad \forall S \in \mathcal{B}_I$$

Given a subgroup of automorphisms, $\Theta \subset \mathcal{G}$, a linear subspace $Q \subset \mathcal{G}_I$ is said to be Θ -*symmetric* if $\theta^*Q \subset Q$ for all $\theta \in \Theta$.

Let Q be a linear subspace of \mathcal{G}_I . An operator $\Phi : Q \rightarrow FA$ is said to be *linear* if $\Phi(\alpha V_1 + V_2) = \alpha\Phi(V_1) + \Phi(V_2)$ $\forall V_1, V_2 \in Q, \alpha \in \mathfrak{R}$. Φ is said to be *positive* if (ΦV) is monotonic for any monotonic V in the domain of Φ . Φ is said to be *efficient* if $\Phi V(I) = V(I)$ $\forall V \in Q$. For a Θ -symmetric space Q , the operator $\Phi : Q \rightarrow FA$ is said to be Θ -*symmetric* if $\Phi\theta^*V = \theta^*\Phi V$, $\forall \theta \in \Theta, V \in Q$.

A Θ -*symmetric axiomatic value operator* on a Θ -symmetric linear space of games Q is a positive, linear, efficient, and Θ -symmetric operator $\Phi : Q \rightarrow FA$; when Θ is the full group \mathcal{G} , we shall name it as *Aumann-Shapley axiomatic value operator*; Aumann and Shapley [1974] proved the existence and uniqueness of this operator axiomatically.

Although in our analysis of random order value we do not use any topological structure on the space of games, to relate our results to the literature, we adopt the following topological concepts from Aumann and Shapley [1974]. A game V is of *bounded variation* if there exist monotonic games U and W such that $V = U - W$. Denote by BV the set of all games of bounded variation. It is known that BV is a linear vector space over \mathfrak{R} . Define a map $\|\cdot\|_{BV} : BV \rightarrow \mathfrak{R}$ by

$$\|V\|_{BV} = \inf \{U(T) + W(T) \mid V = U - W, U \text{ and } W \text{ are monotonic games}\}$$

for each $V \in BV$. It can be shown that $\|\cdot\|$ is a well defined norm on BV and with this norm BV is a Banach space (see Aumann and Shapley [1974, Corollary 4.2, and proposition 4.3]. The following notation is standard in the literature:

NA = set of non-atomic measures on (I, \mathcal{B}_I)

NA¹ = set of probability measures in NA

pNA = $\|\cdot\|_{BV}$ - closure of linear space spanned by powers of $\mu \in NA$

bv¹NA = $\|\cdot\|_{BV}$ - closure of linear space spanned by $f \circ \mu$, where $f : I \rightarrow \mathfrak{R}$ is of bounded variation, continuous at 0 and 1, and $f(0) = 0$, and μ is a non-negative, non-atomic probability measure on (I, \mathcal{B}_I)

It is known that FA, and NA and pNA are all closed subspaces of BV.

2.1. Generation of Random orders

Two main features of the random order approach to values of games with finite set of players are that (1) each automorphism ² generates a distinct ordering of players, i.e., the set of orders is the same as as the group of automorphisms; (2) for all games, the mathematical expectation of the random marginal contribution set function is symmetric with respect to the group of automorphisms if and only if each random ordering of players is equally likely (see Raut [1993]). In the finite players case, the main reason why the expected marginal contribution set function becomes symmetric for any game and with respect to the full group of permutations is that every player is equally likely to form a coalition with a set of players of any size and names in a random order. We want to adopt these two features to the continuum case.

Each $\theta \in \Theta$ generates a binary relation, $\succ_{\theta} \subset I \times I$ defined by

$$\text{for any } s, t \in I, \quad s \succ_{\theta} t \Leftrightarrow \theta(s) > \theta(t)$$

Recall that an order \succ on a set X is a *linear order*, which is also known as *total order* if for any $x, y \in X$, $x \neq y$, either $x \succ y$ or $y \succ x$, for no $x \in I$, $x \succ x$, and for any $x, y, z \in I$, $x \succ y, y \succ z \Rightarrow x \succ z$. A total order is a particular type of preference order. We will refer to a total order in this paper simply as an *order*. It is easy to verify that the binary relation \succ_{θ} generated by an automorphism θ is an order on I and that each $\theta \in \Theta$ generates a distinct order. Let $\bar{I} = I \cup \{\infty\}$, and for all $\theta \in \mathcal{G}$, we assume that $\theta(\infty) = \infty$. For an order \succ_{θ} , $\theta \in \mathcal{G}$, and a $s \in \bar{I}$, define an *initial segment* $I(s, \theta)$ by $I(s, \theta) = \{t \in I \mid \theta(s) > \theta(t)\}$. We view $I(s, \theta)$ as the set of players who are before player s in the order \succ_{θ} .

In the continuum case, however, two Borel automorphisms of I may generate the same ordering of I . For instance, $\theta \in \mathcal{G}$, defined by $\theta(x) = x^2$, $x \in I$ and the identity element $e \in \mathcal{G}$, defined by $e(x) = x, x \in I$, both generate the standard order \succ_e . Thus the set of orderings of players and the group of Borel automorphisms of players are not the same set. We derive the set of orders Ω that are generated by a given group of automorphisms Θ as follows:

²in the finite players case an automorphism is known as permutation.

Define an equivalence relation \sim on $\Theta \times \Theta$ by,

$$\theta_1 \sim \theta_2, \text{ for } \theta_1, \theta_2 \in \Theta \Leftrightarrow \theta_1, \theta_2 \text{ generate the same order on } I$$

Let $\Theta_e = \{\theta \in \Theta \mid \theta \sim e\}$. It can be easily shown that Θ_e is a subgroup of Θ and the set of distinct orders, Ω , generated by the automorphisms in Θ is the set of right cosets given by

$$\Omega \equiv \Theta / \Theta_e \equiv \{\Theta_e \theta \mid \theta \in \Theta\}$$

In the finite player case, the set of automorphisms of players is finite and for finite sets the concept of equal likelihood is intuitive. In the continuum case, however, the set of automorphisms of the players is uncountable. Analogue of the equal likelihood in the continuum set is the following concept of an invariant measurable group or invariant measure, which requires the underlying space to have a group structure:

Definition 1 : A measure space $(\Theta, \mathcal{A}, \Gamma)$ is said to be an *invariant measurable group* if Θ is a group, the map $(\theta_1, \theta_2) \rightarrow \theta_1 \theta_2^{-1}$ from $(\Theta \times \Theta, \mathcal{A} \times \mathcal{A})$ onto (Θ, \mathcal{A}) is measurable, and Γ is σ -finite, not identically zero, and right invariant, i.e., $\Gamma(E\theta) = \Gamma(E)$, for all $E \in \mathcal{A}$, and $\theta \in \Theta$, where $E\theta \equiv \{\sigma\theta \mid \sigma \in E\}$. Γ is known as *right invariant measure*.³ When Γ is furthermore a probability measure, a measurable group $(\Theta, \mathcal{A}, \Gamma)$ is said to be a *right invariant probability measurable group*.

In general Θ_e is not a normal⁴ subgroup of Θ and hence Ω is not necessarily a group. To see this, let $\theta \in \mathcal{G}$ and $\theta_e \in \mathcal{G}^*$ be defined by

$$\theta(x) = \begin{cases} 1-x & \text{if } 0 \leq x < 1/2 \\ x-1/2 & \text{if } 1/2 \leq x \leq 1 \end{cases} \quad \theta_e(x) = \begin{cases} .01x & \text{if } 0 \leq x < .8 \\ .008 + 4.96(x-.8) & \text{if } .8 \leq x \leq 1 \end{cases}$$

Let $t = .4$ and $s = .3$. Thus $\theta_e(s) < \theta_e(t)$, but $(\theta^{-1}\theta_e\theta)(s) = .507 > .506 = (\theta^{-1}\theta_e\theta)(t)$, thus $\theta^{-1}\theta_e\theta \notin \mathcal{G}_e$.

³When Θ is a locally compact topological group, and \mathcal{A} is the Borel σ -algebra, such that $\Gamma(U) > 0$, for every non-empty open set $U \subset \Theta$, then the Borel measure Γ is known as Haar Measure.

