Simultaneous Auctions with Synergies

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Abstract

Motivated by recent auctions of licenses for the radio-frequency spectrum, we consider situations where multiple objects are auctioned simultaneously by means of a second-price, sealed-bid auction. For some buyers, called global bidders, the value of multiple objects exceeds the sum of the objects' values separately. Others, called local bidders, are interested in only one object. In a simple independent private values setting, we (a) characterize an equilibrium that is symmetric among the global bidders; (b) show that the addition of bidders often leads to less aggressive bidding; and (c) compare the revenues obtained from the simultaneous auction to those from its sequential counterpart.

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1 Introduction

In July 1994 the United States Federal Communications Commission (FCC) began a series of sales of PCS licenses.[1] Prior to this, the FCC had granted licenses for use of the radio frequency spectrum either by "comparative hearings" or by means of lotteries. A novel format was adopted for the new series of sales following consultations with prominent auction theorists, among others: licences would be sold in batches, and in each batch by means of open ascending bid auctions that would be conducted simultaneously (McMillan (1994) and Cramton (1994)). Thus bidders interested in more than one license in a batch would have to bid simultaneously in more than one auction.

The design problem the FCC faced was extremely complicated, and one of the most important factors cited in both the theorists' advice and the FCC's decision was the explicit recognition that there are increasing returns (alternatively referred to in the literature as synergies, or superadditive values) of two kinds associated with owning multiple licenses: economies of scale in the amount of spectrum covering a particular geographic area; and economic advantages of various types associated with owning licenses that collectively cover large and/or contiguous geographic areas. There are, however, no equilibrium models in the auction literature of simultaneous auctions of objects having such synergies,[2] so the theorists' advice was based on insights gained from models of a different character.[3]

The synergies are most significant for bidders intending to establish large PCS networks. Other bidders, uninterested in such networks, might still be willing to outbid the network providers, however, even without the external benefit, if they possessed local cost advantages. This means that bidder asymmetries, a source of severe complications in the auction literature,[4] may be rather important in the context of PCS auctions.

In this paper we explore a simple model that seeks to capture interactions of

[1]Personal Communications Services (PCS) is the name given to a new generation of wireless telephones, faxes, and paging services.

[2]See, however, Gale (1990).

[3]There are models containing decreasing returns (for instance, Engelbrecht-Wiggans and Weber (1979) and Lang and Rosenthal (1991)); and, to be sure, there were some decreasing returns imposed by the rules of the PCS auctions since there were limits to the number of licenses any one bidder was permitted to win, with penalties if a bidder exceeded his allotment. But the design problems stemmed mostly from increasing returns, and the models in the literature with decreasing returns are unhelpful for the analysis of positive synergies.

[4]See Maskin and Riley (1994), for example.

the following three elements:

1. the simultaneous sale of multiple items at auction;
2. the presence of two kinds of bidders, whom we call local and global; and
3. increasing returns for the global bidders.

Thus we attempt to deal with the main complicating factors of the PCS auctions. To retain tractability, we shall have to abstract away from other special features of these auctions, however, and we discuss these issues in more detail later. Below, we outline a simple model and characterize its equilibria. These characterizations are part of the contribution of this paper; but we are also able to draw some interesting conclusions about these equilibria that are suggestive of unexpected general qualitative features of simultaneous auctions. We show, for instance, that having more competitors often produces less aggressive bidding. Furthermore, some examples that we examine suggest that whether a simultaneous format raises more revenue than a sequential format depends on how strong the increasing returns are: in our examples the simultaneous format raises greater revenues when the increasing returns are strong.[5]

Our general model has the following constituent elements. There are m objects to be auctioned simultaneously through a second-price, sealed-bid format.[6] There are two kinds of bidders, called local and global. Each local bidder is interested in only one of the objects, while the global bidders are interested in multiple objects. For each of the m objects there are n interested local bidders. Each local bidder has a privately-known valuation for the object in question. These nm private valuations are assumed to be independently distributed on $[0, 1]$ according to the cumulative distribution function F_L. Each of k global bidders also observes a signal that is distributed on $[0, 1]$ according to the distribution function F_G, independently of all other valuations and signals. This signal is his valuation for a single object, but if he wins more than one object, the total value received is more than the sum of his individual valuations. For instance (and for most of the paper), if there are two objects and if the signal a global bidder receives is x, then the value from winning either single object is exactly x but the total value from winning both objects is

[5]Hausch (1986) compares the two formats in a common value setting without synergies.

[6]As usual, the second-price, sealed-bid format is intended to be a proxy for an ascending first-price (or English) auction. When multiple objects are auctioned simultaneously, however, whether combinations of bids on multiple objects are permitted and how the auction should close are delicate matters. After much debate, the FCC chose for the PCS auctions a multiple-round simultaneous bid format in which no combinatorial bids are allowed but in which bidding remains open on all licenses as long as there is activity on any one license.

$2x + \alpha$, where α is a fixed, publicly-known, positive number and is the same for all global bidders. Of course, the local bidders all have (weakly) dominant strategies: to bid their valuations. We therefore assume they do this and concentrate on the game this induces among the global bidders.

We begin in Section 2 with the case where there is a single global bidder ($k = 1$). For expository ease we also assume that there are only two objects ($m = 2$). The equilibrium is then simply the solution to an optimization problem for the global bidder, but as we shall see, it is generally not a concave problem. Its solution contains a few surprises and serves as a benchmark for the analysis when there are multiple global bidders, the case we take up in Section 3. For that case, we characterize an equilibrium in which all global bidders behave symmetrically. Comparative statics for the two-object model are then studied in Section 4. Not surprisingly, we find that increases in α always lead to more aggressive bidding. More surprising is the finding that increases in k, the number of global bidders, always leads to *less* aggressive bidding. For increases in n, the situation is mixed. Extensions to more than two objects are then carried out in Section 5. We consider two models. In one, all global bidders are interested in all the objects. In the second, global bidders are interested in different, but overlapping, subsets of the objects. In Section 6, we first characterize the equilibrium of an auction in which the objects are auctioned sequentially rather an simultaneously. We then compare the expected revenue raised from the simultaneous auction with that from the sequential auction. Since a general analysis appears to be difficult, we study examples having the uniform distribution and small numbers of bidders. We use Monte Carlo methods to calculate the expected revenues accruing from the two formats. Our simulations suggest that whether the simultaneous or sequential auction is superior depends on the extent of the synergies. We find that when the synergies are weak, the sequential auction results in greater revenue. When the synergies are strong, the simultaneous auction is superior. Section 7 concludes.

2 Single Global Bidder, Two Objects

We begin by considering the case where there is a single global bidder and two objects for sale. This will serve as a useful benchmark for the analysis in the next section.

Assume that the distribution function of the local bidders' values, F_L, admits a density, f_L, that is strictly positive on $[0, 1]$. Let L denote the distribution function of the maximum of the n local valuations, so $L(v) \equiv (F_L(v))^n$, and let $l \equiv L'$, the corresponding density. Recall that the local bidders bid their respective values, so the global bidder's expected profit (or payoff) from making the bid pair $(b_1, b_2) \in [0,1] \times [0,1]$ in the two auctions when his signal is x is[7]

$$\begin{aligned}\Pi(b_1, b_2; x) &= L(b_1) L(b_2) (2x + \alpha - E(p \mid b_1) - E(p \mid b_2)) \\ &\quad + L(b_1)(1 - L(b_2))(x - E(p \mid b_1)) \\ &\quad + L(b_2)(1 - L(b_1))(x - E(p \mid b_2))\end{aligned} \tag{1}$$

where $E(p \mid b)$ denotes the expected price that the global bidder pays for an object when he wins it with a bid of $b \in (0,1]$; that is,

$$E(p \mid b) = \frac{1}{L(b)} \int_0^b pl(p)dp.$$

The first term on the right-hand side of (1) is the expected payoff from winning both objects, and the second and third terms are the respective expected payoffs from winning either of the objects separately. Simplifying, (1) becomes

$$\Pi(b_1, b_2; x) = \alpha L(b_1) L(b_2) + L(b_1)(x - E(p \mid b_1)) + L(b_2)(x - E(p \mid b_2)).$$

Suppose $b_1 > b_2$ and $\Pi(b_1, b_2; x) > \Pi(b_2, b_2; x)$. Then, since

$$(L(b_1))^2 - L(b_1)L(b_2) > L(b_1)L(b_2) - (L(b_2))^2,$$

it follows that $\Pi(b_1, b_1; x) > \Pi(b_1, b_2; x)$. Consequently, we may restrict attention to equal-bid pairs and rewrite the payoff function as

$$\Pi(b, x) = (L(b))^2 \alpha + 2L(b)x - 2\int_0^b pl(p)dp,$$

where $b \in [0,1]$ is the same bid in both auctions.

