ON THE CONVERGENCE OF FICTITIOUS PLAY

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Abstract

We study the continuous time Brown-Robinson fictitious play process for non-zero sum games. We show that, in general, fictitious play cannot converge cyclically to a mixed strategy equilibrium in which both players use more than two pure strategies.

1 Introduction

This paper studies the "fictitious play" (FP) learning process due to Brown [1] and Robinson [11]. The FP process was originally proposed as a computational tool for determining the value of a two-person zero-sum game. However, it can also be interpreted as a learning process for boundedly rational agents in which each player plays a myopic best response in each period, on the assumption that the opponent's future actions will resemble the past.

Robinson [11] established the result that the FP process converges in finite two-person zero-sum games. Miyasawa [9] showed the convergence of FP in 2×2 games. However, the convergence cannot be guaranteed in general non-zero sum games as an example due to Shapley [14] shows.

We find it convenient to work with a continuous time formulation of fictitious play (referred to as CFP) rather than the discrete time formulation proposed by Brown [1]. While the convergence results cited above were for the discrete process, both hold for the continuous time process also, as does

Shapley's counterexample. Our main result (Theorem 3) is that CFP almost never converges cyclically to a mixed strategy equilibrium in which both players use more than two pure strategies. In a recent paper, Hofbauer [6] has made a related conjecture: if CFP converges to a regular mixed strategy equilibrium, then the game is zero-sum.

As is well-known, the interpretation of mixed strategy equilibria is problematic (see, for instance, Rubinstein [13]). In two person zero sum games a justification for mixed strategies is that the "correct" probabilities provide the best defense against the opponent. But in non-zero sum games a justification on defensive grounds cannot be made. A point of view originating with Harsanyi [4] takes the position that the equilibrium probabilities represent only the subjective beliefs of other players about the behavior of a particular player; thus it is not necessary to assume that players actually choose randomized strategies. Fictitious play and associated learning procedures suggest a way in which such beliefs can form over time by means of a gradual process. However, learning procedures can serve to justify mixed strategy equilibria only in circumstances in which the procedures converge to an equilibrium. Our result shows that, in general non-zero sum games, mixed strategy equilibria cannot be limits of a fictitious play process and thus are inherently unstable.

The behavior of dynamical processes in the presence of mixed equilibria has previously been examined in a related context by Crawford [2]. Crawford [2] studies a class of learning procedures in which (a) players have a finite memory; and (b) play mixed strategies which are adjusted in response to the difference in payoffs from playing a particular pure strategy and the mixed strategy against the actual play in the recent past. Crawford [2] then shows that mixed strategy equilibria are generally unstable. The procedures considered do not include the CFP; they are more akin to evolutionary processes like the so called "replicator dynamics."

Evolutionary dynamical systems are considered in more detail by Hofbauer and Sigmund [7]. Their results also suggest that mixed strategy equilibria are unstable in general (asymmetric) bimatrix games (see Section 27.5 of [7]).

In other, more closely related work, Fudenberg and Kreps [3] study interpretational issues concerning mixed strategies and learning processes like FP. They propose some alternative systems based on ideas stemming from Harsanyi's [4] purification theorem and derive convergence results for 2×2

games. Jordan [8] points out other difficulties in interpreting the convergence of learning processes to mixed equilibria. In particular, he points out that the convergence concerns players' expectations and not strategies or payoffs.

2 Fictitious Play

Let $G = (A, B)$ be a two-player *game* where A and B are $I \times J$ matrices. We will refer to $I = \{1, 2, \ldots, I\}$ and $J = \{1, 2, \ldots, J\}$ as the sets of pure strategies available to players 1 and 2 respectively. As usual, if player 1 chooses strategy i and player 2 chooses strategy j, the payoff to player 1 is a_{ij} and the payoff to player 2 is b_{ij}. The sets of mixed strategies are denoted by $\Delta(I)$ and $\Delta(J)$ respectively. Let $\delta_i \in \Delta(I)$ be the mixed strategy that assigns weight 1 to i. We will identify i with δ_i and write $i \in \Delta(I)$ instead of $\delta_i \in \Delta(I)$.

