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Asynchronous Evolution of Pairs
How spatial evolution leads to
inequality

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Abstract

We present a simple model of spatial evolution that avoids several problems that arise with more complex networks of players. We consider a world where pairs of players are matched forever. These players learn from the whole population but they are more likely to learn to strategies used by their partners. Thus, several features of spatial evolution are captured while nonlinearities that would arise with more complex networks are avoided. We can identify characteristics of evolution in networks such as stable cooperation in prisoners' dilemma games and long run exploitation among different strategies. We further discuss evolution of repeated game strategies in this framework comparing synchronous models with asynchronous ones.

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1 Introduction

The analysis of evolution of local interaction has become a challenging and fascinating subject during the last years. Unfortunately local evolution leads often to highly nonlinear problems (see Axelrod (1984, p. 158ff) or May and Nowak (1992)). Thus, analysis is often based on simulations involving often several assumptions and parameters.

In this paper we try to model an evolutionary environment that captures some properties of 'real' local interaction but that has more tractability. We try to concentrate on the evolution of pairs of players. We can reproduce features of evolution of networks such as stable cooperation in prisoners' dilemma games and long run survival of strategies with different payoffs (exploitation).

While there are lots and lots of different models of global evolutionary dynamics, most of them have one thing in common: Once a population is in equilibrium all remaining

strategies achieve the same payoff. If strategy a would achieve higher payoff than strategy b then the population share of a would increase, this of course cannot be an equilibrium.

In this paper we want to show how equity is linked to the assumption of global evolution. Maynard Smith and Price (1973) inspired the development of game theory during the last decades. They used the paradigm of randomly matched animals that, if they are equipped with a ‘wrong’ strategy, produce less offspring and die with higher probability than others. Success of a strategy and speed of replication is measured globally. For randomly matched players it is further appropriate to assume that strategies have no memory since repeated interactions are unlikely.

Once we apply evolutionary game theory to behavior of humans or to the behavior of human controlled organizations both the assumption of random matching and consequently the assumption of no memory have to be reconsidered. Let us first interpret evolution of patterns of human interaction as learning. The interpretation of ‘death’ and ‘birth’ of animals is replaced by humans dropping seemingly unsuccessful strategies and adopting more successful ones. Let us assume that players remain in interaction for several periods. Thus, memory matters already at this stage. While players update their strategies, they often remain in uninterrupted interaction with their opponents. Very often such a change of strategy is not announced publicly such that opponents can only deduce from a player’s actions that his strategy might have changed. Thus, players’ memory matters yet *after* an ‘evolutionary step’ (learning). This is not the case in several biological contexts. There after an evolutionary step (death of a player) the relation between this player and his opponent breaks totally apart. Unfortunately this persistence of memory over evolutionary steps forbids separating the evolutionary from the game’s dynamics. Thus, analysis becomes much more complicated.

Further such an update might not — as in a global model — be determined by the state of the whole population but to a higher degree by the state of his environment. Models of spatial interaction often assume that the state of a player’s neighborhood determines his learning behavior. These models¹ may assume highly structured neighborhoods, such as players living on circles while neighborhoods are defined by low distances between players. In these kinds of models, relations between players can be complex. Most players are not neighbors in the sense of the above definition, but still their fates are linked through chains of intermediate neighborhoods. Due to the multitude of different relations between players, evolution of these models is highly nonlinear and hard to analyze.

In section 2.1 we describe a population where only two types of relations between players exist. Players either are interacting with each other or are not interacting with each other. Learning is based on global performance of a strategy but may be biased toward the strategies of one’s interaction partners. In section 2.3 and 2.4 we analyze stability of equilibria in this setting. Section 3 turns to repeated game strategies and presents a formal model of asynchronous learning and interaction. Section 4 compares then for a given game a synchronous dynamic with an asynchronous dynamic. Section 5 draws some conclusions.

¹see, e.g. Axelrod (1984, p. 158ff), Eshel, Samuelson and Shaked (1994), Hegselmann (1994), Kirchkamp (1994), Nowak and May (1992), Routledge (1993), Sakoda (1971), Schelling (1971).

2 Evolution of Pairs

2.1 The model

Imagine a population playing the following symmetric 2×2 game:

		Player II		
		C	D	
Player I	C	$\pi_{a,a}$	$\pi_{b,a}$	(1)
	D	$\pi_{a,b}$	$\pi_{b,b}$	

We will assume all payoffs to be positive.

In the following we will discuss a modified (local) replicator dynamics in terms of *pairs of players*. We assume a population where each player is matched with another player to play the game 1. The resulting pairs are matched *forever* but players may sometimes change their strategy. For convenience of notation we normalize size of the population to 1. In a symmetric 2×2 game there are pairs $\langle a, a \rangle$, $\langle a, b \rangle$ and $\langle b, b \rangle$ that appear in proportions $x_{\langle a, a \rangle}$, $x_{\langle a, b \rangle}$ and $x_{\langle b, b \rangle}$. Total shares of strategies a and b are

$$x_a := 2x_{\langle a, a \rangle} + x_{\langle a, b \rangle} \quad (2)$$

$$x_b := 2x_{\langle b, b \rangle} + x_{\langle a, b \rangle}. \quad (3)$$

Average payoffs of strategies are

$$\pi_a := \frac{2x_{\langle a, a \rangle}\pi_{a,a} + x_{\langle a, b \rangle}\pi_{a,b}}{x_a} \quad (4)$$

$$\pi_b := \frac{2x_{\langle b, b \rangle}\pi_{b,b} + x_{\langle a, b \rangle}\pi_{b,a}}{x_b}. \quad (5)$$

As stated above we will analyze an environment where players are matched to play a game forever and might change their strategy while remaining in uninterrupted interaction with their partner. Evolutionary game theory makes very often the opposite assumption: Players change their strategy less frequently but break up relationships with their partners fairly often. The latter kind of interaction between players is often called ‘random matching’.

Definition 1 (random matching) *A state of a population is randomly matched if for any two distinct strategies a and b the total amount of pairs $\langle a, a \rangle$ is x_a^2 and of pairs $\langle a, b \rangle$ is $2x_a x_b$.*

A common evolutionary population dynamics, ‘standard replicator dynamics’ (as defined by Taylor and Jonker (1978) and Zeeman (1981)), relies on random matching. Therefore we will sometimes refer to ‘standard replicator dynamics’ also as ‘global replicator dynamics’:

Definition 2 (standard (global) replicator dynamics) *If the payoff of strategy a is π_a , its population share x_a and the average payoff of the population $\bar{\pi}$ then the speed of increase dx_a/dt in the population share of strategy a follows a standard replicator dynamic iff there is a positive constant ρ such that*

$$\frac{dx_a}{dt} \equiv \rho \cdot (\pi_a - \bar{\pi})x_a . \quad (6)$$

Global interaction — an alternative interpretation of ‘standard replicator dynamics’:

The biological motivation for replicator dynamics assumes animals that produce offspring proportional to their present share in the population and proportional to their payoff. For social interaction we reinterpret this story. We imagine players sampling randomly other players from the population. Players then switch to the observed players’ strategy proportional to their respective success.

Assume that during a period of length Δt a player with strategy e.g. a has with probability $\rho' \Delta t$ the opportunity to observe a randomly sampled player, his strategy and his payoff. Since there are x_b players with strategy b in the population which has size one, an observing player a observes a b with probability x_b . We assume that the probability to adopt the observed strategy is proportional to the observed payoff. Of course the payoff of the observed strategy b need not be the average payoff π_b . Some switching as might observe a b with a high payoff, others might observe bs with low payoffs. But if all observing players adopt the observed strategy proportional to its payoff π (say with probability $\rho'' \pi$) then the total probability that an a becomes a b can be expressed as $\rho \Delta t x_b \pi_b$ (where $\rho := \rho' \rho''$). Since there are x_a many a players the total flow from a to b is $\rho \Delta t x_a x_b \pi_b$.

Similarly we can derive the flow from b to a as $\rho \Delta t x_a x_b \pi_a$. Thus the total change in the amount of a -playing players is

$$\Delta x_a = \rho \Delta t x_b x_a (\pi_a - \pi_b) . \quad (7)$$

Let us now move gradually from a model where time is discrete to a model where time is continuous. Dividing by Δt and taking the limit $\Delta t \rightarrow 0$ gives the time derivative.

$$\frac{dx_a}{dt} = \rho \cdot (\pi_a - \pi_b) x_b x_a \equiv \rho \cdot (\pi_a - \bar{\pi}) x_a \quad (8)$$

Now compare equation 8 with equation 6. Both dynamics are equivalent. We see that the story of sampling and switching players is one possible interpretation of standard replicator dynamics.

Local interaction — evolution of pairs: We model *local interaction* assuming that players are not randomly matched each period again but that they are matched forever. As with the global model above let us again consider the discrete time model, but now focus on the dynamics of pairs.