The first-order condition for a maximum of $\Pi(\cdot, x)$ is

$$\begin{aligned}\frac{\partial \Pi(b;x)}{\partial b} &= 2L(b)\, l(b)\, \alpha + 2l(b)\, x - 2bl(b) \\ &= 0.\end{aligned}$$

[7] Of course, in this setting, bidding above one is equivalent to bidding one.

It is convenient to define

$$\varphi(x, b) \equiv 2\alpha L(b) - 2b + 2x.$$

so that interior local extrema of $\Pi(\cdot; x)$, if there are any, occur where $\varphi(x, b) = 0$.

Figure 1 is a schematic depiction of the locus of solutions to this equation for a situation where $\alpha < 1$ and L is a convex function on $[0, 1]$. In the region to the left of the curve, $\varphi(x, b) < 0$, and to its right, $\varphi(x, b) > 0$. It follows that for $x < 1 - \alpha$, the point $b^{-}(x)$ on the curve is the unique global maximizer of $\Pi(\cdot; x)$; and for $x > \bar{x}$ the global maximizer is 1. Between $1 - \alpha$ and $\bar{x}$, the smaller of the two points on the curve, $b^{-}(x)$, is a local maximizer while the other is not; but 1 is also a local maximizer. Define

$$\widehat{x} \equiv \max\left\{x : \Pi(1; x) - \Pi(b^{-}(x); x) \leq 0\right\}. \tag{2}$$

Since

$$\begin{aligned} \frac{d}{dx}[\Pi(1; x) - \Pi(b^{-}(x); x)] &= 2 - 2L(b^{-}(x)) - 2\frac{d}{dx}l(x)\varphi(x, b^{-}(x)) \\ &> 0 \end{aligned}$$

it follows that if $\Pi(1; \widehat{x}) - \Pi(b^{-}(\widehat{x}); \widehat{x}) = 0$, then for all $\widehat{x} < x < \bar{x}$,

$$\Pi(1; x) - \Pi(b^{-}(x); x) > 0$$

and it is better to bid 1. Note that all this goes through more generally; the only special features used in the argument are that the function

$$x(b) \equiv b - \alpha L(b)$$

(which solves $\varphi = 0$ for x in terms of b) is nondecreasing on $[0, \bar{b}]$ and nonincreasing on $[\bar{b}, 1]$; which holds if $x(b)$ is quasi-concave. Assuming that $x(b)$ is indeed quasi-concave the following result is immediate.

Theorem 1 *Suppose $k = 1$. The following constitutes an equilibrium of the simultaneous auction. (i) All local bidders bid their respective values; and (ii) the global bidder follows the strategy:*

$$b^{*}(x) = \begin{cases} b^{-}(x) & \text{if} \quad 0 \leq x \leq \widehat{x} \\ 1 & \text{if} \quad \widehat{x} < x \leq 1 \end{cases}$$

where $b^{-}(x)$ is the smaller of two roots of $\varphi(x, b) = 0$ and $\widehat{x}$ is determined by (2).

Note that if $x(b)$ is increasing on $[0, 1]$ (as in the uniform case with $n = 1$ and $\alpha < 1$), $\widehat{x} = 1$ and there is no discontinuity in the optimal bid function. If $x(b)$ is quasi-concave, $n = 1$, and $f_L(0) \leq \frac{1}{\alpha}$, then $x'(0) \leq 0$. In this case, $\widehat{x} = 0$ and the optimal bid is 1 for all values of x.

Quasi-concavity of $x(\cdot)$ is not easy to characterize usefully in terms of the primitives of the model, but the stronger hypothesis that $x(\cdot)$ is concave is obviously equivalent to convexity of $(F_L(\cdot))^n$. When $n = 1$, this is the rather strong condition that f_L is nondecreasing; but for $n > 1$ it is a considerably weaker hypothesis. In fact, it is easy to see that if $(F_L(\cdot))^n$ is convex then so is $(F_L(\cdot))^{n+1}$; and thus the restriction becomes progressively weaker as more local bidders are added to the model. If $x(\cdot)$ is not quasi-concave, the situation becomes more complicated and not worth pursuing here, since quasi-concavity of $x(\cdot)$ will be needed for the analysis of the next section when there are multiple global bidders.

3 Multiple Global Bidders, Two Objects

For $k \geq 2$, if $\alpha \geq 2$, then for all $x > 0$, the average or per-unit value of an object to a global bidder exceeds 1. In this case, the local bidders will be completely shut-out of the auction and the analysis is straightforward (mimicking the case $x > x_\alpha$ below). So suppose that $\alpha < 2$, and consider a global bidder who receives a signal of $x_\alpha \equiv \left(1 - \frac{\alpha}{2}\right)$. Such a bidder bids exactly 1. This removes the local bidders from the picture; and so when at least one global bidder has a signal of x_α or greater, a standard second-price auction prevails among the global bidders for the two object "bundle." In such situations, it is a best response for a global bidder to bid $x + \frac{\alpha}{2}$, that is, half his total value for the "bundle," and this will be part of our equilibrium construction.

To analyze bidding behavior for $x \in (0, x_\alpha]$ assume that the density corresponding to F_G, f_G, is strictly positive on $[0, 1]$. Adopting notation parallel to that for the local bidders, let G denote the distribution function of the maximum of $k - 1$ signals of the global bidders, that is, $G(y) \equiv (F_G(y))^{k-1}$; and let $g \equiv G'$ be the corresponding density function. We retain the assumption made in Section 2 that the distribution function of the maximum of the values of the n local bidders, L, is convex, or more generally, that $b - \alpha L(b)$ is quasi-concave on $[0, 1]$.

Suppose that $k - 1$ global bidders follow the same partial strategy β, which assigns the same bid $\beta(x) \in [0, 1]$ in both auctions upon receipt of the signal

$x \in (0, x_\alpha]$.[8] For now, suppose that β is increasing (this will be verified later), though not necessarily continuous. The expected payoff to a global bidder, say 1, who receives a signal of $x \in [0,1]$ and bids an amount b is

$$\begin{aligned} \Pi(b;x) &= (L(b))^2 G\left(\beta^{-1}(b)\right)(2x+\alpha-2E(p \mid b)) \\ &\quad +2L(b)(1-L(b))\,G\left(\beta^{-1}(b)\right)(x-E(p \mid b)) \end{aligned} \tag{3}$$

where $\beta^{-1}(b) = \sup\{x : \beta(x) \leq b\}$ and $E(p \mid b)$ is the expected price paid by global bidder 1 when he wins with a bid of b and will be calculated presently. As before, the first term is the expected payoff from winning both objects, and the second term is the expected payoff from winning one of the objects.

Let H denote the distribution function of the highest bid among n local bidders and $k-1$ global bidders when the locals bid their values and global bidders use the strategy β. Then

$$H(p) = L(p)\,G\left(\beta^{-1}(p)\right), \tag{4}$$

and the corresponding density (for $p \in [0,1]$) is

$$h(p) = l(p)\,G\left(\beta^{-1}(p)\right) + L(p)\,g\left(\beta^{-1}(p)\right)\frac{1}{\beta'\left(\beta^{-1}(p)\right)} \tag{5}$$

assuming that β' exists.

Thus,

$$\begin{aligned} E(p \mid b) &= \frac{1}{H(b)}\int_0^b ph(p)dp \\ &= \frac{1}{L(b)\,G\left(\beta^{-1}(b)\right)}\int_0^b ph(p)dp. \end{aligned} \tag{6}$$

Substituting from (6) into (3) we obtain

$$\Pi(b;x) = \alpha(L(b))^2 G\left(\beta^{-1}(b)\right) + 2xL(b)\,G\left(\beta^{-1}(b)\right) - 2\int_0^b ph(p)dp. \tag{7}$$

Maximizing with respect to b yields the first-order condition:

$$\frac{\partial\Pi}{\partial b}(b;x) = 2\alpha L(b)\,l(b)\,G(\beta^{-1}(b)) + \alpha\,(L(b))^2\,g(\beta^{-1}(b))\frac{1}{\beta'\left(\beta^{-1}(b)\right)}$$

[8] If other global bidders follow a strategy that assigns equal bids in the two auctions, it is optimal for a global bidder to do the same. The argument the same as in Section 2.

$$
\begin{aligned}
&+2xl(b)\,G(\beta^{-1}(b)) + 2xL(b)\,g(\beta^{-1}(b))\frac{1}{\beta'(\beta^{-1}(b))} \qquad (8)\\
&-2bh(b)\\
=\ & 0.
\end{aligned}
$$

Using (5) and the fact that in equilibrium $b = \beta(x)$ and rearranging (8) yields the following differential equation:

$$\beta' = \frac{-L(\beta)\,g(x)}{l(\beta)\,G(x)}\left(\frac{\alpha L(\beta) - 2\beta + 2x}{2\alpha L(\beta) - 2\beta + 2x}\right) \tag{9}$$

for bids β in $[0, 1]$. Thus, the lower part of the equilibrium strategy will be characterized by the differential equation (9) together with the boundary condition:

$$\beta(x_\alpha) = 1. \tag{10}$$

However, the differential equation (9) together with the boundary condition (10) need not have a continuous solution on $(0, x_\alpha]$ since it may be that $2\alpha L(\beta) - 2\beta + 2x = 0$ for some $x \in (0, x_\alpha]$. Thus, we proceed as follows: First, we construct a particular, monotonically increasing and piecewise-continuous function β that satisfies (9) and (10) on $(0, x_\alpha]$. We then show that the function β so obtained indeed completes the construction of an equilibrium strategy for the global bidders.