⁴ N is a normal subgroup of G if for all $\theta \in G$, we have $\theta^{-1}\nu\theta \in N$ for all $\nu \in N$.

This we do not have the group structure on Ω that we needed to extend the concept of equal likelihood of orderings in Ω . But Ω is a homogeneous space acted on by the group Θ , and for homogeneous spaces there is a natural concept of invariant measure (see Parthasarathy [1977, section 55]; or Segal and Kunz [1978, section 7.4]). In our set-up, however, we can use the *natural map* $\Pi : \Theta \rightarrow \Omega$ defined by $\Pi(\theta) = \Theta_e \theta$ to induce an invariant probability measure structure $(\Omega, \mathcal{B}, \mu)$ on the homogeneous space Ω of induced orders. $(\Omega, \mathcal{B}, \mu)$ will be referred to as a *set of random orders*.

2.2. Connection with Aumann-Shapley measurable orders

Aumann and Shapley [1974, pp. 94-95] defined an order \mathcal{R} on I to be *measurable* if the σ -algebra generated by the set of initial segments $\{I(s, \mathcal{R}) | s \in \bar{I}\}$ coincides with \mathcal{B}_I . An order \succ_θ generated by a Borel automorphism $\theta \in \mathcal{G}$ is measurable in the Aumann-Shapley sense, but not every order measurable in the sense of Aumann and Shapley can be represented by a Borel automorphism. To see this, let $u : I \rightarrow I \cup \{2\}$ be a Borel isomorphism⁵ and define an order \succ_u on I by $x, y \in I, x \succ_u y \Leftrightarrow u(x) > u(y)$. It is easy to see that \succ_u is an Aumann-Shapley measurable order but it cannot be induced by an automorphism. The difference between an Aumann-Shapley measurable order and an order generated by an automorphism can be seen from the following complete characterization of these orders.⁶

An order \succ is said to be *strongly separable* if there is a countable set $Z \subset I$ so that for any $x, y \in I, x \succ y$, implies there is a $z \in Z$ and $x \succ z \succ y$. An order \succ is said to be complete⁷ if any non-empty subset of $E \subset I$, which is bounded above, has a least upper bound ($l.u.b.$) in I . Let \succ on $I = [0, 1]$ be a strongly separable complete order with the Z being the countable subset associated with the definition of strong separability of \succ . Let \succ_e denote the standard order on I . Let Q_I be the set of rational numbers that lie in $(0, 1)$. It is well known that \succ_e is strongly separable on $(0, 1)$ with respect to Q_I , and that $(0, 1)$ is complete. For ease of exposition,

⁵The Borel isomorphism theorem states that for any two sets of the same cardinality if both sets are Borel subsets of complete and separable metric spaces, then there exists a Borel isomorphism between these two sets, i.e., there exists an one-one and onto map between the sets such that both the map and its inverse are Borel measurable with respect to the relative Borel σ -algebras of the sets. Notice that both I and $I \cup \{2\}$ are Borel subsets of \mathfrak{R} , hence there exists a Borel isomorphism between these two sets.

⁶I am grateful to an anonymous referee for pointing out this to me.

⁷This is sometimes also referred as order complete and it is distinct from the completeness axiom used in defining preference relation in utility theory.

let (X, \succ) to denote the set X ordered by \succ . Let \bar{I} denote the ordered set (I, \succ) after its first and last ordered elements being removed. Without loss of generality we can assume that $Z \subset \bar{I}$. An *order isomorphism* between two ordered sets is an one-one and onto map between the sets which preserves the orders of the sets. By Cantor's theorem we know that there exists an order isomorphism $h : (Z, \succ) \rightarrow (Q_I, \succ_e)$. For each $x \in (I, \succ)$, let $R(x) = \{h(z) | z \in Z \text{ and } x \succ z\}$ which is a subset of Q_I . It is easy to note that $R(x)$ is non-empty and bounded above, and hence has a *l.u.b.* Define the map $f : I \rightarrow (0, 1)$ by $f(x) \equiv l.u.b. R(x)$. Strong separability of \succ implies that f is order preserving and hence one-one. Completeness of \succ implies that f is an onto map. Now we extend the map f to (I, \succ) by letting it map the first and last elements of (I, \succ) respectively to 0 and 1. Notice that the initial segments $I(s, \succ), s \in \bar{I}$ under the order \succ are all of the form $I(s, \succ) = f^{-1}[0, x)$, where $x = f(s)$, hence initial segments generate \mathcal{B}_I and f^{-1} is measurable; since f is one-one, the Borel isomorphism theorem assures that f is measurable, and hence f is a Borel automorphism.

Conversely, an order generated by a Borel automorphism is clearly strongly separable and complete. For, let θ be an automorphism. Taking $Z = \theta^{-1}(Q_I)$ in the definition of strong separability, it is easy to note that \succ_θ is strongly separable. For any non-empty $E \subset (I, \succ_\theta)$ one can show that $\theta^{-1}(\sup_{t \in E} \theta(t))$ is the *l.u.b.* of E .

An order \succ is said to be *weakly separable* if there is a countable set $Z \subset I$ so that for any $x, y \in I, x \succ y$, implies there is a $z \in Z$ and $x \succeq z \succeq y$. Aumann-Shapley [1974, p.107] have shown that \succ is an Aumann-Shapley measurable order if and only if \succ is weakly separable and all initial segments are measurable. We summarize these facts in the following proposition.

Proposition 1 :

- (i) An order \succ on I arises from a Borel automorphism if and only if \succ is strongly separable and complete.
- (ii) An order \succ on I is Aumann-Shapley measurable if and only if \succ is weakly separable and all initial segments are measurable.

2.3. Θ -symmetric random order value operator

Given a game V , and an order \succ_{θ} , $\theta \in G$, a *marginal contribution set function*, $\phi^{\theta}V$ on (I, \mathcal{B}_I) is a measure on (I, \mathcal{B}_I) such that

$$(\phi^{\theta}V)(I(s, \theta)) = V(I(s, \theta)), \quad \forall s \in \bar{I} \quad (1)$$

Notice that for any $\theta, \theta' \in \Theta$, such that $\theta \sim \theta'$, we have $I(s, \theta) = I(s, \theta')$; hence it follows from (1) that $\phi^{\theta}V(S) = \phi^{\theta'}V(S)$ for all $S \in \mathcal{B}_I$. This allows us to unambiguously define $(\phi^{\omega}V)(S) = (\phi^{\theta}V)(S)$ where θ is such that $\omega = \Theta_{\theta}$.

The *expected marginal contribution set function* for a game V is a set function $\Phi_{\Gamma}V$ defined by

$$\begin{aligned} (\Phi_{\Gamma}V)(S) &= \int_{\Omega} (\phi^{\omega}V)(S) d\mu(\omega) \\ &= \int_{\Theta} (\phi^{\theta}V)(S) d\Gamma(\theta), \quad S \in \mathcal{B}_I \end{aligned} \quad (2)$$

The second equality follows from the change of variable formula for Lebesgue integrals and the facts in the previous paragraph. Let us define the space of games:

$$L1(\Theta, \Gamma) = \{V \in G_{\Gamma} \mid \phi^{\theta}V(S) \text{ is integrable for all } S \in \mathcal{B}_I\} \quad (3)$$

The main issues that arise for the above framework to provide a meaningful random order approach and our main results concerning them are stated in the next section.

3. Statement of the main results

For the operator Φ_{Γ} defined in (2) to yield a random order value operator in the continuum case, three basic facts must be established. **First**, for any game V and any order \succ_{θ} , $\theta \in \Theta$, if there exists a measure $\phi^{\theta}V$ satisfying (1), it should be unique so that for each $S \in \mathcal{B}_I$, $\phi^{\theta}V(S)$ is a function of θ . Proposition 2 ensures this. **Second**, in order for the operator Φ_{Γ} to be Θ -symmetric with respect to a given subgroup of automorphisms, $L1(\Theta, \Gamma)$ defined in (3) must be a Θ -symmetric linear subspace of G_{Γ} , this is shown

to be true in proposition 4. **Third**, the approach is of little use if for a given symmetry group of automorphisms, Θ , two different probability measurable structures assign two different finitely additive set functions to a game. The second part of Theorem 1 ensures that the mathematical expectation in (2) depends only on the fixed subgroup of automorphisms, Θ but not on a specific invariant probability measurable group structure on Θ .