Consider a pair $\langle a, a \rangle$. Both members of the pair have the possibility to become a b . Thus a pair $\langle a, a \rangle$ changes with probability $2\rho \Delta t \cdot (1 - \rho' \Delta t) x_b \pi_b$ to become a pair

$\langle a, b \rangle$ and with probability $\rho^2 \Delta t^2 x_b^2 \pi_b^2$ to become a $\langle b, b \rangle$. Similarly we can also derive the dynamics for $\langle a, b \rangle$ and $\langle b, b \rangle$ and thus calculate $\Delta x_{\langle a, a \rangle}$, $\Delta x_{\langle a, b \rangle}$ and $\Delta x_{\langle b, b \rangle}$.

Now we move again from the discrete model to a model where time is continuous. Divide $\Delta x_{\langle a, a \rangle}$, $\Delta x_{\langle a, b \rangle}$ and $\Delta x_{\langle b, b \rangle}$ by $2\Delta t$ and take the limit $\Delta t \rightarrow 0$ to obtain the time derivative.

$$\begin{pmatrix} \dot{x}_{\langle a, a \rangle} \\ \dot{x}_{\langle a, b \rangle} \\ \dot{x}_{\langle b, b \rangle} \end{pmatrix} = \begin{pmatrix} -x_b \pi_b & \frac{1}{2} x_a \pi_a & 0 \\ x_b \pi_b & -\frac{1}{2} (x_a \pi_a + x_b \pi_b) & x_a \pi_a \\ 0 & \frac{1}{2} x_b \pi_b & -x_a \pi_a \end{pmatrix} \cdot \begin{pmatrix} x_{\langle a, a \rangle} \\ x_{\langle a, b \rangle} \\ x_{\langle b, b \rangle} \end{pmatrix} \rho. \quad (9)$$

Global evolution: Notice that above we implicitly assumed that players are *equally likely* to learn either to a strategy they already know from their interaction pair or to any other strategy.

We call this equal likelihood to learn to a foreign or nearby (chosen from the interaction partner) strategy ‘global evolution’.

Local evolution: We propose to model *local* evolution as a higher probability to switch to strategies that are already known from the interaction partner. Thus a pair $\langle a, a \rangle$ still becomes an $\langle a, b \rangle$ with probability $2\rho \Delta t \cdot (1 - \rho' \Delta t) x_b \pi_b$.

But now we assume that the b -player in a pair $\langle a, b \rangle$ becomes not only with probability $\rho \Delta t \cdot (1 - \rho' \Delta t) x_a \pi_a$ an a and thus makes the pair an $\langle a, a \rangle$ but with a higher probability $2e \rho \Delta t \cdot (1 - \rho' \Delta t) x_a \pi_a$ where $e \geq \frac{1}{2}$. We argue that the former b -player might be more ready to switch to the ‘nearby’ strategy a that he already knows from his interaction partner, more ready to switch at least than the members of an $\langle a, a \rangle$ are if they are to switch to b which is not present in their environment. Thus we characterize local evolution as

$$\begin{pmatrix} \dot{x}_{\langle a, a \rangle} \\ \dot{x}_{\langle a, b \rangle} \\ \dot{x}_{\langle b, b \rangle} \end{pmatrix} = \begin{pmatrix} -x_b \pi_b & e x_a \pi_a & 0 \\ x_b \pi_b & -e \cdot (x_a \pi_a + x_b \pi_b) & x_a \pi_a \\ 0 & e x_b \pi_b & -x_a \pi_a \end{pmatrix} \cdot \begin{pmatrix} x_{\langle a, a \rangle} \\ x_{\langle a, b \rangle} \\ x_{\langle b, b \rangle} \end{pmatrix} \rho. \quad (10)$$

with $e \geq 1/2$. The global case is approached as $e = 1/2$.

Comparison with standard replicator dynamics: To compare the above dynamics (which is written in terms of pairs) with standard (global) replicator dynamics (which is expressed in strategies) let us first focus on the relation between local dynamics and random matching:

With random matching the state of the population is only determined by the total amount of a and b players. In a model where players are matched forever in pairs, the state of the population is described by the respective amount of pairs $x_{\langle a, a \rangle}$, $x_{\langle a, b \rangle}$ and $x_{\langle b, b \rangle}$. Nonetheless it turns out that the dynamics of a randomly matched population are reflected in the permanently matched population.

The left part of figure 1 shows the possible states of the population that can be achieved through random matching in the simplex of pairs.

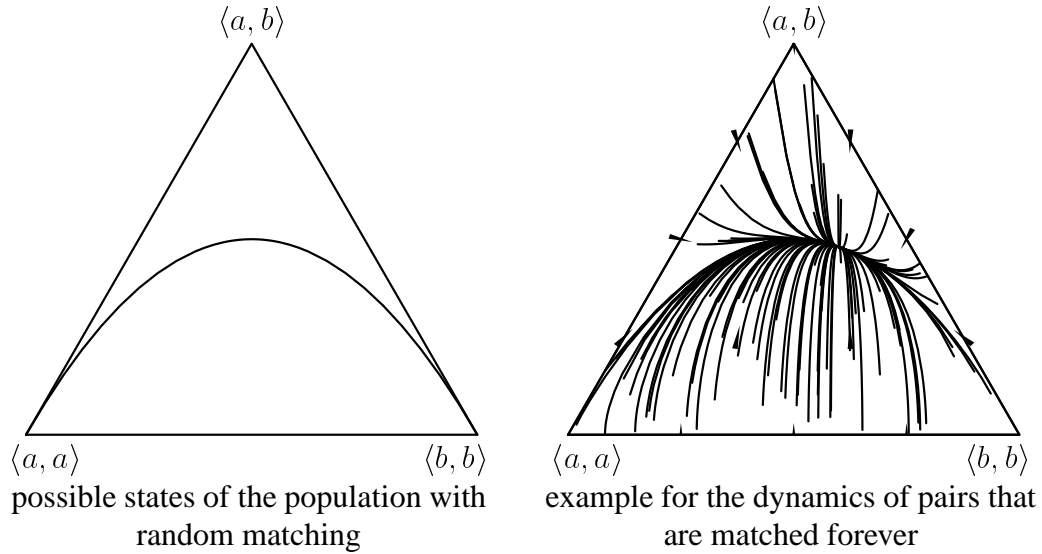


Figure 1: Global learning of permanently matched pairs leads to a randomly matched population.

The right part of figure 1 shows the dynamics of a permanently matched population with global learning ($e = 1/2$).²

While a population that is matched forever might start at a state that can never be attained with random matching (e.g. to have only pairs $\langle a, b \rangle$ is excluded by random matching) the learning dynamics moves always to a state that could also be attained with random matching. The population which is permanently matched behaves as if it were randomly matched since the learning process is still driven by random sampling.

Once such a state is achieved the learning dynamics of a population matched forever are equivalent to the dynamics of a randomly matched population. We summarize these findings in the following two propositions:

Proposition 1 (Convergence to random matching) *A completely mixed population of permanently matched pairs converges under the dynamics of equation 9 to a state of a population of randomly matched players.*

For a proof see appendix A on page 22.

Proposition 2 *For a randomly matched population equation 9 describes a standard replicator dynamics.*

The proof follows immediately from assuming random matching ($x_{\langle a, a \rangle} = x_a^2$, $x_{\langle a, b \rangle} = 2x_a \cdot (1 - x_a)$, $x_{\langle b, b \rangle} = (1 - x_a)^2$) and summing up the dynamics of equation 9 on page 5 to \dot{x}_a and \dot{x}_b .

The above two propositions lead to the following corollary:

²The precise game is given as game 14 on page 13.

Corollary 3 (Equivalence with standard replicator dynamics) *A completely mixed population of permanently matched pairs converges under the dynamics of equation 9 to the behavior of a population of randomly matched pairs under standard replicator dynamics.*

No equivalence between random matching and permanent matching in the case of local evolution: Can we find a similar property for local evolution, i.e., for the dynamics given in equation 10 with $e > 1/2$? Perhaps we can find a special matching technology that, even if it does not meet the requirements of random matching given in definition 1 on page 3, still is a replicator dynamic on the level of total amounts of strategies x_a and x_b . If we compare equation 6 and 10 it turns out that at least sometimes such a matching technology exists. Unfortunately these matchings are not stable. Not only our learning dynamics do not converge to such a matching, even if we would start with such a matching learning dynamics would immediately move away to a different matching that no longer leads to standard replicator dynamics.

Proposition 4 *For $e \neq \frac{1}{2}$ there is no stable matching scheme such that the local dynamics given in 10 on page 5 is a standard replicator dynamics for all generic games.*

Thus, standard replicator dynamics can only be used in an environment with global evolution.