Construction of Lower Part of the Equilibrium Strategy It is useful to define the functions $\psi(x, b) \equiv \alpha L(b) - 2b + 2x$ and (as in Section 2) $\varphi(x, b) \equiv 2\alpha L(b) - 2b + 2x$ so that (9) can be rewritten as

$$\beta' = \frac{-L(\beta)\,g(x)}{l(\beta)\,G(x)}\left(\frac{\psi(x, \beta)}{\varphi(x, \beta)}\right) \tag{11}$$

Next observe that for $b > 0, \varphi(x, b) > \psi(x, b)$, and thus if

$$S \equiv \left\{(x, b) \in R^2 : \psi(x, b) \leq 0 \text{ and } \varphi(x, b) \geq 0\right\},$$

then whenever $(x, \beta(x)) \in S$, $\psi(x, \beta(x))$ and $\varphi(x, \beta(x))$ have opposite signs and thus $\beta'(x) > 0$. The set S consists of the set of points lying between the curves $\varphi(x, b) = 0$ and $\psi(x, b) = 0$. (See Figure 2 for an illustration.)

Now observe that since $\beta(x_\alpha) = 1$ we have that $\psi(x_\alpha, \beta(x_\alpha)) = 0$ and $\varphi(x_\alpha, \beta(x_\alpha)) = \alpha > 0$. Hence $\beta'(x_\alpha) = 0$. So there exists an $\epsilon > 0$ such that for all $x \in (x_\alpha - \epsilon, x_\alpha]$,

$(x, \beta(x)) \in S$. Thus (11) has a continuous monotonic solution in the interval $(x_\alpha - \epsilon, x_\alpha]$. By increasing ϵ, extend the local solution β as much as possible; say, to the interval $(\underline{x}, x_\alpha]$. Notice that the curve $(x, \beta(x))$ can leave the set S only by crossing the boundary defined by $\varphi(x, b) = 0$, since the boundary $\psi(x, b) = 0$ repels the curve back into S.

Now define the point $\overline{x}$ as the maximized value of the quasi-concave function $x(b) = b - \alpha L(b)$. If $\underline{x} > 0$, we must have that $\varphi(\underline{x}, \beta(\underline{x})) = 0$, or equivalently, $\underline{x} = \beta(\underline{x}) - \alpha L(\beta(\underline{x}))$, and thus $\overline{x} \geq \underline{x}$. If $\underline{x} = 0$, then $\varphi(\underline{x}, 0) = 0$ and thus again $\overline{x} \geq \underline{x}$. Now if $\overline{x} = \underline{x}$ we obtain a continuous and monotonic solution to (11) and (10) on $(0, x_\alpha]$. So suppose $\overline{x} > \underline{x}$. For all $x \in [\underline{x}, \overline{x}]$, define $b^+(x) = \beta(x)$ as the solution to (11) and (10) and $b^-(x) = \min\{b : \varphi(x, b) = 0\}$. (See Figure 3 which depicts a situation where $\underline{x} > 0$. Note that the curve β hits the curve $\varphi(x, b) = 0$ where it bends backwards.)

Next consider the function K on the interval $[\underline{x}, \overline{x}]$ defined by

$$K(x) = \int_{b^-(x)}^{b^+(x)} \varphi(x, b)\, l(b)\, db. \tag{12}$$

Lemma 1 *K is an increasing function on* $[\underline{x}, \overline{x}]$.

Proof. Differentiating (12) we obtain:

$$\begin{aligned} K'(x) &= \int_{b^-(x)}^{b^+(x)} \frac{\partial \varphi}{\partial x}(x, b)\, l(b)\; db \\ &\quad + \varphi(x, b^+(x))\, l(b^+(x))\, b^{+\prime}(x) - \varphi\left(x, b^-(x)\right) l\left(b^-(x)\right) b^{-\prime}(x) \end{aligned}$$

But by definition, $\varphi(x, b^-(x)) = 0$ and $b^+(x) = \beta(x)$, the solution to (11) and (10). Furthermore, $\frac{\partial \varphi}{\partial x}(x, b) = 2$. Thus, we have

$$\begin{aligned} K'(x) &= 2L(b^+(x)) - 2L\left(b^-(x)\right) + \varphi(x, \beta(x))\, l(\beta(x))\, \beta'(x) \\ &= 2L(b^+(x)) - 2L\left(b^-(x)\right) - \psi(x, \beta(x)) \frac{L(\beta(x))\, g(x)}{G(x)} \\ &> 0 \end{aligned}$$

using (11) and recalling that when $x \in [\underline{x}, \overline{x}]$, $\psi(x, \beta(x)) \leq 0$.

Now observe that if $\underline{x} > 0$, then $K(\underline{x}) < 0$ since in that case $\varphi(\underline{x}, b) < 0$ for all $b \in (b^-(\underline{x}), b^+(\underline{x}))$. The discontinuity in the solution to the differential equation, when there is one, occurs at the point $\widehat{x}$, defined by

$$\widehat{x} \equiv \begin{cases} \max\{x \leq x_\alpha : K(x) \leq 0\} & \text{if} \quad K(\underline{x}) \leq 0 \\ \underline{x} & \text{if} \quad K(\underline{x}) > 0 \end{cases} \tag{13}$$

(Notice that if $K(\underline{x}) < 0$, then $\widehat{x}$ is well defined since K is continuous and by Lemma 1 if $K(\widehat{x}) = 0$, then for all $x \in [\widehat{x}, \overline{x}]$, $K(x) > 0$.) Now, at $\widehat{x}$ terminate the upper leg of β and restart the differential equation at $(\widehat{x}, b^-(\widehat{x}))$. So for $x \in (0, \widehat{x}]$ let β be the solution to the differential equation (11) together with the boundary condition:

$$\beta(\widehat{x}) = b^-(\widehat{x}) \tag{14}$$

The complete construction is summarized as follows.

Theorem 2 *The following constitutes an equilibrium of the simultaneous auction. (i) All local bidders bid their respective values; and (ii) all global bidders follow the strategy:*

$$b^*(x) = \begin{cases} \beta(x) & \textit{if} \quad 0 \leq x \leq x_\alpha \\ x + \frac{\alpha}{2} & \textit{if} \quad x_\alpha < x \leq 1 \end{cases}$$

where $\beta(x)$ is the solution to (11) *and* (14) *on the interval* $(0, \widehat{x}]$ *and is the solution to* (11) *and* (10) *on the interval* $(\widehat{x}, x_\alpha]$.

Proof. Clearly, it is a (weakly) dominant strategy for the local bidders to bid their values.

Suppose $k-1$ global bidders are following the strategy b^*. As in (7) the expected payoff of a global bidder with a signal of x who bids b is

$$\Pi(b; x) = \alpha (L(b))^2 G(b^{*-1}(b)) + 2xL(b) G(b^{*-1}(b)) - 2\int_0^b ph(p)dp \tag{15}$$

where $b^{*-1}(b) = \sup\{x : b^*(x) \leq b\}$.

Before checking that no deviations are profitable, we will first compute the slope of the payoff function (15). Since b^* is not necessarily differentiable (or even continuous) we need to consider four regions separately.

Region 1: $b \in (0, b^-(\widehat{x})]$.

In this case, $b^{*-1}(b) = \beta^{-1}(b)$ and thus the derivative of $\Pi(b;x)$ is the same as in (8). Using the definition of $h(b)$ from (5) and collecting terms we can write:

$$\begin{aligned}\frac{\partial \Pi}{\partial b}(b;x) &= l(b)\,G(\beta^{-1}(b))\left[2\alpha L(b) - 2b + 2\beta^{-1}(b)\right] \\ &\quad + L(b)\,g(\beta^{-1}(b))\left[\alpha L(b) - 2b + 2\beta^{-1}(b)\right]\frac{1}{\beta'(\beta^{-1}(b))} \\ &\quad + 2h(b)\left(x - \beta^{-1}(b)\right)\end{aligned} \tag{16}$$

From (11) the first two terms in (16) vanish and it follows that for $b \in (0, b^{-}(\widehat{x})]$,

$$\frac{\partial \Pi}{\partial b}(b;x) = 2h(b)\left(x - \beta^{-1}(b)\right). \tag{17}$$

Region 2: $b \in \left(b^{-}(\widehat{x}), b^{+}(\widehat{x})\right]$.