Theorem 1 : *Let $(\Theta, \mathcal{A}, \Gamma)$ be an invariant probability measurable group structure on a fixed subgroup of automorphisms $\Theta \subset G$. Then the operator Φ_{Γ} defined in (2) is positive, linear, efficient and Θ -symmetric on $L1(\Theta, \Gamma)$ or any Θ -symmetric linear subspace of $L1(\Theta, \Gamma)$.*

Furthermore, suppose $(\Theta, \mathcal{A}', \Gamma')$ is another invariant probability measurable group structure on Θ , then $\Phi_{\Gamma'} = \Phi_{\Gamma}$ on the linear space of games $L1(\Theta, \Gamma) \cap L1(\Theta, \Gamma')$.

For a given fixed subgroup of automorphisms $\Theta \subset G$ and a Θ -symmetric linear subspace of games $Q \subset G_{\Gamma}$, the operator $\Phi_{\Gamma} : Q \rightarrow F_{\mathcal{A}}$ defined in (2) with respect to an invariant probability measurable group structure $(\Theta, \mathcal{A}, \Gamma)$ on Θ such that $Q \subset L1(\Theta, \Gamma)$ is said to be a *Θ -symmetric random order value operator* on Q . Theorem 1 assures that when such an operator exists, it is independent of a specific invariant probability measurable group structure on Θ and it coincides with the Θ -symmetric axiomatic value operator on Q . In fact, a Θ -symmetric random order value operator is a particular characterization of the Θ -symmetric axiomatic value operator. In particular, therefore, if we take Θ to be the full automorphism group G , then the existence of an Aumann-Shapley axiomatic value operator on Q can be reduced to the question of the existence of an invariant probability measurable group structure, (G, \mathcal{B}, Γ) , with the property that $Q \subset L1(G, \Gamma)$.

In our random order approach, the set of orders induced by G is a proper subset of the set of measurable orders considered by Aumann and Shapley in their random order approach. Therefore we may recast the Aumann-Shapley impossibility issue and expect a possibility result in our framework:

Question 1: *Could there exist an invariant probability measurable group structure on G , such that $\text{pNA} \subset L1(G, \Gamma)$?*

Indeed, on any group Θ , there always exists a right invariant probability measurable group structure, for instance, the trivial, coarsest σ -algebra,

$\mathcal{B} = \{\emptyset, \Theta\}$ with a trivial probability measure that assigns 0 to empty set and 1 to the whole set. The coarser the σ -algebra, the meager are the sets of measurable and integrable functions, and hence fewer games belong to $L1(\Theta, \Gamma)$ which may not include games in pNA. If the answer is still negative, we may find a positive answer when we restrict the question to other possibly smaller class of economically important games than pNA. I guess the proof of Aumann-Shapley impossibility theorem could be adopted to the present framework to produce a negative answer to the above question. Very little is known about the structure of the group \mathcal{G} that can shed light on the above issues, and I have not pursued these issues any further in this paper either. Instead, I carry out my investigation along the following line:

Question 2: *If we restrict the symmetry of the value operator to a "reasonable" proper subgroup of automorphisms Θ , could we construct a probability measurable group structure, $(\Theta, \mathcal{A}, \Gamma)$ on Θ such that $L1(\Theta, \Gamma)$ contains a large set of games including pNA and on a large class of economically important games including pNA, Θ -symmetric random order value operator coincides with the Aumann-Shapley axiomatic value operator?*

In section 4, I construct an uncountably large subgroup $\check{\Theta}$ of the group of l.m.p. automorphisms and the invariant probability measurable group structure, $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$. We carry out our above type of enquiry for the $\check{\Theta}$ -symmetric random order value operator $\Phi_{\check{\Gamma}}$.

In the context of restricting the symmetry group, two types of issues are addressed in the axiomatic approach to value. **First**, The largest subspace of BV on which Aumann-Shapley established the existence and uniqueness of \mathcal{G} -symmetric axiomatic value operator is bv'NA, and they also showed that there cannot exist a \mathcal{G} -symmetric axiomatic value operator on all of BV. Ruckle [1982] addressed the question: If the symmetry of the value operator is restricted to a proper subgroup, could there exist an axiomatic value operator on all of BV? Ruckle has shown that if Θ is taken to be any "locally finite"⁸ group then there exists a Θ -symmetric axiomatic value operator on all of BV. A group is said to be *locally finite* if for any finite number of elements from the group there is a finite subgroup which contains these elements. Monderer and Ruckle [1990] gave further refinement of this result. We have an analogous result as follows:

Definition 2 : A set function $V \in \mathcal{G}_I$ is said to be *normalized* if (i) $V(A_n) \rightarrow 0$ as $n \rightarrow \infty$ for any sequence of sets, $A_n \in \mathcal{B}_I$, $A_n \uparrow \emptyset$ as $n \rightarrow \infty$,

and (ii) $V(A_n) \rightarrow V(A)$ as $n \rightarrow \infty$ for any sequence of sets, $A_n \in \mathcal{B}_I$, $A_n \uparrow A$ as $n \rightarrow \infty$, where $A \in \mathcal{B}_I$.

Let us denote by NBV = the set of normalized set functions from BV. It can be easily seen that NBV is a linear space.

Theorem 2 : *There exists a unique $\check{\Theta}$ -symmetric random order value $\Phi_{\check{\Gamma}}$ on NBV.*

Second, Monderer [1986], [1989] addressed the question: If Θ is a proper subgroup of \mathcal{G} , then on which linear subspace of pNA, does there exist a unique Θ -symmetric value operator? Or equivalently, on which subspace of games in pNA, Θ -symmetric value operator coincides with the Aumann-Shapley axiomatic value operator? He has shown that when Θ is taken to be the group of μ -measure ($\mu \in NA$) preserving automorphisms and the linear space of games restricted to pNA(μ) (the closed subspace of pNA spanned by a single probability measure $\mu \in NA^1$),⁸ then there exists a unique Θ -symmetric axiomatic value operator satisfying dummy axiom on pNA(μ). We have even a stronger result along this line in the following theorem:

Theorem 3 : *Let $f : I \rightarrow \mathfrak{R}$ be absolutely continuous, and $\mu \in NA^1$, then the unique $\check{\Theta}$ -symmetric random order value of the scalar measure game $f \circ \mu$ yields the following diagonal formula:*

$$\Phi_{\check{\Gamma}}[f \circ \mu](S) = \mu(S) \int_0^1 f'(t) d\lambda(t) \quad (4)$$

Thus, the $\check{\Theta}$ -symmetric random order value operator $\Phi_{\check{\Gamma}}$ coincides with the Aumann-Shapley axiomatic value operator on all of pNA.

3.1. Further remarks

Remark 1 : There are economically important non-smooth games which neither belong to bv'NA, MIX, nor even to ASYMP. Mertens [1988] extended the diagonal formula for value to a very powerful closed subspace

⁸See his economic justifications for restricting the symmetry group and the class of games in this particular way so that his theory applies to a large class of market games.

of games in BV, known as *Mertens space*, on which the extended diagonal formula provides a value operator of norm 1 and the Mertens space was shown to include all well known spaces such as bvNA, ASYMP, DIFF and DIAG. J.F. Mertens and Abraham Neyman suggested to me to examine if Mertens space belongs to $L1(\check{\Theta}, \check{\Gamma})$. I have not tried to get a general answer to this question, instead I show that the $\check{\Theta}$ -symmetric random order value exists for the non-smooth game of "n-handed gloves markets" considered in example 19.2 of Aumann and Shapley [1974, p.136]: $V(S) = \min \{\mu_1(S), \mu_2(S), \dots, \mu_n(S)\}$, $\mu_i \in NA^1$, $i = 1, 2, \dots, n$, and $S \in \mathcal{B}$. This kind of non-smooth games arise in economies with strong complementarities. Aumann and Shapley showed that this game did not belong even to ASYMP when $n > 2$. One of the motivations for Mertens [1988] to extend the diagonal formula to the Mertens space was to include such games in the space. Notice that V is of bounded variation. Since each μ_i is a non-atomic probability measure, the game $V(S)$ is normalized and hence belongs to NBV. Thus there exists a unique $\check{\Theta}$ -symmetric random order value for V .