2.2 Equality

Equality is a pleasant and standard byproduct of global evolutionary models. All strategies that survive in the long run under (global) replicator dynamics yield the same payoff. This is no longer true with local dynamics. Assume that a population is in a mixed state, i.e., $x_{\langle a,a \rangle}, x_{\langle a,b \rangle}, x_{\langle b,b \rangle}$ are all positive. If this state is a fixed point of the local dynamic defined in equation 10 on page 5 it must be in the null space of the transition matrix from equation 10 on page 5. Take $x_a \pi_a$ and $x_b \pi_b$ as given then the nullspace consists of a single vector. Thus, we can calculate x_a and x_b and then derive the payoff difference of the two strategies

$$\pi_a - \pi_b = 2(-1 + 2c)(-x_a \pi_a + x_b \pi_b) \frac{(ex_a \pi_a^2 + x_a \pi_a x_b \pi_b + ex_b \pi_b^2)}{(2ex_a \pi_a + x_b \pi_b)(x_a \pi_a + 2cx_b \pi_b)} \quad (11)$$

Since the fraction in the above equation is always positive the following proposition follows immediately:

Proposition 5 *In the case of local dynamics ($e > \frac{1}{2}$) a strategy that has a larger than an average population share in equilibrium has less than average payoff. Equal payoffs require equal population shares.*

Notice that equal payoffs constitute almost never an equilibrium.

Corollary 6 *Generically equality of equilibrium payoffs implies global dynamics (i.e., $e = \frac{1}{2}$).*

2.3 Stability of pure strategies

Any pure strategy is a fixed point of dynamics defined in equation 10 on page 5. If there are only pairs $\langle a, a \rangle$ in the population, there is no way that a player learns to become a b . In this section we discuss stability of these kinds of fixed points. Assume that players play a symmetric $n \times n$ game. Consider the state of the population where each pair plays $\langle a, a \rangle$. Then we can calculate the eigenvalues of the jacobian of the above dynamics. We know that an asymptotically stable point of the dynamics has the property that the real parts of all eigenvalues are negative. Thus, we derive the following stability condition:

Proposition 7 (asymptotic stability of pure strategies) *In a symmetric $n \times n$ game a state of the population where each pair of players plays $\langle a, a \rangle$ is asymptotically stable if all deviation payoffs $(\pi_{b,a})$ are smaller than $2e$ times the equilibrium payoff $(\pi_{a,a})$.*

For the details of the proof see appendix B on page 23.

Notice that asymptotic stability of a population playing only $\langle a, a \rangle$ coincides with $\langle a, a \rangle$ being a Nash equilibrium if $e = 1/2$, i.e., again for the global case.

Notice further that *any* pure strategy can be the only one played in an asymptotically stable state if locality (e) is only strong enough.

An example — a prisoners’ dilemma: To illustrate the above proposition, let us consider the following prisoners’ dilemma.

		Player II		
		a	b	
Player I	a	3	4	(12)
	b	1	2	
		4	2	

Figure 2 on the following page shows the dynamics in the simplex $\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle$ for various values of e . Global dynamics is equivalent to $e = \frac{1}{2}$. Here only the (noncooperative) Nash equilibrium $\langle b, b \rangle$ remains stable. Now let us increase the parameter for locality gradually. A critical point is $e = \frac{2}{3}$. For $e > \frac{2}{3}$ the cooperative pair $\langle a, a \rangle$ becomes a second asymptotically stable equilibrium. The ‘irrational’ cooperative outcome can be justified if any pure pair evolves with the ‘normal speed’ $\frac{1}{2}$ while any mixed pair evolves with a slightly advanced speed larger than $\frac{2}{3}$.

Why does more locality stabilize an otherwise unstable outcome? To discuss this point let us come back to global dynamics as described in equation 6 on page 4. Assume that most of the population uses strategy a while simultaneously b yields more payoff. Global dynamics can be split into two parts. First a large population is more ‘attractive’. A member of the small b population is more likely to sample an a than a b . If payoff differences are small then any player that actually samples another player is more likely to sample an a (and thus switch to a) if the population share of a is only large enough. Second a large population is more ‘leaky’. With a large fraction of a most of the players that sample another player are actually as . With global dynamics these two effects cancel

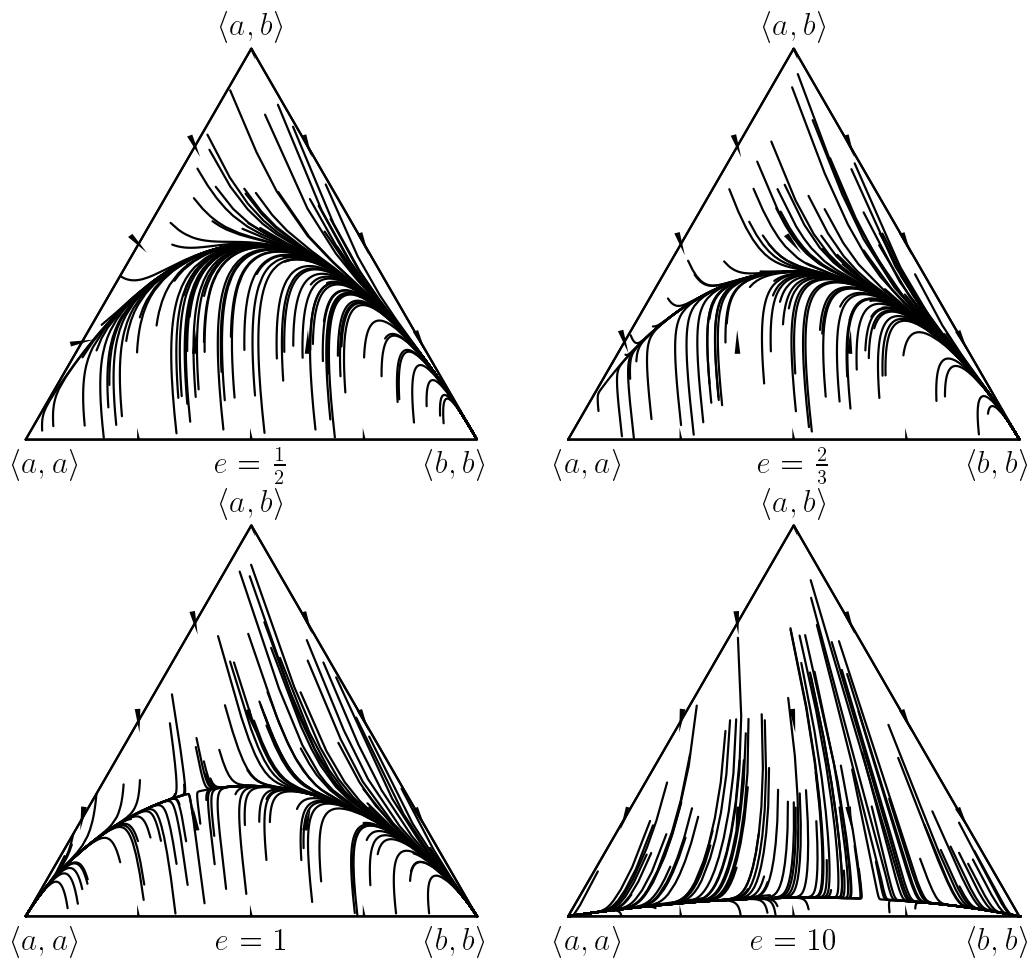


Figure 2: Dynamics in a prisoners' dilemma for various values of e .

and only payoff differences determine whether a strategy increases its population share or whether it decreases.

With local evolution dynamics change. Assume that locality (e) is very large. Then mainly mixed pairs change their state. Thus, even if a population contains mostly users of strategy a , dynamics are determined mostly by mixed pairs $\langle a, b \rangle$. Therefore ‘leakiness’ of the predominant strategy a can be arbitrary small if e is only large enough. On the other hand ‘attractiveness’ of a does not necessarily diminish.

The prisoners’ dilemma example shows that with large e mixed pairs $\langle a, b \rangle$ become more and more unstable, their number shrinks and average payoffs of strategies a and a are mainly determined by the success of ‘pure pairs’ $\langle a, a \rangle$ and $\langle b, b \rangle$. Since the $\langle a, a \rangle$ s cooperate with each other, the average payoff of an a is now higher than the payoff of the mutually defecting $\langle b, b \rangle$. This explains the relatively large basin of attraction of $\langle a, a \rangle$ for large values of e .