Since other global bidders do not bid in this range, $G(b^{*-1}(b)) = G(\widehat{x})$ and $h(b) = G(\widehat{x})l(b)$. So we obtain from (15) that for $b \in \left(b^{-}(\widehat{x}), b^{+}(\widehat{x})\right]$,

$$\begin{aligned}\frac{\partial \Pi}{\partial b}(b;x) &= 2\alpha L(b)\,l(b)\,G(\widehat{x}) + 2xl(b)\,G(\widehat{x}) - 2G(\widehat{x})\,bl(b) \\ &= G(\widehat{x})\varphi(x,b)\,l(b).\end{aligned} \tag{18}$$

Region 3: $b \in (b^{+}(\widehat{x}), 1]$.

The calculations here are the same as in Region 1 and thus again we obtain that for $b \in (b^{+}(\widehat{x}), 1]$,

$$\frac{\partial \Pi}{\partial b}(b;x) = 2h(b)\left(x - \beta^{-1}(b)\right). \tag{19}$$

Region 4: $b \in (1, 1 + \frac{\alpha}{2}]$.

In this case, $L(b) = 1$ and it is easy to see that for $b > 1$,

$$\frac{\partial \Pi}{\partial b}(b;x) = 2g\left(b - \frac{\alpha}{2}\right)\left(x - b + \frac{\alpha}{2}\right). \tag{20}$$

We are now ready to verify that there are no profitable deviations from $b^{*}(x)$. The arguments for the three cases (A) $x \leq \widehat{x}$, (B) $\widehat{x} < x \leq x_{\alpha}$; and (C) $x_{\alpha} < x \leq 1$ are slightly different and in each case are broken down according to the four possible regions of deviations identified above.

CASE A: $x \leq \widehat{x}$, so that $b^{*}(x) \in (0, b^{-}(\widehat{x})]$.

A1. $b \in (0, b^- (\widehat{x})]$.

From (17), $\frac{\partial \Pi}{\partial b} > 0$ for all $b < \beta(x) = b^*(x)$ and it follows that it does not pay to deviate and bid $b < b^*(x)$. Similarly, (17) implies that it does not pay to bid b satisfying $b^*(x) < b \leq b^-(\widehat{x})$.

A2. $b \in (b^-(\widehat{x}), b^+(\widehat{x}))$.

From (18),

$$\Pi(b^+(\widehat{x}), x) - \Pi\left(b^-(\widehat{x}), x\right) = G(\widehat{x}) \int_{b^-(\widehat{x})}^{b^+(\widehat{x})} \varphi(x, b)\, l(b)\, db;$$

and

$$\begin{aligned} \Pi(b^+(\widehat{x}), \widehat{x}) - \Pi\left(b^-(\widehat{x}), \widehat{x}\right) &= G(\widehat{x}) \int_{b^-(\widehat{x})}^{b^+(\widehat{x})} \varphi(\widehat{x}, b)\, l(b)\, db \\ &\leq 0 \end{aligned}$$

by construction. Since $\varphi(x, b)$ is increasing in x, this implies that if $x \leq \widehat{x}$, then

$$\Pi(b^+(\widehat{x}), x) \leq \Pi\left(b^-(\widehat{x}), x\right). \tag{21}$$

From (18), $\frac{\partial \Pi}{\partial b}$ has the same sign as $\varphi(x, b)$. By definition, $\varphi(x, b^-(x)) = 0$ and hence $\frac{\partial \Pi}{\partial b}(x, b^-(x)) = 0$ also. Now since $x \leq \widehat{x}$, $b^-(x) \leq b^-(\widehat{x})$ and as b increases from $b^-(\widehat{x})$ to $b^+(\widehat{x})$, $\frac{\partial \Pi}{\partial b}(x, b)$ is first negative and then positive. Thus, for all $b \in (b^-(\widehat{x}), b^+(\widehat{x}))$,

$$\begin{aligned} \Pi(b; x) &\leq \max\left\{\Pi\left(b^-(\widehat{x}), x\right), \Pi(b^+(\widehat{x}), x)\right\} \\ &= \Pi\left(b^-(\widehat{x}), x\right) \\ &\leq \Pi(b^*(x); x) \end{aligned}$$

using (21) and A1.

A3. $b \in (b^+(\widehat{x}), 1]$.

From (19), for all $b \in (b^+(\widehat{x}), 1]$, $\Pi(b; x) \leq \Pi(b^+(\widehat{x}); x) \leq \Pi(b^-(\widehat{x}); x) \leq \Pi(b^*(x); x)$.

A4. $b \in (1, 1 + \frac{\alpha}{2}]$.

From (20), for all $b \geq 1$, $\Pi(b; x) \leq \Pi(b^+(\widehat{x}); x) \leq \Pi(b^*(x); x)$.

CASE B: $\widehat{x} < x \leq x_\alpha$, so that $b^*(x) \in (b^+(\widehat{x}), 1]$.

B3. $b \in (b^+(\widehat{x}), 1]$.

From (19), in this region $\frac{\partial \Pi}{\partial b}$ is positive for $b < b^*(x)$ and negative for $b > b^*(x)$ and $\Pi(b;x)$ is continuous in b. Thus, for all $b \in (b^+(\widehat{x}), 1]$, $\Pi(b^*(x);x) \geq \Pi(b;x)$. By continuity, we also have that $\Pi(b^*(x);x) \geq \Pi(b^+(\widehat{x});x)$.

B4. $b \in (1, 1 + \frac{\alpha}{2}]$.

From (20), in this region $\frac{\partial \Pi}{\partial b}$ is negative for $b > b^*(x)$ and $\Pi(b;x)$ is continuous at 1. Thus, again there are no profitable deviations in this region.

B2. $b \in (b^-(\widehat{x}), b^+(\widehat{x}))$.

Again, from (18) $\frac{\partial \Pi}{\partial b}$ has the same sign as $\varphi(x,b)$. Now since $x > \widehat{x}$, $b^-(x) > b^-(\widehat{x})$ and as b increases from $b^-(\widehat{x})$ to $b^+(\widehat{x})$, $\frac{\partial \Pi}{\partial b}(x,b)$ is first positive, is 0 at $b^-(x)$ and then negative. Thus, for all $b \in (b^-(\widehat{x}), b^+(\widehat{x}))$,

$$\Pi\left(b^-(x);x\right) \geq \Pi(b;x).$$

We now show that $b^-(x)$ is not a profitable deviation. Since $x > \widehat{x}$, $b^-(x) > b^-(\widehat{x})$ and thus for all $b \in (b^-(x), b^+(\widehat{x}))$, $\frac{\partial \Pi}{\partial b}$ is given by (18). Similarly, $b^+(x) > b^+(\widehat{x})$ and thus for all $b \in (b^+(\widehat{x}), b^+(x))$, $\frac{\partial \Pi}{\partial b}$ is given by (19). Thus, we can write for $x > \widehat{x}$,

$$\begin{aligned}
\Delta(x) &= \Pi(b^+(x);x) - \Pi\left(b^-(x);x\right) \\
&= \int_{b^-(x)}^{b^+(\widehat{x})} G(\widehat{x})\varphi(x,b)\,l(b)\,db + \int_{b^+(\widehat{x})}^{b^+(x)} 2\beta'(\beta^{-1}(b))h(b)\left(x - \beta^{-1}(b)\right)\,db.
\end{aligned}$$

Differentiating Δ we obtain:

$$\begin{aligned}
\Delta'(x) &= G(\widehat{x})\int_{b^-(x)}^{b^+(\widehat{x})} \frac{\partial \varphi(x,b)}{\partial x} l(b)\,db - G(\widehat{x})\varphi\left(x, b^-(x)\right) l\left(b^-(x)\right) b^{-\prime}(x) \\
&\quad +2\int_{b^+(\widehat{x})}^{b^+(x)} \beta'(\beta^{-1}(b))h(b)\,db \\
&\quad +2\beta'\left(\beta^{-1}(b^+(x))\right) h(b^+(x))\left(x - \beta^{-1}(b^+(x))\right) \\
&= G(\widehat{x})\int_{b^-(x)}^{b^+(\widehat{x})} \frac{\partial \varphi(x,b)}{\partial x} l(b)\,db + 2\int_{b^+(\widehat{x})}^{b^+(x)} \beta'(\beta^{-1}(b))h(b)\,db
\end{aligned}$$

since by definition, $\varphi(x, b^-(x)) = 0$ and $b^+(x) = b^*(x) = \beta(x)$. Since $\frac{\partial \varphi(x,b)}{\partial x}$ and β' are both positive, $\Delta'(x) > 0$. But, since $\widehat{x} < x_\alpha$ implies that $K(\widehat{x}) = 0$, we have that $\Delta(\widehat{x}) = G(\widehat{x})K(\widehat{x}) = 0$, and so for all $x > \widehat{x}$, $\Delta(x) > 0$.