Remark 2 : An important issue regarding the reformulated random order approach of this paper is: What characteristics of the group $\check{\Theta}$ that makes the random order value coincides with the axiomatic value on pNA? In section 4 I argue that a random order generated according to the probability model $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$ has the characteristics that the random set of players that is placed before any given player is equally likely to be of any size $s \in [0, 1]$; the anonymous games in which worth of a coalition depends only through its size not names such as games in pNA, each player gets the average of the set of all possible marginal contributions and thus average is fully symmetrized in the sense that the value thus obtained is symmetric with respect to the full group of automorphisms. Locally finite groups of automorphisms may not do the job, as we have illustrated in section 4. The games that arise in most economic applications are anonymous. However, for an wider applicability of the present approach, we must construct a larger invariant probability measurable group structure $(\Theta, \mathcal{B}, \Gamma)$ than $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$, so that the random order value $\Phi_{\Gamma}(V)$ with respect to it also fully symmetrizes non-anonymous games.

Remark 3 : Robert Aumann pointed out to me that for an alternative reformulation of random order approach to value, one might give up the

measure theoretic model of the player set, i.e., (I, \mathcal{B}_I) , and instead consider the player space to be torus or other topological spaces with more well-behaved automorphism groups. It should be noted that there can exist only two orders on any topological space that is connected. This, for instance, will greatly simplify our analysis of random order value. We do not know, however, what kind of fairness such symmetry group entails and what kind of economic situations are appropriate for such models; most of the economic models with a continuum of agents, however, have employed measure theoretic structures for the space of agents, and thus we must begin to imagine the nature and study the implications of economic models with a topologically space of agents.

We keep the above unresolved issues for future research to shed more light on.

4. On the symmetry subgroup $\check{\Theta}$

The essence of the success of the random order approach to characterize the Shapley value of games with finite set of players ask the following question: *Does there exist an uncountably large subgroup of automorphisms in \mathcal{G} that can be equipped with an invariant probability measurable group structure, $(\Theta, \mathcal{A}, \Gamma)$, such that in a randomly selected order every player is equally likely to have placed before it a set of players of any size containing players from anywhere in the unit interval?* This notion beckons to the strongly mixing automorphisms. To fix ideas, let us consider mixing with respect to Lebesgue measure λ . A Lebesgue measure preserving automorphism, $\theta \in \mathcal{G}$, is said to be *strongly mixing* if $\lim_{n \rightarrow \infty} \lambda(\theta^{-n}E \cap F) = \lambda(E \cap F)$, for all $E, F \in \mathcal{B}_I$. In essence a strongly mixing automorphism, θ , allows thorough mixing of any set of players $t \in E$ with any other player in the unit interval I by producing an orbit $Q(E) = \{\theta^n t | t \in E, n = 0, 1, \dots\}$ which is dense and uniformly spread all over I . A l.m.p. automorphism is *weakly mixing* if $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \lambda(\theta^{-j}E \cap F) = \lambda(E \cap F)$. It is known that with respect to the "weak topology", the set of such automorphisms is of the first category and the set of weakly mixing automorphisms is of the second category. This means that generically a measure preserving automorphism is a weakly mixing but not strongly mixing. Aumann [1967]⁹ has shown that it is impossible to find an invariant probability measurable group structure

⁹I am grateful to Professor Robert Aumann for drawing my attention to this result.

on the whole group of l.m.p. automorphisms which satisfies further condition that the real valued function $f(\theta) \equiv \lambda(E \cap \theta F)$ is measurable for all $E, F \in \mathcal{B}_I$.

Thus in our approach, we seek to achieve thorough mixing of players with the help of actions of an uncountably large subgroup $\check{\Theta}$ of l.m.p. automorphisms. This we obtain as a (projective) limit of an increasing sequence of "carefully constructed" increasing finite subgroups, Θ_n , $n \geq 0$ of l.m.p. automorphisms. We describe the construction of $\check{\Theta}$ briefly in this section, (for details, see Raut [1996]). It is, however, interesting to note that in our construction we get thorough mixing with the help of automorphisms in Θ_n 's which are all recurrent of period n .

We also want to equip $\check{\Theta}$ with a fine enough measurable group structure $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$ so that it allows sufficiently rich set of games in $L1(\check{\Theta}, \check{\Gamma})$. We impose the following separability requirement on our measurable group structure:

Definition 3 : A measurable group $(\Theta, \mathcal{A}, \Gamma)$ is *separated*¹⁰ if $\forall \theta \in \Theta$, $\theta \neq e$, there exists $E \in \mathcal{A}_\theta$ such that $0 < \Gamma(E) < \infty$ and $\Gamma(E\theta\Delta E) > 0$, where Δ is the symmetric difference operator between two sets.

We define the increasing sequence of finite groups, Θ_n , $n \geq 0$ each containing l.m.p. automorphisms that are discontinuous at only a finite number of points of I . In the n -th automorphism group, Θ_n , the discontinuities of the automorphisms are at the points $\frac{k}{2^n}$, $k = 1, \dots, 2^n - 1$. These $2^n - 1$ points in I determine 2^n dyadic subintervals of I : $I_k = [\frac{k}{2^n}, \frac{k+1}{2^n}]$, $k = 0, 1, \dots, 2^n - 1$. In order for the automorphisms to be Lebesgue measure preserving, we assume that in each subinterval I_k , the automorphisms are linear with slope ± 1 .

Let $N_n = \{0, 1, 2, \dots, 2^n - 1\}$. We represent each automorphism in Θ_n by a pair of functions, π_n and \mathcal{O}_n such that $\pi_n : N_n \rightarrow N_n$ is a permutation of N_n and $\mathcal{O}_n : N_n \rightarrow \{-1, 1\}$ is a map as follows: For each $k \in N_n$, $\pi_n(k)$ specifies which subinterval of the unit interval the image of the k th subinterval be mapped to, and $\mathcal{O}_n \circ \pi_n(k)$ specifies the slope of the automorphism that the image subinterval will take. Sometimes we will refer to \mathcal{O}_n as the *slope map*. We denote such an automorphism as described above

¹⁰This separation notion for measurable groups is the analogue of Hausdorff separation axiom for topological spaces, see Halmos [1950, pp.273].

by the symbol

$$\theta_n = (\pi_n(k), \mathcal{O}_n \circ \pi_n(k))_{k=0}^{2^n-1} \quad (5)$$

An equivalent description of the above automorphism that we will often use is the following:

$$\theta_n(x) = \begin{cases} \frac{\pi_n(k)}{2^n} - \frac{k}{2^n} + x & \text{if } x \in I_k \text{ and } \mathcal{O}_n(\pi_n(k)) = +1 \\ \frac{\pi_n(k)+1}{2^n} - \frac{k}{2^n} - x & \text{if } x \in I_k \text{ and } \mathcal{O}_n(\pi_n(k)) = -1 \end{cases} \quad (6)$$

$k = 0, 1, \dots, 2^n - 1$

We use θ_n to mean the representation (5) and $\theta_n(\cdot)$ or $\theta_n(x)$ to mean the representation (6) of an element in Θ_n .

Let $\mathfrak{S} = \{+1, -1\}$ denote the set of slopes. With the usual multiplication operation of real numbers and with $+1$ as the identity element, it is trivial to show that \mathfrak{S} is a group.

Let

$$\Theta_n = \left\{ \begin{array}{l} \theta_n : I \rightarrow I, \text{ defined by (6) } | \pi_n \text{ is a permutation of } N_n \\ \text{and } \mathcal{O}_n \text{ is an orientation of the } 2^n \text{ sub-intervals} \end{array} \right\}$$

There are $(2^{2^n} \cdot 2^n!)$ total number of elements in Θ_n . It can be easily seen that Θ_n is a subgroup of l.m.p. automorphisms. For instance, when $n = 0$, there is no subdivision of I , and thus

$$\Theta_0 = \{\theta_0 = (\pi_0(0), \mathcal{O}_0(0)) \mid \mathcal{O}_0(0) \in \mathfrak{S}\}$$

. Note that Θ_0 has only two elements.