Comparison with cellular automata: Let us compare the findings so far with the following simulation result. We did, in a spatial framework similar to the one used by Axelrod ((Axelrod 1984, p. 158ff) or Nowak and May (1992) (for a detailed description how we implemented this framework see Kirchkamp (1994)), simulations on a torus of size 21×21 where players interacted with their 8 surrounding neighbors but learned from a possibly different neighborhood. Figure 3 on page 12 shows the amount of cooperation for various games. Games are parametrized by payoffs g and h with the underlying game

		Player II		
			C	
Player I	C	g	1	(13)
	D	h	0	

Each of the ten pictures in figure 3 on page 12 shows the result of 800 simulations, each of them with a different game and initial configuration. The amount of cooperation in a population after 2000 rounds is proportional to the areas of the black circles whose positions indicate the payoffs g and h of the respective games. The absence of circles indicates the absence of cooperation for the respective simulation. In each of the ten rounds we chose a different learning radius. A learning radius r_l means that players learn from those neighbors whose horizontal and whose vertical distance is both less or equal r_l . We see that with increasing size of the learning neighborhood the range of games where cooperation persists decreases. We interpret an environment with a large learning radius (players learn from almost the whole population) as less local than an environment with a small learning radius. Thus, more ‘locality’ involves more cooperation. And this is what we have found above. Given the dynamics defined in equation 10 on page 5 a pair of strategies $\langle a, a \rangle$ is asymptotically stable in the game 13 on the page before if $2eg > 1$. Thus, with increasing e the range of prisoners’ dilemmas where cooperation is asymptotically stable becomes larger. Here we establish in a simple model a property that

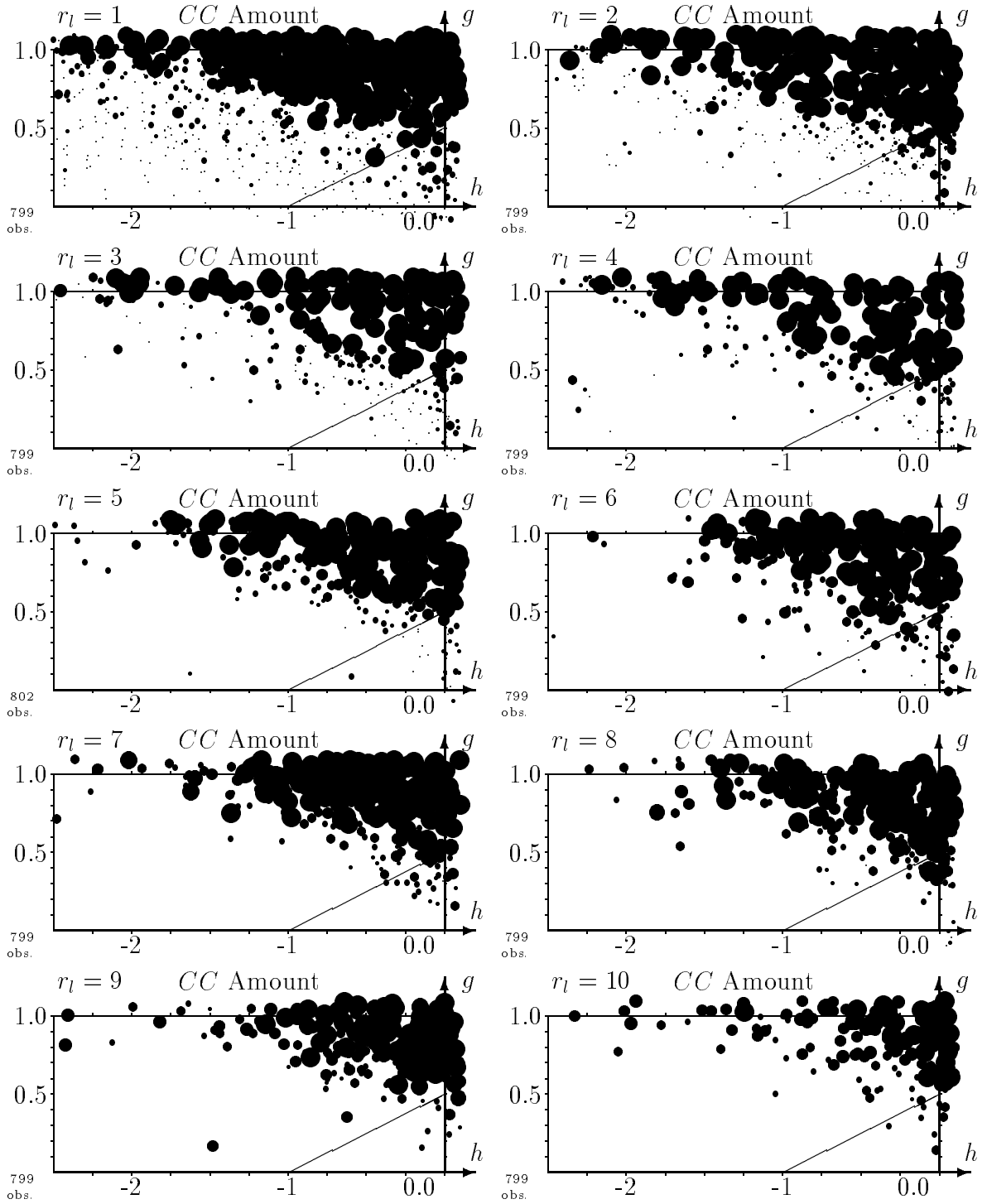


Figure 3: The influence of local learning on the amount of cooperation.

is both qualitatively equivalent to findings of more elaborated and complicated simulations and that is further a standard feature in the context of social conflicts.

2.4 Mixed equilibria

Proposition 8 *For 2×2 games the dynamics given in equation 10 have no more than one interior fixed point.*

Proof: Equation 10 on page 5 defines the dynamics of local evolution with three equations, all of them are polynomials of degree two. One of the equations is redundant. Looking for fixed points of the dynamic means looking for zeroes of the remaining two equations. We construct the resultant of these two polynomials that is a polynomial of degree four with four (possibly complex) zeroes. Two of these zeroes are the pure fixed points ($\langle a, a \rangle$ and $\langle b, b \rangle$) of the dynamic and can be eliminated, yielding a polynomial of degree two with now only two zeroes. One of these zeroes can be shown to be never in the strategy simplex — therefore we remain with one possible candidate for a mixed fixed point. ■

Proposition 9 *An interior fixed point given by the dynamics in equation 10 exists and is asymptotically stable iff $\frac{1}{2} < e < \min \left\{ \frac{\pi_{b,a}}{2\pi_{a,a}}, \frac{\pi_{a,b}}{2\pi_{b,b}} \right\}$.*

The above proposition follows again from the negativity of the eigenvalues of the jacobian of the dynamic.

Proposition 10 *The interval given in proposition 9 is nonempty if the game has only one symmetric Nash equilibrium that is mixed.*

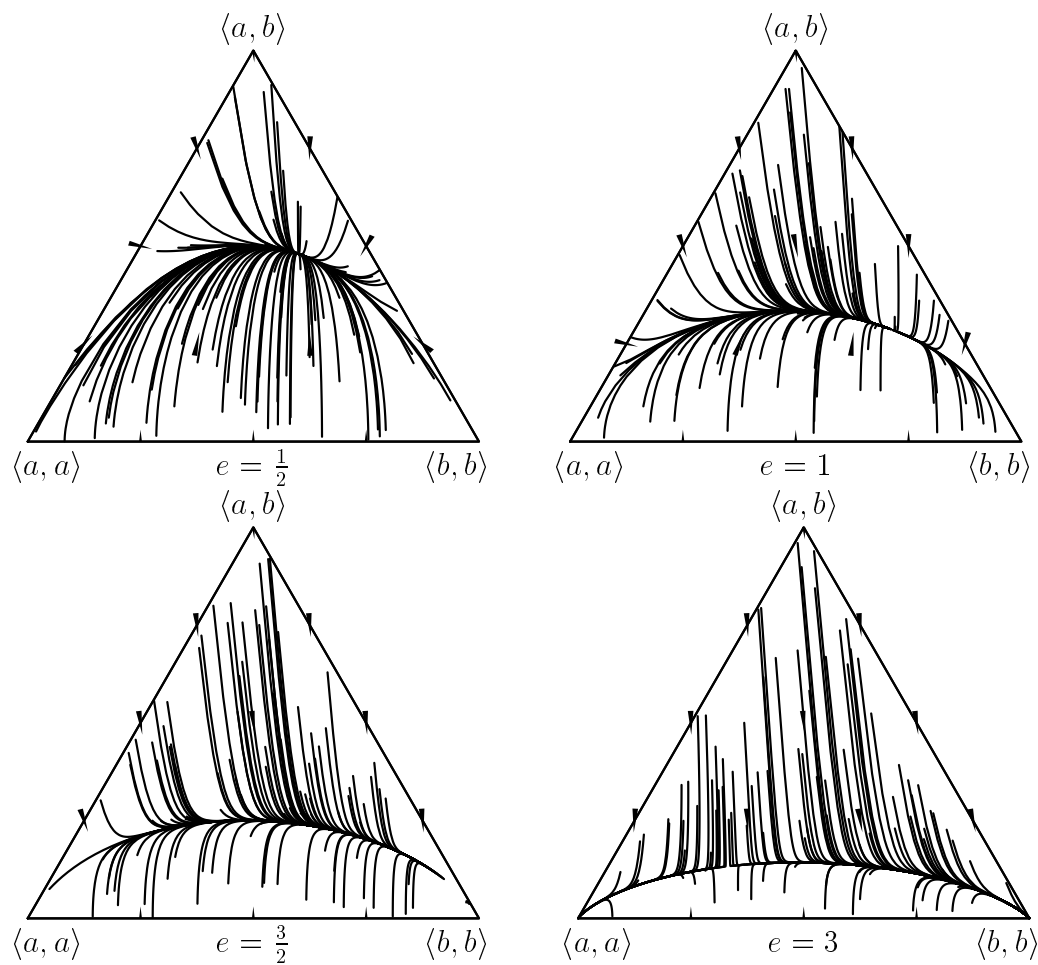
The motivation for proposition 10 is obvious: We are considering a symmetric 2×2 game. The asymptotic stability of a mixed fixed point under standard (global) replicator dynamic is generically equivalent to the lack of pure symmetric Nash equilibria. As standard replicator dynamic is a special case of the (local) dynamic given in equation 9, we know that for this case the stability conditions must coincide.