Hence,

$$\Pi(b^*(x);x) > \Pi\left(b^-(x);x\right).$$

B1. $b \in (0, b^{-}(\widehat{x})]$.

From (17), $\frac{\partial \Pi}{\partial b} > 0$ in this region and thus for all $b < (0, b^{-}(\widehat{x})]$, $\Pi\left(b^{-}(\widehat{x}); x\right) \geq \Pi(b; x)$. But we have already shown that $b^{-}(\widehat{x})$ is not a profitable deviation.

CASE C: $x_{\alpha} < x \leq 1$, so that $b^{*}(x) \in (1, 1 + \frac{\alpha}{2}]$.

C4. $b \in (1, 1 + \frac{\alpha}{2}]$.

From (20), in this region $\frac{\partial \Pi}{\partial b}$ is positive for $b > b^{*}(x)$ and negative for $b > b^{*}(x)$. Thus, there are no profitable deviations in this region. In particular, $\Pi(b^{*}(x); x) \geq \Pi(1; x)$.

C3. $b \in (b^{+}(\widehat{x}), 1]$.

From (19), in this region $\frac{\partial \Pi}{\partial b}$ is positive for $b < b^{*}(x)$ and thus for all $b \in (b^{+}(\widehat{x}), 1]$, $\Pi(b; x) \leq \Pi(1; x)$. Since 1 is not a profitable deviation, there are no profitable deviations in this region.

C2. $b \in \left(b^{-}(\widehat{x}), b^{+}(\widehat{x})\right)$.

If $x < \overline{x}$, the argument is the same as in Case B2.

If $x \geq \overline{x}$, then for all $b \in \left(b^{-}(\widehat{x}), b^{+}(\widehat{x})\right)$, $\varphi(x, b) \geq 0$ and thus from (18), Π is non-decreasing. This implies that $\Pi(b; x) \leq \Pi(b^{+}(\widehat{x}); x)$. Since $b^{+}(\widehat{x})$ is not a profitable deviation, neither is any b in this region.

C1. $b \in (0, b^{-}(\widehat{x})]$.

From (17), in this region $\frac{\partial \Pi}{\partial b}$ is positive and thus for all $b < (0, b^{-}(\widehat{x})]$, $\Pi\left(b^{-}(\widehat{x}); x\right) \geq \Pi(b; x)$. But we know that $b^{-}(\widehat{x})$ is not a profitable deviation.

We have verified that no deviations are profitable at any $x \in [0, 1]$.

Finally, note that the equilibrium payoff of a global bidder who receives a signal of x is

$$\Pi(b^{*}(x); x) = \int_{0}^{b^{*}(x)} \frac{\partial \Pi}{\partial b}(b; x)\, db \geq 0.$$

Thus each bidder wants to participate in the auction. This completes the proof.

Structure of the Equilibrium Strategy Some observations about the symmetric equilibrium strategy of the global bidders are in order.

First, observe that while the strategy is monotonically increasing, it may be discontinuous. In that case, the quasi-concavity of $x(b) = b - \alpha L(b)$ ensures that there is a single discontinuity at $\widehat{x}$ and:

$$\lim_{x \nearrow \widehat{x}} b^{*}(x) = b^{-}(\widehat{x}) < b^{+}(\widehat{x}) = \lim_{x \searrow \widehat{x}} b^{*}(x).$$

By construction, a global bidder with signal $\widehat{x}$ is indifferent between bidding $b^{-}(\widehat{x})$ and $b^{+}(\widehat{x})$, although we have chosen $b^{*}(\widehat{x}) = b^{-}(\widehat{x})$.

Second, consider the behavior of $b^{*}(x)$ when x is close to x_{α}. The two conditions $b^{*}(x_{\alpha}) = \beta(x_{\alpha}) = 1$ and $\beta'(x_{\alpha}) = 0$ together imply that there is an interval $(x - \epsilon, x_{\alpha})$ such that for all $x \in (x - \epsilon, x_{\alpha})$, $b^{*}(x) > x + \frac{\alpha}{2}$. Thus when the signal is close to x_{α}, the global bidders bid more than half the value of the two-object bundle. To interpret such "overbidding," think of $x+\alpha$ as the marginal value of a second object to a global bidder who has already won one object. A win with a bid just under 1 is very likely to be accompanied by a win in the other auction (since the other global bidders are surely beaten), and so the expected marginal value is close to $x + \alpha$, though bidding that much in both auctions would not be a good idea, as the two expected marginal values sum to much more than $2x + \alpha$.[9]

Third, if $\alpha \leq 1$ and $n > 1$, both the curves $\varphi = 0$ and $\psi = 0$ are positively sloped at the origin; indeed both have a slope of 1. Since $b^{*}(x)$ lies between the curves $\varphi = 0$ and $\psi = 0$ when x is close to 0, it is also the case that $b^{*}(0) = 0$ and $b^{*\prime}(0) = 1$. Thus for x close to 0, $b^{*}(x) < x + \frac{\alpha}{2}$. To interpret this "underbidding" observe that when x is close to 0, there is little chance that a winning bid in one auction will be accompanied by a win in the other (even though the other global bidders will surely be beaten), so with high probability, the marginal value of the object is only x.

Fourth, when $\alpha > 1$, it is possible that $\lim_{x \to 0} b^{*}(x) > 0$. This is because now the $\varphi = 0$ curve hits the vertical axis at a height less than 1 and thus the solution to the differential equation (9) and (10) may hit the vertical axis unhindered. Intuitively, even for bidders with signals close to 0, the remote, but attractive possibility of winning both objects, leads to high bids.

Finally, as an example, suppose that all signals and values are uniformly distributed. For the case $k = 2, n = 1$ and $\alpha = 1$, a closed-form solution for b^{*} is available:

$$b^{*}(x) = \begin{cases} \frac{4x}{1+4x^{2}} & \text{if } \; 0 \leq x \leq \frac{1}{2} \\ x + \frac{1}{2} & \text{if } \; \frac{1}{2} \leq x \leq 1 \end{cases}$$

and is depicted in Figure 4. Of course, the fact that b^{*} is concave over the interval $[0, x_{\alpha}]$ (or even that it is continuous) does not generalize.

[9] Robert Wilson pointed out to us that with synergies, a second price simultaneous auction has some of the flavor of a war of attrition in that *ex post* losses are a real possibility. Bids that exceed the value occur with positive probability in equilibria of the war of attrition (see Krishna and Morgan (1994)).

4 Comparative Statics

In this section, we consider the effects of changes in the three parameters α, k, and n separately on the equilibria from Sections 2 and 3.

Varying the Synergy Parameter When $k = 1$, increasing α decreases $x(b) \equiv b - \alpha L(b)$, hence increases the smallest positive root of $\varphi(x, \cdot)$, $b^-(x)$, which is the bid before any jump. Now the jump to 1, if there is one, occurs where $\Pi(1, \widehat{x}) = \Pi(b^-(\widehat{x}), \widehat{x})$; but, from the Envelope Theorem,

$$\frac{d}{d\alpha}[\Pi(1, \widehat{x}) - \Pi(b^-(\widehat{x}), \widehat{x})] = 1 - (L(b^-(\widehat{x}))^2 > 0.$$

So the jump comes at a smaller value of $\widehat{x}$ when α is larger, and the optimal bid is therefore nondecreasing in α for every x.