For further illustration of the above concepts, in panel (a) of figure 1, we have shown the graph of $\theta_2(x)$ corresponding to the permutation, $\pi_2(1) = 2$, $\pi_2(2) = 4$, $\pi_2(3) = 1$ and $\pi_2(4) = 3$, and the orientation, $\mathcal{O}_2(1) = +1$, $\mathcal{O}_2(2) = -1$, $\mathcal{O}_2(3) = +1$, and $\mathcal{O}_2(4) = -1$.

Let us examine the kind of randomization of the players that are performed by the random orders in Θ_n for large n . For illustration purpose, let us consider the automorphism $\theta_2 \in \Theta_2$ that is depicted in panel (a) of

An element θ_2 of Θ_2
 Intersections of dashed lines with y-axis are values of $\theta(t)$ for all $\theta \in \Theta_2$

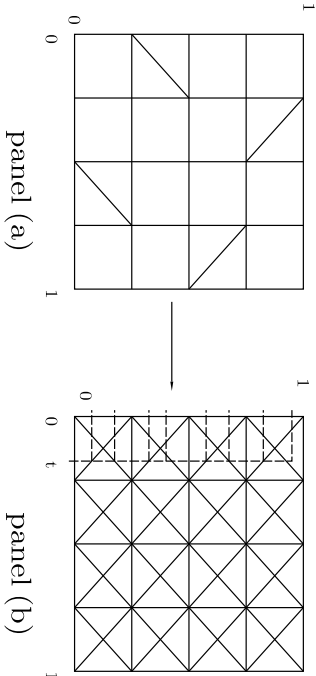


Figure 1:

figure 1. Note that the set of players that are placed before player t , $t \in I$ in the random order $\theta_2 \in \Theta_2$ is given by

$$I(\{t\}, \theta_2) = \begin{cases} [0, t) \cup I_3 & \text{if } t \in I_1 \\ I_1 \cup (t, \frac{1}{2}) \cup I_3 \cup I_4 & \text{if } t \in I_2 \\ [\frac{1}{2}, t) & \text{if } t \in I_3 \\ I_1 \cup I_3 \cup (t, 1] & \text{if } t \in I_4 \end{cases}$$

Note that the nature of randomization produced by an element of Θ_n depends on the permutation π and the orientation \mathcal{Q} . Let us fix a $t \in I$ and suppose $t \in I_1$. Consider the initial segments of player t in each of the random orders $\theta_n \in \Theta_n$, that has the same value for t , say $\theta_n(t) = t_0$. All of these random orders will have either positive orientation or negative orientation. Let us assume that they have positive orientation. Let us denote by $[x]$, $x \in \mathbb{R}$, as the greatest integer in x . The way the initial segments are randomized by these random orders is that $[\frac{2^n}{t_0}]$ sub-intervals from the set of all sub-intervals except I_1 are randomly selected and then placed before the set of points $[0, t)$ in all possible permutations. For very large n , size of each sub-intervals is very small, and hence for large n , all those θ_n 's with fixed value of $\theta_n(t) = t_0$ are placing almost any infinitesimally small sub-intervals of I that can fit in an interval of size $[0, t_0]$. The size of the interval also varies as we vary t_0 in the orbit $\mathcal{Q}_n = \{\theta(t_0) \mid \theta \in \Theta_n\}$.

In panel (b) of figure (1), we have graphed all the elements of Θ_2 . The set \mathcal{Q}_2 is shown as the intersection of the dash lines with the y-axis. It is trivial to note that $\Theta_n \subset \Theta_{n+1}$ and as $n \rightarrow \infty$, the number of elements in \mathcal{Q}_n becomes large and are spread uniformly over I .

It can be easily shown that the limiting set, $\Theta \equiv \bigcup_{n=1}^{\infty} \Theta_n = \lim_{n \rightarrow \infty} \Theta_n$ is a group of l.m.p. automorphisms, but contains only countable number of elements, and thus cannot admit an invariant probability measurable structure. It is interesting to note, however, that Θ is locally finite. Thus we have an example of a locally finite symmetry group Θ such that there does not exist a Θ -symmetric random order value operator on any space of games at all.

We can, however, construct a large subgroup of l.m.p. automorphisms with a different kind of limit, namely, the projective limit of the above sequence of Θ_n limiting further the elements in each of them as follows: We use $\hat{\Theta}_n$ to denote the modified Θ_n . $\hat{\Theta}_n, n \geq 0$ are recursively constructed as follows:

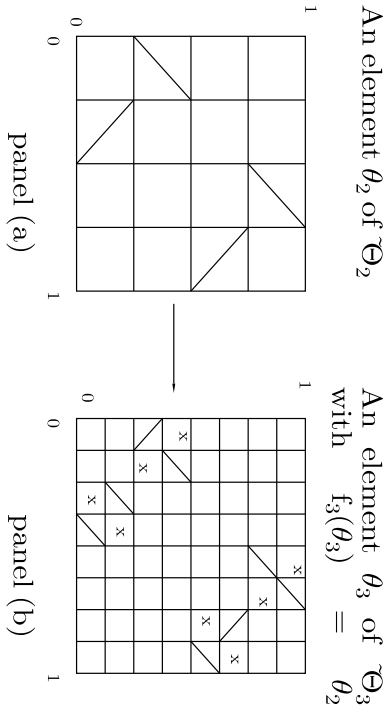


Figure 2:

For $n = 0$, take $\hat{\Theta}_0 = \Theta_0$. To define $\hat{\Theta}_1$, note that we have two dyadic sub-intervals of I denoted as I_0 and I_1 . Each $\theta_0 \in \hat{\Theta}_0$, induces a unique permutation π_{1,θ_0} of $N_1 = \{0, 1\}$ defined by

$$\pi_{1,\theta_0}(j) = i \quad \text{if for all } x \in I_j, \quad \theta_0(x) \in I_i, \quad i, j \in N_1$$

Given $\theta_0 \in \hat{\Theta}_0$, let us denote by

$$A_1(\theta_0) = \left\{ \theta_1 = (\pi_{1,\theta_0}(k), \mathcal{O}_1(\pi_{1,\theta_0})(k))_{k=0}^1 \mid \mathcal{O}_1(\pi_{1,\theta_0})(j) \in \mathfrak{S}, j = 0, 1 \right\}$$

We now define $\hat{\Theta}_1$ by

$$\hat{\Theta}_1 = \bigcup_{\theta_0 \in \hat{\Theta}_0} A_1(\theta_0)$$

Note that each $A_1(\theta_0)$ has $2 \times 2 = 4$ elements and hence $\hat{\Theta}_1$ has $2 \times 4 = 8$ elements.

Let us suppose that we have already defined $\hat{\Theta}_{n-1}$. We now define $\hat{\Theta}_n$ from $\hat{\Theta}_{n-1}$.

Let the 2^n dyadic sub-intervals at stage n be denoted as I_0, \dots, I_{2^n-1} . Each $\theta_{n-1} \in \hat{\Theta}_{n-1}$ induces a unique permutation $\pi_{n,\theta_{n-1}}$ of the set N_n defined by

$$\pi_{n,\theta_{n-1}}(j) = i \text{ if for all } x \in I_j, \quad \theta_{n-1}(x) \in I_i, \quad i, j \in N_n \quad (7)$$

For each $\theta_{n-1} \in \hat{\Theta}_{n-1}$, we define

$$A_n(\theta_{n-1}) = \left\{ \theta_n = (\pi_{n,\theta_{n-1}}(k), \mathcal{O}_n(\pi_{n,\theta_{n-1}})(k))_{k=0}^{2^n-1} \mid \mathcal{O}_n(\pi_{n,\theta_{n-1}})(i) \in \mathfrak{S} \forall i \in N_n \right\}$$

and

$$\hat{\Theta}_n = \bigcup_{\theta_{n-1} \in \hat{\Theta}_{n-1}} A_n(\theta_{n-1})$$

For each $n \geq 1$, we define the multiplication operation in $\hat{\Theta}_n$ as the composition of functions, namely, for $\hat{\theta}_n, \hat{\theta}'_n \in \hat{\Theta}_n$, we define $\hat{\theta}_n \hat{\theta}'_n = \hat{\theta}_n(\hat{\theta}'_n(x))$, $x \in I$, and for each $n \geq 1$, we define the projection maps $f_n : \hat{\Theta}_n \rightarrow \hat{\Theta}_{n-1}$, $f_n(\hat{\theta}_n) = \hat{\theta}_{n-1}$, where $\hat{\theta}_{n-1}$ is such that $\theta_n \in A_n(\hat{\theta}_{n-1})$.