With increasing locality e (inside this interval) the mixed fixed point moves gradually from the mixed Nash equilibrium to $x_{\langle a,a \rangle}$ if $\frac{\pi_{b,a}}{2e\pi_{a,a}} < \frac{\pi_{a,b}}{2e\pi_{b,b}}$ and to $x_{\langle b,b \rangle}$ otherwise.

To give an example for the evolution of mixed equilibria consider an asymmetric battle of the sexes given in game 14

		Player II		
		a	b	
Player I	a	1	4	(14)
	b	3	1	

Figure 4 shows the dynamics in the simplex $\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle$ for various values of e . In the global case ($e = \frac{1}{2}$) only the mixed fixed point (which coincides with the Nash



1

Figure 4: Dynamics in an asymmetric battle of the sexes game for various values of e

equilibrium) is asymptotically stable. With increasing locality e this mixed fixed point is given by the dynamics of equation 10 on page 5 such that pairs $\langle a, a \rangle$, $\langle a, b \rangle$, $\langle b, b \rangle$ appear in proportions $e \cdot (2e - 3)^2 : (2e - 3)(2e - 4) : e \cdot (2e - 4)^2$. For growing e the equilibrium proportion of a decreases to zero as $e \rightarrow \frac{3}{2}$. Notice that for $e \in (\frac{1}{2}, \frac{3}{2})$ payoffs for a and b differ: $\pi_a = (e^2 - 3)/(e^2 - e - 1)$ while $\pi_b = 4(e^2 - 3)/(4e^2 - 6e - 3)$. This difference increases with increasing locality e .

For $e > \frac{3}{2}$ the mixed fixed point disappears and $\langle b, b \rangle$ remains the only asymptotically stable point until $e = 2$. For $e > 2$ both $\langle a, a \rangle$ and $\langle b, b \rangle$ are asymptotically stable. The mixed fixed point that appears for $e > 2$ is unstable.

3 Repeated game strategies and asynchronous learning

So far we have always assumed that players use stage game strategies. We have used this assumption in a model where players are matched forever. Since the general idea of the model is one of players who for whatever reason do not think deeply to find clever strategies in games, we might argue that stage game strategies fit perfectly well into this context. Nevertheless there might still be some simple repeated game strategies that could be used even by an unsophisticated player.

In the remainder of the paper we will give an example of how to apply the above model to a more complex problem — modelling the evolution of repeated game strategies. We describe how to find conditions for equilibria of an evolutionary process if the space of repeated game strategies can be restricted to a finite number of automata. Even with this restriction in mind the analysis becomes substantially more complex than the analysis of stage game strategies.

We consider a continuous population of size 1 where all players are matched in pairs to play a repeated symmetric game. The stage game has stage game strategies $s \in \mathcal{S}$. If a player uses stage game strategy s against s' then he receives a payoff of $\pi_{s,s'}$.

Automata For simplicity we describe players' strategies as finite automata a that are selected from a set of finite automata \mathcal{A} . Repeated game strategies are represented as automata.

An automaton a can be in different states $n \in \mathcal{N}(a)$. We will denote an automaton a in state n with the symbol a_n . A state determines (1) the stage game strategy s_n^a to play and (2) a 'transition function' $t_n^a(\cdot)$ that describes how to react on the opponent's choice of stage game strategy s' , i.e., to move to a possibly different state $t_n^a(s') \in \mathcal{N}(a)$ if the opponent used stage game strategy s' . Further an automaton has an 'initial state' a_0 that is the state that the automaton uses for its first interaction. A complete description of an automaton is thus given by an initial state a_0 and a vector of states $(\dots, (s_n^a, t_n^a(\cdot)), \dots)$.

Delegates We also will identify the combination of an automaton a plus a state n as a 'delegate' a_n . Each player controls one delegate³. Games are played by pairs of delegates $\langle a_n, b_m \rangle$. The set of all possible delegates will be denoted by $\mathcal{D} := \{c_k | c \in \mathcal{A} \wedge k \in \mathcal{N}(c)\}$

³In a setting where each player has several opponents he controls of course more delegates, each of them using the same automaton, but not necessarily in the same state.

Order of delegates Notice that due to symmetry of the game there is no a priori difference between the pair $\langle a_n, b_m \rangle$ and the symmetric pair $\langle b_m, a_n \rangle$. Since we do not *want* to make any distinction between such two pairs and for convenience of notation we make the following assumption:

Assumption 1 (notation of pairs of delegates) *The set of delegates is completely ordered by \prec such that for any two delegates a_n and b_m the following holds:*

- if $a \neq b$ then either $\forall_{\substack{n \in \mathcal{N}(a) \\ m \in \mathcal{N}(b)}}} a_n \prec b_m$ or $\forall_{\substack{n \in \mathcal{N}(a) \\ m \in \mathcal{N}(b)}}} a_n \succ b_m$.
- if $a = b$ then either $a_n \prec b_m$ or $n = m$ or $a_n \succ b_m$.

We further write $a_n \preceq b_m$ iff either $a_n \prec b_m$ or $a_n = b_m$.

We assume that the ‘higher order’ automaton is always the ‘second player’, i.e., whenever we look at a pair $\langle a_n, b_m \rangle$ we do it such that $a_n \preceq b_m$.

State of the Population We describe the proportion of players using automata of type a as x_a with $\sum_{a \in \mathcal{A}} x_a = 1$.

Players that use automata of type a might face opponents that use various automata. Let us denote the number of players that use automata of type $a \in \mathcal{A}$ and face a user of $b \in \mathcal{A}$ as opponent with $x_{\langle a, b \rangle}$. Notice that assumption 1 implies $a \succ b \Rightarrow x_{\langle a, b \rangle} = 0$. Notice further that $\sum_{b \in \mathcal{A}} x_{\langle a, b \rangle} \equiv x_a$.

Let us finally denote the number of pairs of delegates where one delegate uses an automaton of type a in state $n \in \mathcal{N}(a)$ while the other delegate uses b in state $m \in \mathcal{N}(b)$ with $x_{\langle a_n, b_m \rangle}$. Of course $\sum_{n \in \mathcal{N}(a)} \sum_{m \in \mathcal{N}(b)} x_{\langle a_n, b_m \rangle} \equiv x_{\langle a, b \rangle}$.

Transitions When a pair of delegates $\langle a_n, b_m \rangle$ plays a game then after playing each delegate applies the transition function of his automaton depending on his own state and his opponent’s strategy and switches to a possibly new state. The pair changes from $\langle a_n, b_m \rangle$ to $\langle a_{t_n^a(s_m^b)}, b_{t_m^b(s_n^a)} \rangle$.

Thus, if each period a fraction i of the pairs actually interact, a given state of the population x changes due to transitions each period such that:

$$\begin{aligned} \Delta_1 x_{\langle a_n, b_m \rangle} = & -i \cdot x_{\langle a_n, b_m \rangle} + \\ & i \cdot \sum_{\substack{\forall a_{n'} \preceq b_{m'} \text{ s.t. } a_n = a_{t_{n'}^a(s_{m'}^b)} \wedge b_m = b_{t_{m'}^b(s_{n'}^a)} \\ \forall a_{n'} \prec b_{m'} \text{ s.t. } a_n = b_{t_{m'}^b(s_{n'}^a)} \wedge b_m = a_{t_{n'}^a(s_{m'}^b)}}} x_{\langle a_{n'}, b_{m'} \rangle} \end{aligned} \quad (15)$$

The first line of equation 15 on the page before takes those pairs a_n, b_m into account that interact and therefore might change their state and thus decrease the value of $x_{\langle a_n, b_m \rangle}$. The second line describes the increase in the value of $x_{\langle a_n, b_m \rangle}$ due to pairs $\langle a_{n'}, b_{m'} \rangle$ that interact, possibly change their state and thus become a pair $\langle a_n, b_m \rangle$. Here two cases can be distinguished:

Assume we analyze the change in the value of $x_{\langle a_5, b_5 \rangle}$. Assume a pair, e.g. $\langle a_3, b_4 \rangle$ that changes after interaction to $\langle a_5, b_5 \rangle$. We know that each round $x_{\langle a_5, b_5 \rangle}$ increases by

$i \cdot x_{\langle a_3, b_5 \rangle}$. This is an example for a transition of pairs $\langle a_{n'}, b_{m'} \rangle$ to $\langle a_n, b_m \rangle$ which is captured by the top line under the sum.