For $k \geq 2$, a similar conclusion is obtained:

Proposition 1 *Assume $k \geq 2$ and $\alpha_1 > \alpha_2$. Let $b_1^*(\cdot)$ and $b_2^*(\cdot)$ be the respective symmetric equilibrium strategies of the simultaneous auction. Then for all x,*

$$b_1^*(x) \geq b_2^*(x).$$

Proof. The conclusion is clearly true for $x \geq (1 - \frac{\alpha_1}{2})$, and $b_1^*(1 - \frac{\alpha_1}{2}) > b_2^*(1 - \frac{\alpha_1}{2})$. Therefore, if there exists an x such that $b_1^*(x) < b_2^*(x)$, either there is a largest value of x, say x^*, where $b_1^*(x^*) = b_2^*(x^*)$ and for all $x \in [x^*, 1]$, $b_1^*(x) \geq b_2^*(x)$, or b_1^* jumps over b_2^* and stays above it. To rule out the first possibility, observe that in the last factor of (11) the numerator is negative and the denominator positive; hence changing from α_1 to α_2 increases the absolute value of that factor, and hence increases $\beta'(x^*)$. But this is inconsistent with the definition of x^*, a contradiction. To rule out the second possibility, we argue first that if b_1^* jumps at, say, $\widehat{x}_1$, then if b_2^* jumps at all, this jump is to the right of $\widehat{x}_1$. To see this, observe that

$$\frac{d}{d\alpha}[\Pi(b_1^+(\widehat{x}_1), \widehat{x}_1) - \Pi(b_1^-(\widehat{x}_1), \widehat{x}_1)] = [(L(b_1^+(\widehat{x}_1))^2 - (L(b_1^-(\widehat{x}_1))^2]G(\widehat{x}_1) > 0$$

from the Envelope Theorem; but then the monotonicity of $K(x)$ (Lemma 1) implies that decreasing α pushes any jump in b to the right. So, for all $x \in (\widehat{x}_1, 1]$, $b_1^*(x) \geq b_2^*(x)$. Now, since the locus satisfying $\varphi_2 = 0$ for α_1 lies to the left of the corresponding locus for α_2, it follows that for all $x \in [0, \widehat{x}_1]$, $b_1^*(x) \geq b_2^*(x)$ as well.

So increases in α unambiguously increase bids (weakly) for all global bidders.

Varying the Number of Global Bidders Next consider changes in k, where the result is perhaps not so intuitive.

Proposition 2 *Let $b_k^*(\cdot)$ and $b_{k+1}^*(\cdot)$ be the symmetric equilibrium strategies of the simultaneous auction when the number of global bidders is k and $k+1$, respectively. Then for all x,*

$$b_{k+1}^*(x) \leq b_k^*(x).$$

Proof. First, consider changes from k to $k+1$, where $k \geq 2$. On $[1-\frac{\alpha}{2}, 1]$, the two equilibrium strategies, b_k^* and b_{k+1}^*, coincide. If neither equilibrium strategy has a jump at $1-\frac{\alpha}{2}$, then for $x < 1-\frac{\alpha}{2}$ but sufficiently close to it, differences in β' for equal values of x are determinative. Substituting for G and g in (11) reveals that k enters only through the factor $(k-1)$ in the numerator. So a change from k to $k+1$ increases β' and hence reduces β (since the two $\beta-$curves meet at $1-\frac{\alpha}{2}$). So, as above, if $b_{k+1}^*(x) > b_k^*(x)$, either there is a largest value of x, say x^*, where $b_{k+1}^*(x^*) = b_k^*(x^*)$ and $b_{k+1}^*(x) < b_k^*(x)$ for all $x \in (x^*, 1-\frac{\alpha}{2})$, or b_k^* jumps over b_{k+1}^* and stays above it until $1-\frac{\alpha}{2}$. The first possibility is ruled out, since at x^* the higher derivative is associated with b_{k+1}^*. To rule out the second possibility, note, again from the monotonicity of $K(x)$, that the jump in b_k^*, say at $\widehat{x}_{k,}$ must occur to the left of any jump in b_{k+1}^*. But, since the $\varphi_2(\cdot,\cdot) = 0$ locus is the same for all k, this means that $b_{k+1}^*(x) < b_k^*(x)$ for all $x \in (0, \widehat{x}_k)$ as well. So b_k^* cannot jump over b_{k+1}^*. (Note that if either strategy jumps at $1-\frac{\alpha}{2}$, the argument is essentially unchanged.)

Now consider changes from $k = 1$ to 2. It is straightforward to check that in all cases b_1^* hits 1 before b_2^* does. (And for still larger signals, we may take the global's bid when $k = 1$ to be as large as we like.) Before its jump, $b_1^*(x)$ follows $b^-(x)$, the curve that defines the upper boundary of S for $k = 2$; so the single global bidder is again more aggressive.

Varying the Number of Local Bidders For changes in n, it appears difficult to say anything in general. We confine ourselves here to reporting a single comparison which illustrates that the equilibrium bid functions b^* may cross. Suppose that all values and signals are uniformly distributed. When $\alpha = 1$ and $k = 2$,

for $n = 1$, $b^*(.24) \cong .78$ and $b^*(.30) \cong .88$;

for $n = 2$, $b^*(.24) \cong .75$ and $b^*(.30) \cong .91$.

5 More Than Two Objects

In this section we consider situations where the number of objects, m, is greater than two. There is more than one way in which the model of Section 3 can be generalized to the case of many objects. Here we study only two of the possible extensions.

The first is a straightforward extension of the two object case: there are m objects and each global bidder is interested in all the objects. We refer to this as a model with "common interests." In the second model, there are m objects and km global bidders with "overlapping interests," as follows: k global bidders are interested in objects 1 and 2; k are interested in objects 2 and 3; k are interested in objects 3 and 4; and so on, ending with k bidders interested in objects m and 1. Thus each global bidder is interested in only two objects and there are exactly $2k$ global bidders who bid on any single object.

5.1 Common Interests

In the model with common interests each of k global bidders is interested in obtaining as many of the m objects as possible. Of course, we assume that the values associated with multiple objects are subject to increasing returns, modelled as follows. Consider the marginal value of an object to a global bidder. The first object has a marginal value of x and, as in Section 3, the marginal value of the second object is $x + \alpha$. To continue in the simplest way, now suppose that the marginal value of the third object is $x+2\alpha$, the marginal value of the fourth object is $x+3\alpha$, and so on. In general, the marginal value of the tth object is $x+(t-1)\,\alpha$, for $t = 1, 2, ..., m$, and thus the total value from obtaining t objects is $tx + \frac{t(t-1)}{2}\alpha$. Notice that the marginal value of additional objects is increasing in the number of objects obtained, and the increasing returns implicit in this formulation are rather strong.

Of course, if $\alpha \geq \frac{2}{m-1}$ then the average or per-unit value associated with m objects is greater than 1 for all $x > 0$, and so the local bidders will be shut-out. To rule out this trivial case, assume that $\alpha < \frac{2}{m-1}$.

If $k-1$ global bidders follow the strategy β and a global bidder with signal x bids b, his expected payoff is

$$\Pi(b;x) = \sum_{t=1}^{m} \binom{m}{t} L^t (1-L)^{m-t}\, G\left[t\,(x - E(p \mid b)) + \frac{t\,(t-1)}{2}\alpha\right] \quad (22)$$

$$
\begin{aligned}
&= \frac{1}{2}m(m-1)\alpha L^2 G + m(x - E(p \mid b))LG && (23)\\
&= \frac{1}{2}m(m-1)\alpha L^2 G + mxLG - m\int_0^b ph(p)\,dp
\end{aligned}
$$

where we have economized on notation by writing $L(b)$ as L and $G\left(\beta^{-1}(b)\right)$ as G and where $E(p \mid b)$ is defined as usual by (6). The tth term in (22) is the payoff to a global bidder from obtaining exactly t of the m objects. The simplification to (23) results from recalling the formulae for the first two moments of the binomial distribution with parameters m and L:

$$\sum_{t=1}^{m}\binom{m}{t}L^t(1-L)^{m-t}t = mL,$$

and

$$\sum_{t=1}^{m}\binom{m}{t}L^t(1-L)^{m-t}t^2 = mL(1-L) + m^2L^2.$$

The first-order condition for the lower part of the equilibrium yields the differential equation

$$\beta' = \frac{-L(\beta)g(x)}{l(\beta)G(x)}\left(\frac{\frac{1}{2}m(m-1)\alpha L(\beta) - m\beta + mx}{m(m-1)\alpha L(\beta) - m\beta + mx}\right), \tag{24}$$

which generalizes (9). The relevant boundary condition is now

$$\beta\left(1 - \frac{m-1}{2}\alpha\right) = 1. \tag{25}$$

As in Section 3, define

$$
\begin{aligned}
\psi_m(x,b) &\equiv \frac{1}{2}m(m-1)\alpha L(b) - mb + mx\\
\varphi_m(x,b) &\equiv m(m-1)\alpha L(b) - mb + mx
\end{aligned}
$$

so that (24) may be rewritten as

$$\beta' = \frac{-L(\beta)g(x)}{l(\beta)G(x)}\left(\frac{\psi_m(x,\beta)}{\varphi_m(x,\beta)}\right), \tag{26}$$

a form analogous to (11).

Observe that for $b > 0$, $\varphi_m(x,b) > \psi_m(x,b)$ and if L is convex, then the function $x(b) \equiv b - (m-1)\alpha L(b)$ which solves $\varphi_m(x,b) = 0$, is concave. Now, as in Section 3, the differential equation (26) together with the boundary condition (25) may be used to construct the lower part of the equilibrium strategy for the global bidders. The upper part is also analogous and the verification that an equilibrium results is the same as in the proof of Theorem 2.