To get an idea about these projection maps, in panel (b) of figure 2, we have drawn a $\theta_3 \in \hat{\Theta}_3$ whose projection using the map f_3 is $\theta_2 \in \hat{\Theta}_2$.

Denote by

$$\check{\Theta} = \left\{ \check{\theta} = (\theta_0, \theta_1, \theta_2, \dots) \mid \theta_n \in \hat{\Theta}_n, \forall n \geq 0 \text{ and } f_n(\theta_n) = \theta_{n-1}, \forall n \geq 1 \right\}$$

For any two elements $\check{\theta} = (\theta_0, \theta_1, \theta_2, \dots)$ and $\check{\theta}' = (\theta'_0, \theta'_1, \theta'_2, \dots)$ from $\check{\Theta}$, define the multiplication operation $\check{\theta} \circ \check{\theta}'$ by

$$\check{\theta} \circ \check{\theta}' = (\theta_0 \theta'_0, \theta_1 \theta'_1, \theta_2 \theta'_2, \dots)$$

By Proposition 3 in Raut[1996], $f_n(\theta_n \theta'_n) = f_n(\theta_n) f_n(\theta'_n) \in \hat{\Theta}_{n-1}$. Hence, $\check{\theta} \circ \check{\theta}' \in \check{\Theta}$. With $\check{\theta}^{-1} = (\theta_0^{-1}, \theta_1^{-1}, \theta_2^{-1}, \dots)$ as the inverse of $\check{\theta} = (\theta_0, \theta_1, \theta_2, \dots)$, and with $\check{\epsilon} = (\epsilon_0, \epsilon_1, \dots)$, where ϵ_n is the identity element of $\hat{\Theta}_n$ as the unit element, we note that $\check{\Theta}$ is a group.

Define for $n \geq 0$ the projection maps $\pi_n : \check{\Theta} \rightarrow \hat{\Theta}_n$ by

$$\pi_n(\check{\theta}) = \theta_n, \text{ where } \check{\theta} = (\theta_0, \theta_1, \theta_2, \dots)$$

Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \pi_n^{-1}(\mathcal{B}_n)$. It can be easily shown that \mathcal{F} is a Boolean algebra. Let $\check{\mathcal{B}}$ be the σ -algebra generated by \mathcal{F} . $(\check{\Theta}, \check{\mathcal{B}})$ is called the *projective limit* of the sequence of measure spaces, $(\hat{\Theta}_n \mathcal{B}_n)$, $n \geq 0$ through the maps f_n , $n \geq 1$.

The results in the following theorem are proved in Raut [1996].

Theorem 4 :[Raut(1996)]

There exists a unique right invariant probability measure, $\check{\Gamma}$ on the projective limit, $(\check{\Theta}, \check{\mathcal{B}})$ of the sequence of measurable groups, $(\hat{\Theta}_n, \mathcal{B}_n, \Gamma_n)_{n=0}^{\infty}$, through the sequence of homomorphisms, $\{f_n\}_{n=0}^{\infty}$, such that

$$(i) \quad \check{\Gamma} \pi_n^{-1} = \Gamma_n$$

(ii) $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$ is an uncountably large separated probability measurable group.

(iii) For each $\check{\theta} = (\theta_0, \theta_1, \dots, \theta_n, \dots) \in \check{\Theta}$, the limit $\check{\theta}(t) = \lim_{n \rightarrow \infty} \theta_n(t)$ exists for all $t \in I$ and the limit function $\theta : I \rightarrow I$ is a Lebesgue measure preserving automorphism.

Two measure spaces, $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$ are said to be *isomorphic* if there exists two sets $N_i \subset X_i$, $\mu_i(N_i) = 0$, $i = 1, 2$ and a Borel automorphism $T : X_1 \setminus N_1 \rightarrow X_2 \setminus N_2$ such that $\mu_1 T^{-1} = \mu_2$. In this paper we prove the following isomorphism theorem for the invariant probability measurable group $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$.

Theorem 5 :[Isomorphism Theorem]

The projective limit group $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$ is isomorphic to the unit interval with Lebesgue measure, $(I, \mathcal{B}_I, \lambda)$.

We will utilize the above two theorems to derive diagonal formula for random order value operator for a larger class of games than in Raut [1996], and to prove theorems 2 and 3 stated earlier.

Proof of Theorem 5: Let $L_2(I)$ be the Hilbert space of square integrable functions with respect to Lebesgue measure on (I, \mathcal{B}_I) . Let \mathcal{U} denote the set of all operators U on $L_2(I)$, such that U is onto and U is isometric, i.e. $(U(f), U(g)) = (f, g)$, $f, g \in L_2(I)$ where (\cdot, \cdot) is the inner-product operation of $L_2(I)$. Such an operator U of $L_2(I)$ is known as *unitary operator*. It is known that with respect to the strong operator topology, i.e., metric of the Banach space of bounded operators on $L_2(I)$, \mathcal{U} is a complete, separable metric space. Each l.m.p. automorphism θ defines a unitary operator $U(\theta) \in \mathcal{U}$ by $(U(\theta)f)(x) = f(\theta(x))$, $f \in L_2(I)$. A Borel space is said to be *standard* if it is Borel isomorphic to the Borel space of a Borel measurable subset of a complete separable metric space. Thus, each $(\hat{\Theta}_n, \mathcal{B}_n, \Gamma_n)$ is standard and hence their countable Cartesian product $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$ is also standard (Mackey [1957, Theorem 3.1]). Notice that for any $\check{\theta} = (\theta_0, \theta_1, \dots, \theta_n, \dots) \in \check{\Theta}$, we have $\{\check{\theta}\} = \lim_{n \rightarrow \infty} \pi_n^{-1}(\theta_n)$. Hence $\check{\Gamma}(\{\check{\theta}\}) = \lim_{n \rightarrow \infty} \check{\Gamma} \pi_n^{-1}(\theta_n) = \lim_{n \rightarrow \infty} \Gamma_n(\{\theta_n\}) = 0$, for all $\check{\theta} \in \check{\Theta}$. Thus,

$(\check{\theta}, \check{\beta}, \check{\Gamma})$ is isomorphic to (I, β_I, λ) (see Parthasarathy [1977, Proposition 26.6]).

Q.E.D.

Let us examine the limitations that are imposed on the pattern of randomization by the projective limit group as compared to the limit group Θ that we discussed earlier. Let us fix a $t \in I$ and suppose $t \in I_1$. Consider the initial segments of player t in each of the random orders $\theta_n \in \check{\Theta}_n$, that has the same value for t , say $\theta_n(t) = t_0$. From figure 2 it is clear that all these random orders place a particular type of sets of Lebesgue measure t_0 before player t ; and the type of the sets depends on t_0 , and the order of the dyadic subdivision, n . For example, suppose $t_0 < 1/2$, then all those random orders do not place any players from the interval $[1/2, 1]$. Thus for a given size t_0 , $0 \leq t_0 \leq 1$, the random orders in $\check{\Theta}$ allow t to form coalition only with certain sets of players of size t_0 but not every (Borel) set of players whose size is t_0 . On the other hand, t gets to have any given player placed before him in a suitable random order and a suitable size t_0 .

For games in which the worth of a coalition depends only through its size but not through any of their other identities, i.e., for anonymous games such for instance as the games in pNA, we expect and we will formally show that the expected marginal contribution of a player with respect to the group of random orders, $(\check{\theta}, \check{\beta}, \check{\Gamma})$, coincides with the axiomatic value for these games as characterized by Aumann and Shapley [1974]. See Theorem 3 for details.