To give an example for the second case, suppose that we analyze the change in the value of $x_{\langle a_4, a_6 \rangle}$. Assume further that the ordering on delegates fulfills $a_3 \prec a_4 \prec a_5 \prec a_6$. Imagine that a delegate a_3 when playing against an a_6 changes its state such that it becomes an a_5 . Assume further that a_6 when playing against an a_3 changes its state such that it becomes an a_4 . Thus, the pair $\langle a_3, a_6 \rangle$ becomes after transition a pair $\langle a_4, a_5 \rangle$. Notice that by assumption 1 on the preceding page our view of the pair has ‘turned’. The bottom line under the sum in equation 15 on the page before captures the effect of ‘turning’ pairs.

Payoffs In each round delegates realize stage game payoffs $\pi_{s_n^a, s_m^b}$ that depend on their own stage game strategy s_n^a and their opponent’s stage game strategy s_m^b .

Below we will assume that average payoffs of automata control their dynamic development of strategies. We define the average payoff of all automata of type a as

$$\pi_a := \frac{1}{x_a} \sum_{b \in \mathcal{A}} \sum_{n \in \mathcal{N}(a)} \sum_{m \in \mathcal{N}(b)} x_{\langle a_n, b_m \rangle} \cdot \pi_{s_n^a, s_m^b} \quad (16)$$

Learning We still use the local variant of *standard replicator dynamics* as described in equation 10 on page 5 for the case of two strategies.

Death of delegates and their birth in the initial state What is the equivalent of a strategy in this model? Should we interpret strategies in equation 6 on page 4 as automata or as delegates? In the following we argue that the birth part $\pi_a x_a$ should apply to automata while the death part $-\bar{\pi} x_a$ should apply to delegates. If a player learns a certain strategy, he does not only learn to use it in a certain state, he learns the complete automaton, *including* its initial state. We think that learning the initial state of an automaton and thus starting next period in this initial state is a consistent interpretation of the definition of an automaton. Lots of automata are similar and can be distinguished only by differences in their initial states. These differences might have influence on success or failure of the automata. Therefore if a certain automaton is successful one factor for its success might have been its initial state. The death part $-\bar{\pi} x_a$ on the other hand applies naturally to all delegates, not only the one where the initial state is used.

We start therefore with the following variant of the replicator dynamics:

$$\frac{dx_{a_n}}{dt} \sim \begin{cases} (\pi_a x_a - \bar{\pi} x_{a_n}) & \text{if } n = a_0 \\ -\bar{\pi} x_{a_n} & \text{otherwise} \end{cases} \quad (17)$$

Proposition 11 *On the level of automata equation 17 on the preceding page describes a standard (global) replicator dynamic as defined in equation 6 on page 4.*

The above proposition follows quickly from summing up over all $n \in \mathcal{N}(a)$.

Quicker learning from partners, slower learning from others As above we neglect here almost all aspects of a player's individual situation. The aspect that we consider is a player's situation as part of a pair. Only the type of his opponent matters. We still assume that the opponent's type determines a player's learning behavior.

Notice that — as argued above — speed of learning to a certain automaton might depend on the composition of a pair. We will say that a player that is a member of the pair $\langle a, b \rangle$ learns with speed $\sigma_{\langle a, b \rangle, c}$ to an automaton c . Of course the probability to become a c is further determined by c 's frequency and payoff.

With standard (global) replicator dynamic of course $\sigma_{\langle a, b \rangle, c}$ is a constant for all a, b, c . In contrast we assume here that learning from interaction partners is carried out with a comparatively higher speed e .

$$\sigma_{\langle a, b \rangle, c} = \begin{cases} e \geq \frac{1}{2} & \text{if } c \in \{a, b\} \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad (18)$$

Evolution of pairs Equation 17 describes the effect of a learning dynamic on the proportion of certain types of delegates. As interaction dynamics are not determined by proportions of delegates, but by proportions of pairs, we have still to distribute the dynamics of equation 17 on pairs.

Let us now split the learning dynamics into several pieces, first considering a discrete time interval of length ϵ and then take the limit $\epsilon \rightarrow 0$.

Take any pair of delegates $\langle a_n, b_m \rangle$. Let us first look at the *decrease* in the proportion of this pair. One delegate among this pair might learn to a target c_0 . This effect is proportional to the proportion of this pair in the society $x_{\langle a_n, b_m \rangle}$, proportional to the success of the target π_c weighted with the proportion of the target x_c and proportional to the 'speed' $\sigma_{\langle a, b \rangle, c}$ that depends on whether c is already present in $\langle a, b \rangle$. We sum this up over all possible targets and weight it with the length of the period ϵ . Notice that both a_n and b_m can switch — therefore we have to consider the double amount of change in proportion.

$$\Delta_1 x_{\langle a_n, b_m \rangle} = -2\epsilon x_{\langle a_n, b_m \rangle} \sum_{c \in \mathcal{A}} \sigma_{\langle a, b \rangle, c} x_c \pi_c \quad (19)$$

Further both delegates of the pair $\langle a_n, b_m \rangle$ might learn simultaneously to a target $\langle c_k, d_l \rangle$. This happens of course only with probability proportional to $2x_{\langle a_n, b_m \rangle} \epsilon^2$ (the 2 is explained by the fact that a_n might learn c_k and b_m might learn to d_l but also vice versa). Further this event is proportional to the weighted success of their targets $x_c \pi_c x_d \pi_d$ and proportional to the speed $\sigma_{\langle a, b \rangle, c}$ and $\sigma_{\langle a, b \rangle, d}$ respectively.

$$\Delta_2 x_{\langle a_n, b_m \rangle} = -2\epsilon^2 x_{\langle a_n, b_m \rangle} \sum_{c \in \mathcal{A}} \sum_{d \in \mathcal{A}} \sigma_{\langle a, b \rangle, c} x_c \pi_c \sigma_{\langle a, b \rangle, d} x_d \pi_d \quad (20)$$

Let us now turn to learning events that lead to an *increase* in the number of the pair $\langle a_n, b_m \rangle$. Somebody could start as a $\langle c_k, b_m \rangle$ or $\langle b_m, c_k \rangle$ and replace the c_k -part of this pair by an a_n . This happens of course only if a_n is an initial state of a and it happens with probability proportional to the success and proportion of a .

$$\Delta_3 x_{\langle a_n, b_m \rangle} = \begin{cases} \epsilon x_a \pi_a \sum_{c_k \in \mathcal{D}} \sigma_{\langle b, c \rangle, a} x_{\langle c_k, b_m \rangle} + \sigma_{\langle b, c \rangle, a} x_{\langle b_m, c_k \rangle} & \text{if } a_n = a_0 \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

Analogously somebody could switch to b_m .

$$\Delta_4 x_{\langle a_n, b_m \rangle} = \begin{cases} \epsilon x_b \pi_b \sum_{c_k \in \mathcal{D}} \sigma_{\langle a, c \rangle, b} x_{\langle a_n, c_k \rangle} + \sigma_{\langle a, c \rangle, b} x_{\langle c_k, a_n \rangle} & \text{if } b_m = b_0 \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

Finally a pair could switch both its members and thus become an $\langle a_n, b_m \rangle$.

$$\Delta_5 x_{\langle a_n, b_m \rangle} = \begin{cases} \epsilon^2 x_a \pi_a x_b \pi_b \left(\sum_{c_k \in \mathcal{D}} \sum_{d_l \in \mathcal{D}} (\sigma_{\langle c, d \rangle, a} + \sigma_{\langle c, d \rangle, b}) x_{\langle c_k, d_l \rangle} + \right. \\ \quad \left. + (\sigma_{\langle c, d \rangle, a} + \sigma_{\langle c, d \rangle, b}) x_{\langle d_l, c_k \rangle} - \right. \\ \quad \left. - \sum_{c_k \in \mathcal{D}} (\sigma_{\langle c, c \rangle, a} + \sigma_{\langle c, c \rangle, b}) x_{\langle c_k, d_l \rangle} \right) & \text{if } a_n = a_0 \wedge b_m = b_0 \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

The total learning dynamics is now

$$\Delta_L x_{\langle a_n, b_m \rangle} := \sum_{i=1}^5 \Delta_i x_{\langle a_n, b_m \rangle} \quad (24)$$

Proposition 12 *Starting from a randomly mixed population the evolutionary dynamics described by equation 24 is a standard (global) replicator dynamic if $\sigma_{\langle a, b \rangle, c}$ is constant for all a, b, c and as $\epsilon \rightarrow 0$.*

Proof Summing up equation 24 over all $a_n \in \{a_n | n \in \mathcal{N}(a)\}$ and over all $b_m \in \mathcal{D}$ gives the total dynamics

$$\Delta_L x_a = -2\epsilon x_a \bar{\pi} - 2\epsilon^2 x_a \bar{\pi} + \epsilon x_a \pi_a + \epsilon \bar{\pi} x_a + 2\epsilon^2 x_a \pi_a \bar{\pi} \quad (25)$$

$$= -\epsilon x_a \bar{\pi} + \epsilon x_a \pi_a - 2\epsilon^2 x_a \bar{\pi} + 2\epsilon^2 x_a \pi_a \bar{\pi} \quad (26)$$

Let now $\epsilon = \Delta t$ then dividing by ϵ and taking limits $\epsilon \rightarrow 0$ gives

$$\frac{dx_a}{dt} = x_a \pi_a - x_a \bar{\pi} \quad (27)$$

that is a standard (global) replicator dynamic. ■

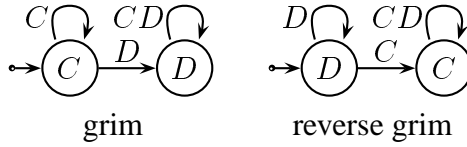


Figure 5: Two example automata.