In the formulation above, the increasing returns have been specified in a particular way: the marginal values are x, $x+\alpha$, $x+2\alpha$, $x+3\alpha$, etc. It can be shown that, if the increasing returns are at least as strong as this, an equilibrium can be constructed along the lines outlined above. If they are weaker, however, there are other complications and the exact nature of the equilibrium strategy is unknown.

5.2 Overlapping Interests

In the model with overlapping interests different global bidders are interested in different pairs of objects. Suppose that there are m types of global bidders and k bidders of each type. For $t = 1, 2, ..., m$, a global bidder of type t is interested in objects #t and #$t+1$ (where $m+1 \equiv 1$). Thus a global bidder of type t competes with the $k-1$ other global bidders of type t and k global bidders of type $t-1$ for object #t. Similarly, he competes with the $k-1$ global bidders of type t and k global bidders of type $t+1$ for object #$t+1$. As always there are also n local bidders who are interested in each object. As in Section 3, for a global bidder with signal x, the value of a single object is x and the value of two objects is $2x+\alpha$.

Suppose all global bidders follow the same strategy β. If a global bidder of type t bids b on both object #t and #$t+1$ after receiving a signal of x, his expected payoff is

$$\begin{aligned}
\Pi(b;x) &= L(b)^2 F_G\left(\beta^{-1}(b)\right)^{3k-1}(2x+\alpha-2E(p \mid b)) \\
&\quad +2L(b)F_G\left(\beta^{-1}(b)\right)^{2k-1}\left(1-L(b)F_G\left(\beta^{-1}(b)\right)^k\right)(x-E(p \mid b)) \\
&= \alpha L(b)^2 F_G\left(\beta^{-1}(b)\right)^{3k-1} \\
&\quad +2L(b)F_G\left(\beta^{-1}(b)\right)^{2k-1}(x-E(p \mid b))
\end{aligned}$$

where the first term in the expression for $\Pi(b;x)$ is the payoff from winning both objects #t and #$t+1$ and the second term is the payoff from winning one of the objects. Once again, $E(p \mid b)$ is the expected price paid by a global bidder who wins with a bid of b; but the distribution function H of the price paid by a global bidder is different from that in Section 3. We now have that

$$H(p) = L(p) F_G \left(\beta^{-1}(b)\right)^{2k-1}.$$

When $b < 1$, the first-order condition for an equilibrium results in the following differential equation

$$\beta' = \frac{-(2k-1) L(\beta) f_G(x)}{l(\beta) F_G(x)} \left(\frac{\left(\frac{3k-1}{2k-1}\right) \alpha L(\beta) F_G(x)^k - 2\beta + 2x}{2\alpha L(\beta) F_G(x)^k - 2\beta + 2x} \right). \tag{27}$$

When $b \geq 1$ and hence $L(b) = 1$, the first-order condition for an equilibrium results in

$$b^*(x) = x + \frac{1}{2} \left(\frac{3k-1}{2k-1} \right) \alpha F_G(x)^k$$

provided that $x \geq x_\alpha$, where

$$x_\alpha + \frac{1}{2} \left(\frac{3k-1}{2k-1} \right) \alpha F_G(x_\alpha)^k = 1.$$

The relevant boundary condition associated with (27) is now

$$\beta(x_\alpha) = 1, \tag{28}$$

so that we have $\beta'(x_\alpha) = 0$.

Define

$$\begin{aligned} \Psi(x,b) &\equiv \left(\frac{3k-1}{2k-1} \right) \alpha L(b) F_G(x)^k - 2b + 2x \\ \Phi(x,b) &\equiv 2\alpha L(b) F_G(x)^k - 2b + 2x, \end{aligned}$$

so that (27) can be rewritten as

$$\beta' = \frac{-(2k-1) L(\beta) f_G(x)}{l(\beta) F_G(x)} \left(\frac{\Psi(x,\beta)}{\Phi(x,\beta)} \right), \tag{29}$$

a form analogous to (11).

First, notice that for $b > 0$, $\Phi(x,b) > \Psi(x,b)$. Second, even though the equation $\Phi(x,b) = 0$ cannot be solved explicitly for x in terms of b, it is still the case that for all x, the set

$$S(x) \equiv \{b : \Phi(x,b) \leq 0\}$$

is convex whenever L is a convex function; so that an argument similar to that in the proof of Theorem 2 goes through.

This allows the construction of the lower part of the equilibrium strategy b^* exactly as in Section 3 and the verification that this constitutes an equilibrium is the same as in Theorem 2.

6 Sequential versus Simultaneous Auctions

In this section we examine the sequential format, that is, when the objects are auctioned off sequentially. Our goal is to compare the revenues raised from the sequential auction to those raised from the simultaneous auction. We assume that there are two objects for sale.

Once again it is convenient to begin with the case of a single global bidder.

6.1 Single Global Bidder

To find the equilibrium in the sequential auction, we work backwards and begin by examining the auction for the second object. As usual, in both auctions it is a dominant strategy for the local bidders to bid their respective values.

6.1.1 Auction #2

Suppose the global bidder received a signal of x. If he won the first auction, the value of the second object is $x + \alpha$ and it is a dominant strategy to bid $x + \alpha$ in the second auction. If he did not win the first auction, the value of the second object is x and it is a dominant strategy to bid x in the second auction.

6.1.2 Auction #1

Let $\pi_1(x)$ denote the expected payoff in the second auction of the global bidder with a signal of x conditional on having *won* the first auction. Having won the first auction, the probability that he will win the second auction with a bid of $x + \alpha$ is $L(x + \alpha)$. His expected payoff can thus be written as

$$
\begin{aligned}
\pi_1(x) &= L(x+\alpha)\left[x+\alpha - E\left(p \mid x+\alpha\right)\right] \\
&= L(x+\alpha)\left[x+\alpha - \frac{1}{L(x+\alpha)}\int_0^{x+\alpha} pl(p)dp\right]
\end{aligned}
$$

$$
\begin{aligned}
&= (x+\alpha)\,L\,(x+\alpha) - \int_0^{x+\alpha} pl(p)dp \\
&= \int_0^{x+\alpha} L(p)dp.
\end{aligned}
$$

Let $\pi_0(x)$ denote the expected payoff in the second auction of the global bidder with a signal of x conditional on having *lost* the first auction. Having lost the first auction, the probability that he will win the second auction with a bid of x is $L\,(x)$. The expected payoff can thus be written as

$$
\begin{aligned}
\pi_0(x) &= L\,(x)\,[x - E\,(p \mid x)] \\
&= \int_0^{x} L(p)dp.
\end{aligned}
$$

In the first auction it is a dominant strategy for the global bidder to bid $\gamma(x)$ where

$$
\begin{aligned}
\gamma(x) &= x + \pi_1(x) - \pi_0(x) \\
&= x + \int_0^{x+\alpha} L(p)dp - \int_0^{x} L(p)dp \\
&= x + \int_x^{x+\alpha} L(p)dp. \qquad (30)
\end{aligned}
$$

The global bidder bids the value of the object, x, plus a premium that represents the difference in the values attached to winning and losing.

We thus obtain:

Theorem 3 *Suppose $k = 1$. The following constitutes an equilibrium of the sequential auction. (i) All local bidders bid their respective values; and (ii) the global bidder with signal x bids as follows:*

$$
\begin{aligned}
&\textit{in auction } \#1: \quad \gamma(x) \\
&\textit{in auction } \#2: \quad \begin{cases} x+\alpha & \textit{if he won auction } \#1 \\ x & \textit{if he lost auction } \#1 \end{cases}
\end{aligned}
$$

where $\gamma(x)$ is given by (30).

6.2 Multiple Global Bidders

We now deal with the case where there are at least two global bidders.

6.2.1 Auction #2

Consider a global bidder, say 1, who has received a signal of x. As before, if this bidder won the first auction, the value of the second object is $x + \alpha$ and it is a dominant strategy to bid $x + \alpha$ in the second auction. If this bidder did not win the first auction, the value of the second object is x and it is a dominant strategy to bid x in the second auction.