5. Details

For equation (2) to be meaningful, the marginal contribution set function $\phi^\theta V$ should be unique so that it is a function of θ not a correspondence. The following proposition proves this, and provides conditions under which $\phi^\theta V$ exists for all $\theta \in \mathcal{G}$ for a large class of games in BV. The second part of proposition 2 is used in the proof of Theorem 2.

Proposition 2 :

- (i) For a game V in GI and an order \succ_{θ_1} , $\theta \in \mathcal{G}$, if a marginal contribution set function $\phi[V](S, \theta)$, $S \in \mathcal{B}_I$ exists, it is unique.

- (ii) For any $V \in NBV$, and for any $\theta \in \mathcal{G}$, the marginal contribution set function, $\phi^\theta V$ exists and it is countably additive; furthermore, for each $\theta \in \mathcal{G}$, $\phi^\theta : NBV \rightarrow NBV$ is a bounded linear operator in the $\|\cdot\|_{BV}$ norm on NBV and for $V \in NBV$, $\|\phi^\theta V\|_{BV} \leq \|V\|_{BV}$ uniformly for all $\theta \in \mathcal{G}$.

Proof of Proposition 2: Let us denote by $[s, t]_\theta = \{j \in I \mid \theta(s) \leq \theta(j) < \theta(t)\}$. Denote by $\mathcal{D}_\theta = \{[s, t]_\theta \mid s \in I, t \in I\}$. One can easily verify that \mathcal{D}_θ is the smallest Boolean semi algebra containing all initial segments $Z_\theta = \{I(s, \theta) \mid s \in I\}$. Without loss of generality assume that V is monotonic. There is a unique extension of $\phi^\theta V$ from Z_θ to \mathcal{D}_θ such that V is finitely additive on \mathcal{D}_θ and equation (1) is satisfied. More precisely, note that for the initial segments in \mathcal{D}_θ , equation (1) defines $\phi^\theta V$, and for all other sets in \mathcal{D}_θ , there is only one way $\phi^\theta V$ can be defined as follows:

$$(\phi^\theta V)([s, t]_\theta) = V(I(t, \theta)) - V(I(s, \theta)) \text{ for } s \in I, t \in \bar{I}.$$

It is known that such a $\phi^\theta V$ can be uniquely extended to a measure on \mathcal{B}_I (see, for instance, corollary 16.9 in Parthasarathy [1977]).

We now prove part (ii). For any $\theta \in \mathcal{G}$, define the real valued function $F_\theta : \bar{I} \rightarrow \mathcal{R}$ by $F_\theta(x) = V(\theta^{-1}([0, x]))$. Note that for any sequence of real numbers x_n , $n \geq 0$ from \bar{I} such that $x_n \downarrow 0$, we have $\theta^{-1}([0, x_n]) \downarrow \emptyset$ as $n \rightarrow \infty$, and since $V \in NBV$, we have that $F_\theta(x_n) = V(\theta^{-1}([0, x_n])) \rightarrow 0$ as $n \rightarrow \infty$. Similarly, for any sequence of real numbers $x_n \uparrow x$, in \bar{I} , we have $F_\theta(x_n) \rightarrow F_\theta(x)$, hence by Rudin [1966, Theorem 8.14], we know that there exists a unique signed measure λ_θ on \mathcal{B}_I such that

$$\lambda_\theta([0, t]) = F_\theta(t) \quad \forall t \in \bar{I}$$

Taking $t = \theta(s)$, $s \in \bar{I}$, noting that $I(s, \theta) = \theta^{-1}[0, \theta(s))$, and defining the measure $\phi^\theta V$ on \mathcal{B}_I by $(\phi^\theta V) \equiv \lambda_\theta \theta^{-1}$, we have

$$(\phi^\theta V)(I(s, \theta)) = V(I(s, \theta)) \quad \forall s \in \bar{I}$$

Hence there exists a unique (uniqueness follows from part (i) of the proposition) marginal contribution measure for $\theta \in \mathcal{G}$. It is easy to check that $\phi^\theta : NBV \rightarrow NBV$ is linear. Since orders generated by automorphisms are also Aumann-Shapley measurable orders, the rest of the proposition follows from their proposition 12.8.

Q.E.D.

The second part of the above proposition establishes that the marginal contribution set function is a measure. The following proposition shows the algebraic interplay of a game V and the actions of any subgroup of automorphisms Θ in the arguments, θ, V, S , of the marginal contribution measure, $(\phi^\theta V)(S)$. The second part of the proposition provides a computational formula for the marginal contribution measure for a large class of scalar measure valued games. First part of proposition 3 is used in proving the Θ -symmetry of the linear space of games, $L1(\Theta, \Gamma)$, in proposition 4 and the Θ -symmetry of the operator Φ_Γ in theorem 1; the second part of proposition 3 is used to establish the diagonal formula (6) in theorem 3.

Proposition 3 :

- (i) Let Θ be any fixed subgroup of automorphisms in G . Suppose for a game $V \in G_I$, the marginal contribution measure $\phi^\theta V$ exists for all $\theta \in \Theta$. Then for any $\pi \in \Theta$, the marginal contribution measure $\phi^\theta(\pi^*V)$ for the game π^*V also exists for all $\theta \in \Theta$, and it is related to the marginal contribution measure of V by,

$$\phi^\theta(\pi^*V)(S) = (\phi^{\theta\pi}V)(\pi^{-1}(S)), \forall S \in \mathcal{B}_I \quad (8)$$

- (ii) Let $f : I \rightarrow \mathfrak{R}$ be absolutely continuous, and θ be any Lebesgue measure preserving automorphism on I , then the marginal contribution measure of the scalar measure valued game $f \circ \lambda$ is given by:

$$\phi^\theta(f \circ \lambda)(S) = \int_S f'(\theta(t))d\lambda(t) \quad (9)$$

The following lemma will be used in proving proposition 3 and other results:

Lemma 1 : Let $S \subset \mathfrak{R}$, and $\theta : S \rightarrow S$, and $\pi : S \rightarrow S$ be two automorphisms of S . Denote by $I(s, \theta) = \{t \in S \mid \theta(t) < \theta(s)\}$ for an automorphism, θ . Then, $\pi^{-1}(I(s, \theta)) = I(\pi^{-1}(s), \theta\pi)$.

Proof of lemma 1 : It follows from the following equivalent statements.

$$x \in \pi^{-1}(I(s, \theta)) \Leftrightarrow \pi(x) \in I(s, \theta)$$

$$\begin{aligned} &\Leftrightarrow \theta(\pi(x)) < \theta(s) \\ &\Leftrightarrow (\theta\pi)(x) < (\theta\pi)\pi^{-1}(s) \\ &\Leftrightarrow x \in I(\pi^{-1}(s), \theta\pi) \end{aligned}$$

Q.E.D.

Proof of Proposition 3: To prove (i) note that for any $s \in I$,

$$\begin{aligned} (\phi^\theta(\pi^*V))(I(s, \theta)) &= (\pi^*V)(I(s, \theta)) \text{ by definition of } \phi^\theta \\ &= V(\pi^{-1}I(s, \theta)) \text{ by definition of } \pi^* \\ &= V(I(\pi^{-1}(s), \theta\pi)) \text{ by lemma 1} \\ &= (\phi^{\theta\pi}V)(I(\pi^{-1}(s), \theta\pi)) \text{ by definition of } \phi^{\theta\pi} \\ &= ((\phi^{\theta\pi}V)\pi^{-1})(I(s, \theta)) \text{ by lemma 1} \end{aligned}$$

Since they agree on the initial segments in I_θ , they agree on \mathcal{B}_I . Thus the measure $(\phi^\theta \pi^*V)$ exists whenever the measure $(\phi^{\theta\pi}V\pi^{-1})$ exists. Since $\theta\pi \in \Theta$, by hypothesis of the proposition $(\phi^{\theta\pi}V)$ exists, and since $(\phi^{\theta\pi}V)\pi^{-1}$ is a measure whenever $(\phi^{\theta\pi}V)$ is a measure, we conclude that $\phi^\theta(\pi^*V)$ exists for all $\theta \in \Theta$ and is given by the right hand side of equation (8).