Combining Interaction and Learning The complete dynamics will be defined as follows. At the beginning of a round we start with any status of a population x . First interaction takes place as described in equation 15 on page 15. This gives a new status of the population $p'()$. Then players may learn as in equation 24 on the preceding page. This gives us the state of the population at the beginning of the next round. We calculate now the limit of this process keeping the interaction probability i fixed while learning becomes more slowly ($\epsilon \rightarrow 0$).

We are studying the limit $e \rightarrow 0$ and keeping i fixed because we are interested in evolution of *repeated game strategies*. The repeated game strategies discussed so far can only be interpreted in a meaningful way if interaction happens substantially more often than learning. Otherwise individuals would, after experiencing the beginning of a repeated game strategy, already switch away to the beginning of a new one. In such a model only initial states of repeated game strategies would matter. We do not deny that in the context of evolution of *stage game strategies* other limits, e.g. fixed ratios between e and i might be appropriate.

4 An example

Consider a population with only two automata, one playing ‘grim’, the other playing ‘reverse grim’. The game is similar to the asymmetric battle of the sexes in 14 on page 13:

		Player II		
		I	II	
Player I	I	1	4	(28)
	II	3	1	
		4	1	

Grim (see figure 5) (which will be called g from now on) plays in its first state g_1 the stage game strategy I and remains there until an opponent plays II . As soon as its opponents plays II , g switches to its second state g_2 where it plays II as well. Once the second state is reached, g remains there forever.

The strategy is called grim because I may be a ‘cooperative’ strategy and II a defective. Then g behaves very much like a real ‘grim’ player.

‘Reverse grim’ (or \bar{g} for short) plays in its first state \bar{g}_1 the stage game strategy II and remains there until it detects its opponent playing I . Then \bar{g} switches to its second state \bar{g}_2 where it remains forever.

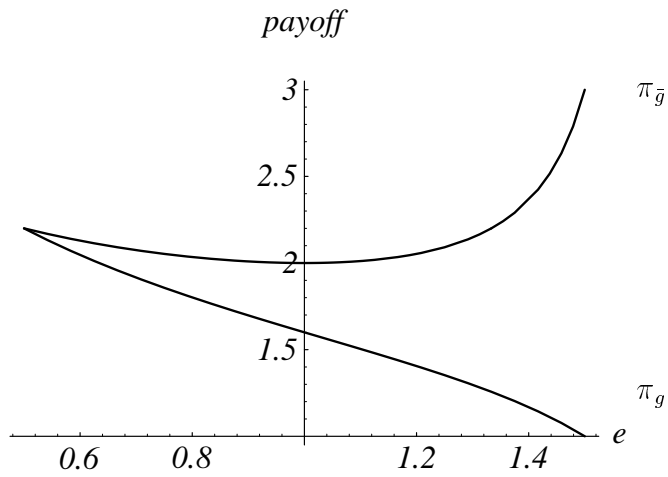


Figure 6: Payoffs with synchronous learning and interaction.

Notice that for the game 28 these two strategies in some way ‘need each other’ to achieve a high payoff.

4.1 Synchronous learning and interaction

Let us first (as a benchmark) consider a population of these two types which is in synchronous interaction. All pairs start simultaneously playing the game 28 on the page before, play the game for some periods, until payoffs are determined, learning takes place and then all pairs start synchronously again playing the first states of their automata. Thus, two grims will always play strategy I against each other, two reverse grims will always both play II , and a pair that consists of a grim that is matched with a reverse grim plays (I, II) once, then both automata switch to their second state to play (II, I) forever.

Then average payoffs are given in the following game:

		Player II		
		g	\bar{g}	
Player I	g	1	3	(29)
	\bar{g}	4	1	

This game is equivalent to the game 14 on page 13. Figure 6 displays the equilibrium payoffs of the two strategies g (lower payoff) and \bar{g} (higher payoff) for values of e where the mixed equilibrium is the only stable one.

4.2 Asynchronous learning and interaction

We hope that the story of synchronous learning and interaction given in section 4.1 was dubious enough to motivate an attempt to solve the same problem in an asynchronous environment. In such an environment one member of a pair might change his strategy

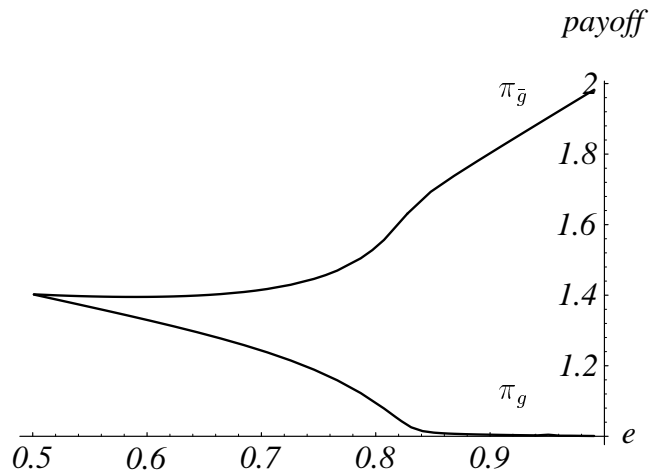


Figure 7: Payoffs with asynchronous learning and interaction.

without the other considering a simultaneous change. We argue that if evolution is applied in a social context synchronous learning is hard to justify. In the following we give an example that shows that synchronous interaction can be an unreliable approximation of asynchronous behavior.

Let us consider the pair of grim and reverse grim who started in the first state. Above we have seen that due to the interaction dynamics they immediately switch both to their second state. Now assume that one member of this pair, e.g. the reverse grim, learns and becomes a grim. Above we have argued that newly learned strategies start in their first state. Thus, we have a pair consisting of one grim in the first state and another grim in the second state. Due to interaction this pair becomes one where both grims are in the second state. Notice that the last two pairs were never mentioned in the above analysis of synchronous learning.

Altogether in a world with only pairs of g and \bar{g} ten different pairs of delegates are possible. Thus, we have ten equations of the form of equation 24 on page 18, one for each possible pair. To find mixed equilibria we must find fixed points of these ten simultaneous quadratic equations. So far we have done only numerical approximations that show three equilibria if e is close enough to $\frac{1}{2}$ (given that negative or complex shares of pairs are not allowed), one where only pairs of g in its first state survive, one for \bar{g} respectively and a mixed one. Figure 7 shows the payoffs of g (lower) and \bar{g} (higher). Up to now we cannot say anything on stability of one of these equilibria.

Notice that in contrast to the synchronous case (compare figure 6 on the preceding page) payoffs of mixed equilibria are substantially lower while payoff-differences are higher. With asynchronous interaction players have to face coordination failures: One member of a ‘coordinated’ pair may decide to learn a new strategy, a new strategy that may be successful in other pairs, but may fail in his particular pair.

With synchronous evolution these coordination failures are excluded, with global evolution inference from global fitness to local expected payoff is always valid. This example shows that with local and asynchronous evolution neither the first nor the second remains valid.

5 Conclusion

In the previous sections we have tried to present a simple model of local evolution and local interaction. We have modeled local interaction as permanent matchings and local evolution as a bias to be more likely to learn a strategy that is already present in the interaction environment. We have tried to point out that it matters whether evolution is modeled synchronously or asynchronously.

We have found that stability both of pure and mixed strategies depends on the degree of locality. We have further found that such a simple model still produces stable evolution of inequality — which must vanish with global evolution. Finally we have seen that here we have already stable evolution of cooperation — even in a model where only stage game strategies are allowed. Here again, in a model of global evolution of stage game strategies of stage game strategies, cooperation would be excluded.

We have the impression that many characteristics of this simple model correspond to very similar characteristics of much more complicated and, in a sense, much more expensive, cellular automata models.

For evolution of repeated game strategies the analysis is so far sketchy. The complexity of the problem restricts a complete and exhaustive analysis. Nevertheless we hope that some insight on stability, at least on stability of pure repeated game strategies, can be gained.

Some of our results are still very limited. Most of them apply only to symmetric 2×2 games. For many cases we claim that these results can be generalized to more complicated games.