6.2.2 Auction #1

Let $\pi_1(x)$ denote the expected payoff in the second auction of a global bidder, say 1, with a signal of x conditional on having *won* the first auction. Having won the first auction global bidder 1 must have outbid the other global bidders and is thus sure to outbid them in the second auction also. Thus the probability of winning the second auction with a bid of $x+\alpha$ conditional on having won the first is exactly $L(x+\alpha)$. Let $M(p) \equiv L(p)G(p)$ and $m \equiv M'$. The expected payoff can thus be written as

$$\begin{aligned}
\pi_1(x) &= L(x+\alpha)\left[x+\alpha - E(p \mid x+\alpha)\right] \\
&= L(x+\alpha)\left[x+\alpha - \frac{1}{M(x+\alpha)}\int_0^{x+\alpha} pm(p)dp\right] \\
&= L(x+\alpha)(x+\alpha) - \frac{1}{G(x+\alpha)}\int_0^{x+\alpha} pm(p)dp \\
&= \frac{1}{G(x+\alpha)}\left[L(x+\alpha)G(x+\alpha)(x+\alpha) - \int_0^{x+\alpha} pm(p)dp\right] \\
&= \frac{1}{G(x+\alpha)}\int_0^{x+\alpha} M(p)dp. \qquad (31)
\end{aligned}$$

Let $\pi_0(x)$ denote the expected payoff in the second auction of a global bidder, say 1, with a signal of x conditional on having *lost* the first auction. Suppose all global bidders follow the strategy γ in the first auction. Having lost the first auction, the payoff from the second auction is positive only if 1 nevertheless outbid all the other global bidders in the first auction. The probability of this event conditional on having lost the first auction is

$$\frac{[1 - L(\gamma(x))]\,G(x)}{1 - G(x)L(\gamma(x))}.$$

The expected payoff in the second auction is thus

$$\begin{aligned}
\pi_0(x) &= \frac{[1-L(\gamma(x))]\,G(x)}{1-G(x)\,L(\gamma(x))} L(x)\,[x-E(p \mid x)] \\
&= \frac{1-L(\gamma(x))}{1-G(x)\,L(\gamma(x))} \left[M(x)x - \int_0^x pm(p)dp\right] \\
&= \frac{1-L(\gamma(x))}{1-G(x)\,L(\gamma(x))} \int_0^x M(p)dp. \qquad (32)
\end{aligned}$$

(Notice it follows from (31) and (32) that $\pi_1(x) \geq \pi_0(x)$.)

A global bidder with signal x should bid $\gamma(x)$, where

$$\begin{aligned}
\gamma(x) &= x + \pi_1(x) - \pi_0(x) \\
&= x + \frac{1}{G(x+\alpha)} \int_0^{x+\alpha} M(p)dp \\
&\quad - \frac{1-L(\gamma(x))}{1-G(x)\,L(\gamma(x))} \int_0^x M(p)dp. \qquad (33)
\end{aligned}$$

The equilibrium strategy is then the solution to a fixed point problem. We now show that such a fixed point always exists. Define the function

$$\chi(x,b) = x + \frac{1}{G(x+\alpha)} \int_0^{x+\alpha} M(p)dp - \frac{1-L(b)}{1-G(x)\,L(b)} \int_0^x M(p)dp,$$

so that the equilibrium bid is one that satisfies $\chi(x,b) = b$. Now notice that since $\chi(x,x) \geq x$ and $\chi(x, x+\pi_1(x)) \leq x + \pi_1(x)$, there exists a $b \in [x, x+\pi_1(x)]$ such that $\chi(x,b) = b$.

Theorem 4 *Suppose $k \geq 2$. The following constitutes an equilibrium of the sequential auction. (i) All local bidders bid their respective values; and (ii) a global bidder with signal x bids as follows:*

$$\begin{array}{ll}
\textit{in auction } \#1: & \gamma(x) \\
\textit{in auction } \#2: & \begin{cases} x+\alpha & \textit{if he won auction } \#1 \\ x & \textit{if he lost auction } \#1 \end{cases}
\end{array}$$

where $\gamma(x)$ is a solution to (33).

6.3 Revenue Comparisons

A general comparison of the revenues from the sequential and simultaneous auctions appears to be rather difficult. This is because the equilibrium strategies for the global bidders are quite complicated, especially in the simultaneous auction. We now report some numerical results on two examples. Suppose that all values and signals are uniformly distributed and that there is only one local bidder at each location, that is, $n = 1$. We ask how the expected revenues from the two auctions vary with the parameter α.

6.3.1 Single Global Bidder

If there is a single global bidder, the revenues from the two auctions can be explicitly computed. Figure 5 depicts the difference between the revenue from the simultaneous auction, R_{SIM}, and the revenue from the sequential auction, R_{SEQ}, as a function of α. When α is small the sequential auction results in higher revenues. For large α, the simultaneous auction is superior.

6.3.2 Two Global Bidders

When there are two global bidders, the revenues from the two auction forms cannot be calculated explicitly and so we report the results of Monte Carlo simulations. For each of twenty different values of α, 1000 (pairs of) auctions were simulated and the resulting expected revenues are depicted in Figure 6.

Again, we find that the sequential auction is revenue superior for low values of α and the simultaneous auction is superior when α is high.

7 Concluding Remarks

We motivated this study by referring to some of the significant aspects of the PCS spectrum auctions. There are other aspects of those auctions for which our model does not provide a good fit, however; and these suggest directions for future research. First, our models assume independent (and within-type identical), private valuations/signals and deterministic, commonly-known synergy terms. Although a decomposition involving an additively separable synergy term is probably not a bad approximation, much more realistic, but also much harder, would be if the signals, valuations, and synergy terms were correlated and, say, affiliated.

Second, the strategic problems that arise in the presence of synergies are very subtle; most likely they are affected by the fine details of the auction rules (for example, in an open-auction format, whether all auctions remain open until there are no active bidders on any, or whether they close individually). The crude, sealed-bid, second-price simultaneous auctions in our model cannot hope to provide more than a rough cut at the strategic problem. By combining the hypothesis of a class of bidders who are unaffected by the synergies with a second-price rule, however, we have been able to accommodate asymmetric bidders with a fairly standard approach. Until more progress is made on single-auction theory with asymmetric bidders, one cannot hope to do too much better.

Third, we have assumed that each global bidder treats the objects as identical *ex post*. More interesting and realistic would be, say, the assumption that each global bidder sees a separate signal for each object. It would then be possible to ask whether a realized signal that is lower relative to its marginal distribution than a companion signal will be more "over-" or "underbid." Based on the discussion at the end of Section 3 about "over-" and "underbidding," we conjecture that when a bidder sees an extremely low signal paired with an extremely high signal in such a setting, in equilibrium he should bid aggressively (relative to the signal) on the object with the low signal and relatively passively on the companion object.

Fourth, since the bidders in the FCC auction are mostly firms that compete in the final market for services, the value of a license depends also on the distribution of licenses to other bidders. Thus the auctions involve endogenous values in the sense studied by K. Krishna (1993) and Gale and Stegeman (1993). In particular, those papers look at sequential auctions when increasing returns are present in a complete information setting. How the presence of incomplete information and simultaneous sales affects their results remains to be seen.

Seemingly less obnoxious are the assumptions of compact supports for the private valuations, identical $[0, 1]-$supports, and strictly positive densities. These assumptions are made for standard technical reasons. The quasi-concavity assumption on $x(b)$ is needed to insure that no more than one jump occurs in equilibrium. We suspect that weakening it will admit the possibility of multiple jumps, but there appear to be additional problems in attempting a straightforward extension of the constructions used in this paper.

Of our results, we call attention once again to three. First, that increasing the number of global bidders always (and increasing the number of local bidders, sometimes) results in less aggressive bidding by the global bidders. The intuition for this is apparently that the more competitors there are, the higher is likely to be the second-highest bid.[10] When "overbidding," this is obviously important. Even when "underbidding," however, this "price-effect" can hurt, for low bids are also too high *ex post* if only one object is won. Evidently this price effect overwhelms the more familiar competitive effect of increasing the number of competitors. Of special interest would be if the decline in aggressiveness were so severe that it could cause a decline in expected revenues from the auctions. (We have not done enough simulations to know whether this is even worth being called a conjecture.) It could also be, however, that the anomalous price effect would be less significant in, say, a first-price auction setting.

Second, our admittedly few simulations suggest that whether simultaneous auctions raise more or less revenues than sequential auctions depends on the strength of the synergies present. When the synergies are strong, the simultaneous auction seems to be revenue superior. The FCC was (by law) not primarily concerned with maximizing revenue in its auction-design decision, but it would likely be important to sellers in other contexts. If more synergy tends to favor the simultaneous design more generally, that would be worth knowing.

Third, in the models of Engelbrecht-Wiggans and Weber (1979) and Lang and Rosenthal (1991), the synergies are negative rather than positive. Those models have two objects and simultaneous sealed-bid auctions, and, in the equilibria, a single player's bids are strongly negatively correlated with each other. The intuition is that one wants to win one object but not both, and severe negative correlation turns out to increase the chances of this when the opponents behave similarly. By contrast here, a global bidder wants to win both but not one (assuming that on average the price will be high). By generating strong positive correlations between his own bids, a global bidder increases the chances of avoiding the bad outcome when all other global bidders behave similarly. This is most obviously seen when the bidding gets high enough so that the local bidders are shut-out of the auction.

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