Part (ii) of the proposition follows from Raut [1996, Proposition 5].

Q.E.D.

Proposition 4 : $L1(\Theta, \Gamma)$ is a linear symmetric subspace of G_I .

Proof of Proposition 4: It is easy to check that $L1(\Theta, \Gamma)$ is a linear space. We show that it is Θ -symmetric. Let $\pi \in \Theta$, and $V \in L1(\Theta, \Gamma)$. We want to show that $\pi^*V \in L1(\Theta, \Gamma)$. From proposition 3(i), we know that $\phi^\theta(\pi^*V)(S)$ exists for all $\theta \in \Theta$ and $S \in \mathcal{B}_I$ and is given by $(\phi^{\theta\pi}V)(\pi^{-1}(S))$. But since $(\Theta, \mathcal{A}, \Gamma)$ is an invariant probability measurable group, it has the property that for any fixed $\pi \in \Theta$ if $h(\theta)$ is integrable then the right translation of the function $h(\theta\pi)$ is also integrable and both have the same integral. Since $(\phi^{\theta\pi}V)(\pi^{-1}(S))$ is integrable by assumption, it follows that $\pi^*V \in L1(\Theta, \Gamma)$. For further details, see the first part of the proof of theorem 1.

Proof of Theorem 1: It is easy to see that Φ_{Γ} is linear, positive and efficient. We want to show that the right invariance of Γ implies the Θ -symmetry of Φ_{Γ} . Note that

$$\begin{aligned} \Phi_{\Gamma}(\pi^*V)(S) &= \int_{\Theta} \phi(\pi^*V)(S, \theta) d\Gamma(\theta) \\ &= \int_{\Theta} (\phi V)(\pi^{-1}(S), \theta\pi) d\Gamma(\theta), \text{ by proposition 3(i)} \\ &= \int_{\Theta} (\phi V)(\pi^{-1}(S), \theta\pi) d\Gamma(\theta\pi) \text{ since } \mu \text{ is right invariant} \\ &= (\Phi_{\Gamma}V)(\pi^{-1}(S)) \\ &= \pi^*(\Phi_{\Gamma}V)(S), S \in \mathcal{B}_I, \text{ and } V \in L1(\Theta, \Gamma) \end{aligned}$$

Hence, $\Phi_{\Gamma}\pi^* = \pi^*\Phi_{\Gamma}$.

To prove the second part of the theorem, suppose $V \in L1(\Theta, \Gamma) \cap L1(\Theta, \Gamma')$. Then $\phi^{\theta}V$ is measurable with respect to the invariant σ -algebra $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{A}'$, and furthermore, the expected value of the marginal contribution set function $(\phi^{\theta}V)(S)$ will be the same with respect to $(\Theta, \mathcal{A}, \Gamma)$ and $(\Theta, \mathcal{A}_0, \Gamma_0)$ where Γ_0 is the restriction of Γ on \mathcal{A}_0 . Note that \mathcal{A}_0 is an invariant σ -algebra on Θ and we have two probability measures Γ_0 and Γ'_0 which are respectively the restrictions of Γ and Γ' to \mathcal{A}_0 . Hence an application of Halmos [1950, Theorem B, section 60, taking his E to be the whole set Θ], we have that $\Gamma_0(F) = \Gamma'_0(F)$ for all $F \in \mathcal{A}_0$. Thus the expected values of $(\phi^{\theta}V)(S)$ with respect to both invariant probability measurable group structures, $(\Theta, \mathcal{A}, \Gamma)$ and $(\Theta, \mathcal{A}', \Gamma')$ are same.

Q.E.D.

Proof of Theorem 2: Let $V \in \text{NBV}$. Let $S \in \mathcal{B}_I$ be an arbitrarily fixed coalition. We know $(\phi^{\theta}V)(S)$ exists for all $\theta \in \check{\Theta}$. Let us denote by $h(\check{\theta}) \equiv (\phi^{\theta}V)(S)$. We want to show that h is integrable with respect to the invariant probability measurable group structure $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$. To that end, for any $\check{\theta} = (\theta_0, \theta_1, \dots, \theta_n, \theta_{n+1}, \dots) \in \check{\Theta}$, define a sequence $\check{\theta}_n, n \geq 0$ of

elements in $\check{\Theta}$ by $\check{\theta}_n \equiv (\theta_0, \theta_1, \dots, \theta_n, \theta_{n+1}, \dots)$, and for any function $h : \check{\Theta} \rightarrow \mathfrak{R}$, define a sequence of functions, $h_n : \check{\Theta} \rightarrow \mathfrak{R}$ by $h_n(\check{\theta}) \equiv h(\check{\theta}_n)$. It is then clear that $\lim_{n \rightarrow \infty} h_n(\check{\theta}) = h(\check{\theta})$ for all $\check{\theta} \in \check{\Theta}$. It is also clear that $h_n(\check{\theta})$ is $\pi_n^{-1}(\mathcal{B}_n)$ measurable, and hence $h_n(\check{\theta})$ is measurable with respect to $(\check{\Theta}, \check{\mathcal{B}})$ for all $n \geq 0$. Furthermore, we have

$$|h_n(\check{\theta})| = |(\phi^{\check{\theta}_n}V)(S)| \leq \|\phi^{\check{\theta}_n}V\|_{\mathcal{B}V} \leq \|V\|_{\mathcal{B}V}$$

Thus by Lebesgue's dominated convergence theorem, the function $h(\check{\theta})$ which is the point-wise limit of a sequence of measurable functions dominated by a constant is integrable with respect to $(\check{\Theta}, \check{\mathcal{B}}, \check{\Gamma})$.

Q.E.D.

Proof of Theorem 3: Let us first take μ to be the Lebesgue measure λ . Note that

$$\begin{aligned} \Phi_{\Gamma}(f \circ \lambda)(S) &= \int_{\check{\Theta}} (\phi^{\check{\theta}}(f \circ \lambda))(S) d\check{\Gamma}(\check{\theta}) \\ &= \int_{\check{\Theta}} \int_S f'(\check{\theta}(t)) d\lambda(t) d\check{\Gamma}(\check{\theta}), \text{ by (9)} \\ &= \int_S \left[\int_{\check{\Theta}} f'(\check{\theta}(t)) d\check{\Gamma}(\check{\theta}) \right] d\lambda(t), \text{ by Fubini's theorem} \\ &= \int_S \left[\int_0^1 f'(x) d\lambda(x) \right] d\lambda(t), \text{ by applying theorem 5 \& since } f' \in L \\ &= \lambda(S) \int_0^1 f'(x) d\lambda(x) \end{aligned}$$

Thus (4) is true when μ is the Lebesgue measure. Note that for any $\xi \in \mathcal{G}$, $\Phi_{\Gamma}(\xi^*(f \circ \lambda))(S) = \Phi_{\Gamma}(f \circ \lambda)(\xi^{-1}(S)) = \lambda(\xi^{-1}(S)) \int_0^1 f'(x) dx = \xi^*\Phi_{\Gamma}(f \circ \lambda)(S)$. Thus $\check{\Theta}$ -symmetric random order value of a game of the form $(f \circ \lambda)(S)$ is symmetric with respect to the full group of automorphisms, \mathcal{G} .

For a general non-atomic measure μ , we know by the isomorphism theorem of measure theory (see Parthasarathy [1977, proposition 26.6]) that

there exists a $\xi \in \mathcal{G}$ such that $\mu\xi^{-1} = \lambda$. Now note that

$$\begin{aligned} \Phi_{\mathbb{R}}(f \circ \mu)(S) &= \Phi_{\mathbb{R}}\xi^{*-1}\xi^*(f \circ \mu)(S) \\ &= \Phi_{\mathbb{R}}\xi^{*-1}(f \circ \mu\xi^{-1})(S) \\ &= \Phi_{\mathbb{R}}\xi^{*-1}(f \circ \lambda)(S) \\ &= \xi^{*-1}\Phi_{\mathbb{R}}(f \circ \lambda)(S) \\ &= \xi^{*-1}\lambda(S) \int_0^1 f'(x) dx \\ &= \mu(S) \int_0^1 f'(x) dx \end{aligned}$$

Q.E.D.

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