A Proof of proposition 1

It is convenient to analyze the dynamics over the amount of the three possible pairs $x_{\langle a,a \rangle}$, $x_{\langle a,b \rangle}$, $x_{\langle b,b \rangle}$ in a two dimensional simplex where the vertical axis describes the amount of $x_{\langle a,b \rangle}$ while the horizontal axis describes $y := x_{\langle b,b \rangle} + x_{\langle a,b \rangle}/2$.

The locus of all random matchings ($x_{\langle a,a \rangle} = x_a^2$, $x_{\langle a,b \rangle} = 2x_a(1 - x_a)$, $x_{\langle b,b \rangle} = (1 - x_a)^2$) is a parabola in this simplex with $x_{\langle a,b \rangle} = 2y \cdot (1 - y)$. Consider now points on other parabolas $x_{\langle a,b \rangle} = ay \cdot (1 - y)$, $a \neq 2$. We prove proposition 1 showing that all loci on a lower parabola move to a higher one while all points above $x_{\langle a,b \rangle} = 2y \cdot (1 - y)$ move to a lower parabola.

A vector that is normal to a tangent on any parabola $x_{\langle a,b \rangle} = ay \cdot (1 - y)$ is $(-x_{\langle a,b \rangle} \cdot (1 - 2y)/(y \cdot (1 - y)), -1)$. \mathcal{M}_e denotes the transition matrix from equation 10 on page 5.

$$\mathcal{M}_e = \begin{pmatrix} -x_b \pi_b & e x_a \pi_a & 0 \\ x_b \pi_b & -e \cdot (x_a \pi_a + x_b \pi_b) & x_a \pi_a \\ 0 & e x_b \pi_b & -x_a \pi_a \end{pmatrix} \quad (30)$$

Call ν the product of this vector and the dynamics of pairs:

$$\nu = \left(\frac{-x_{\langle a,b \rangle} \cdot (1 - 2y)}{y \cdot (1 - y)}, -1 \right) \cdot \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 1 & 0 \end{pmatrix} \mathcal{M}_e \begin{pmatrix} x_{\langle a,a \rangle} \\ x_{\langle a,b \rangle} \\ x_{\langle b,b \rangle} \end{pmatrix} \quad (31)$$

Proposition 1 is equivalent to requiring $\nu > 0$ if we start below the ‘random matching parabola’ $x_{\langle a,b \rangle} = 2y \cdot (1 - y)$ and $\nu < 0$ if we start above. For a game

		Player II	
		I	II
Player I	I	a a	c b
	II	b c	d d

(32)

ν can be calculated as

$$\nu = \frac{x_{\langle a,b \rangle} - 2y \cdot (1 - y)}{4y \cdot (y - 1)} \left(ay \cdot (2 - x_{\langle a,b \rangle} - 2y) + byx_{\langle a,b \rangle} + cx_{\langle a,b \rangle} \cdot (1 - y) + d \cdot (2y - x_{\langle a,b \rangle})(y - 1) \right) \quad (33)$$

The fraction is positive if we start from below $x_{\langle a,b \rangle} = 2y \cdot (1 - y)$ and negative if we start above. The expressions following a, b, c, d are all positive. Thus, ν has the required properties. ■

B Proof of proposition 7

Consider any symmetric $n \times n$ game with the set of strategies \mathcal{A} . The set of unordered pairs of strategies is denoted by \mathcal{P} . We identify each unordered pair of strategies $\langle a, b \rangle$ with $\langle b, a \rangle$. Assume any ordering \prec over elements of \mathcal{P} . The transition matrix \mathcal{M} has dimension $|\mathcal{P}| \times |\mathcal{P}|$. Rows and columns of \mathcal{M} follow the ordering of \prec . x_a and π_a describe the total amount of strategy a in the population and its average payoff. The elements of

$$\mathcal{M} \equiv \begin{pmatrix} & \vdots & \\ \cdots & m_{\langle a,b \rangle, \langle c,d \rangle} & \cdots \\ & \vdots & \end{pmatrix} \quad (34)$$

are

$$\begin{aligned} m_{\langle a,a \rangle, \langle a,a \rangle} &= - \sum_{i \neq a} x_i \pi_i \\ m_{\langle a,a \rangle, \langle a,b \rangle} &= e x_a \pi_a \\ \forall a, b, c, d \text{ mutually distinct : } & \\ m_{\langle a,b \rangle, \langle a,a \rangle} &= x_b \pi_b \\ m_{\langle a,b \rangle, \langle a,b \rangle} &= (1 - e)(x_a \pi_a + x_b \pi_b) - \sum_i x_i \pi_i \\ m_{\langle a,b \rangle, \langle a,c \rangle} &= \frac{1}{2} x_b \pi_b \\ \text{otherwise } m_{\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle} &= 0 \end{aligned} \quad (35)$$

Be x a vector describing the proportions of pairs in the population. $x_{\langle a,b \rangle}$ is an element of x and denotes the amount of pairs $\langle a, b \rangle$ in the population. Elements of x are ordered as

in \mathcal{M} . Then the dynamics is described by

$$\dot{x} = \mathcal{M}x. \quad (36)$$

Payoffs of the pure strategy a against the pure strategy b will be denoted $\pi_{a,b}$. Then terms like $x_a \pi_a$ can be expressed as

$$x_a \pi_a = \frac{1}{2} x_{\langle a,a \rangle} \pi_{a,a} + \frac{1}{2} \sum_b x_{\langle a,b \rangle} \pi_{a,b} \quad (37)$$

We check now whether a state of the population where each pair of players plays $\langle a, a \rangle$ is asymptotically stable. Using $\mathbf{1}'x = 1$ we can eliminate $x_{\langle a,a \rangle}$. We calculate now the Jacobian $J = \nabla_x(\mathcal{M}x)$. Since we have already eliminated $x_{\langle a,a \rangle}$ we can drop that row of J that corresponds to $\langle a, a \rangle$. The remaining matrix is of size $(|\mathcal{P}| - 1) \times (|\mathcal{P}| - 1)$. Checking stability of the pure strategy a we consider a state of the population with only pairs $\langle a, a \rangle$. Thus, we substitute for $x_{\langle a,a \rangle} = 1$ and $\forall \langle b,c \rangle \neq \langle a,a \rangle : x_{\langle b,c \rangle} = 0$. Then the elements of the jacobian

$$J \equiv \begin{pmatrix} & \vdots & \\ \cdots & j_{\langle b,c \rangle, \langle d,e \rangle} & \cdots \\ & \vdots & \end{pmatrix} \quad (38)$$

are

$$\begin{aligned} j_{\langle b,b \rangle, \langle b,b \rangle} &= j_{\langle b,c \rangle, \langle b,c \rangle} = -\pi_{a,a} \\ j_{\langle a,b \rangle, \langle b,b \rangle} &= \pi_{a,a} + \pi_{b,b} \\ \forall b, c, d \text{ mutually distinct} : \quad j_{\langle a,b \rangle, \langle a,b \rangle} &= -e\pi_{a,a} + \frac{1}{2}\pi_{b,a} \\ j_{\langle a,b \rangle, \langle b,c \rangle} &= \frac{1}{2}(\pi_{a,a} + \pi_{b,c}) \\ \text{otherwise } j_{\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle} &= 0 \end{aligned} \quad (39)$$

Since J has dimension $(|\mathcal{P}| - 1) \times (|\mathcal{P}| - 1)$ we should now expect to find $|\mathcal{P}| - 1$ eigenvectors, one for each pair, except for the pair $\langle a, a \rangle$ that we dropped above.

- There is one eigenvector for each pair $\langle a, b \rangle$ ($b \neq a$) with $x_{\langle a,b \rangle} = 1$ and all other elements zero. The eigenvalues are $-e\pi_{a,a} + \frac{1}{2}\pi_{b,a}$.
- There is one eigenvector for each pair $\langle b, b \rangle$ ($b \neq a$) with $x_{\langle b,b \rangle} = (-\frac{1}{2}\pi_{b,a} - (1 - e)\pi_{a,a})/(\pi_{a,a} + \pi_{b,b})$, $x_{\langle a,b \rangle} = 1$ and all other elements zero. The eigenvalues are $-\pi_{a,a}$.
- There is one eigenvector for each pair $\langle b, c \rangle$ (a, b, c mutually distinct) with $x_{\langle b,b \rangle} = \frac{1}{2}(-\pi_{a,a} + \pi_{c,b})/(\pi_{a,a} + \pi_{b,b})$, $x_{\langle c,c \rangle} = \frac{1}{2}(-\pi_{a,a} + \pi_{c,b})/(\pi_{a,a} + \pi_{c,c})$ and all other elements zero. The eigenvalues are again $-\pi_{a,a}$.

Eigenvalues in the last two categories are negative by definition. Eigenvalues in the first category are negative if all deviation payoffs $\pi_{b,a}$ (for all $b \neq a$) are smaller than $2e$ times the payoff of the tested pure strategy a . But if all eigenvalues are negative, then the state of a population that consists of pairs where everybody plays a is asymptotically stable. ■

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