For all $q \in \Delta(J)$, let $BR(q)$ be the set of *pure strategy best responses* for player 1 and denote by $\mathit{supp}\, q = \{j : q_j > 0\}$ the *support* of q. The mixed strategy pair (p^*, q^*) is a *Nash equilibrium* if $\mathit{supp}\, p^* \subseteq BR(q^*)$ and $\mathit{supp}\, q^* \subseteq BR(p^*)$.

For $t = 0, 1, 2, ...$, the sequence $(p(t), q(t))$ is a *discrete time fictitious play process (DFP)* if

$$(p(0), q(0)) \in \Delta(I) \times \Delta(J);$$

and for all $t \geq 0$,

$$p(t+1) = \frac{tp(t) + i(t)}{t+1}, \;\; q(t+1) = \frac{tq(t) + j(t)}{t+1}$$

where $i\,(t) \in BR\,(q\,(t))$ and $j\,(t) \in BR\,(p\,(t))$.

Thus, $p(t+1)$ is a weighted average of $p(t)$ and $i\,(t)$ where the weights are $\frac{t}{t+1}$ and $\frac{1}{t+1}$. New strategies are chosen in each "period." Now suppose $\delta > 0$ is the time between adjustments. Replacing the weights by $\frac{t}{t+\delta}$ and $\frac{\delta}{t+\delta}$, we get

$$p(t+\delta) = \frac{tp(t) + \delta i\,(t)}{t+\delta}$$

As $\delta \rightarrow 0$, we obtain

$$\frac{dp(t)}{dt} = \frac{i\,(t) - p(t)}{t}$$

This is not defined for $t = 0$, so the continuous time version should start at some $t_0 > 0$, say $t_0 = 1$. This leads to the following definition:

For $t \geq 1$, the path $(p(t), q(t))$ is a *continuous time fictitious play process (CFP)* if

$$(p(1), q(1)) \in \Delta(I) \times \Delta(J);$$

and

$$\frac{dp(t)}{dt} = \frac{i\,(t) - p(t)}{t}, \; \frac{dq(t)}{dt} = \frac{j\,(t) - q(t)}{t}$$

where $i\,(t) \in BR\,(q\,(t))$ and $j\,(t) \in BR\,(p\,(t))$.

The discrete time fictitious play process (DFP) is also known as the "Brown-Robinson Learning Process" ([1], [11]). In this paper, we find it convenient to work with its continuous time version (CFP). We hope to explore whether our results continue to hold for the discrete process (DFP) later.

It is well known that if the DFP (or CFP) $(p(t), q(t))$ converges to (p^*, q^*), then (p^*, q^*) is a Nash equilibrium of G.

Cyclic Play Under fictitious play, each player always plays a best response against the empirical distribution of the opponent's play. Under CFP, therefore, when a player switches from one pure strategy to another he is precisely indifferent between these two strategies. This fact is crucial for our analysis. Let $t_0 = 1$ and let $(t_1, t_2, t_3, ...)$ be the times when some player switches his/her strategy. (In an exceptional case, both players may switch at the same instant.) Let

$$(i_{t_n}, j_{t_n}) \equiv (i(t), j(t)) \quad \text{for} \quad t \in (t_n, t_{n+1})$$

denote the choices in the interval (t_n, t_{n+1}). The interval (t_n, t_{n+1}) consists of a string of consecutive plays of (i_{t_n}, j_{t_n}), referred to as a *run*. The *run-length* is $t_{n+1} - t_n$.

The *sequence of play* is the sequence of pure strategy combinations:

$$(i_{t_0}, j_{t_0}), (i_{t_1}, j_{t_1}), (i_{t_2}, j_{t_2}), ..., (i_{t_n}, j_{t_n}), ...$$

The sequence of play is eventually *cyclic* if there is K and N such that $(i_{t_n}, j_{t_n}) = (i_{t_{n+K}}, j_{t_{n+K}})$ for all $n > N$. Cyclic play has been called "quasi-periodic" play by Rosenmüller [12]. If the CFP converges to some Nash equilibrium (p^*, q^*) and the sequence of play is eventually a cycle, we will refer to it as *cyclic convergence.*

We wish to alert the reader that cyclic play refers to the fact that pure strategy combinations are played in a fixed pattern and not that the trajectory $(p(t), q(t))$ reaches a limit cycle. As a simple example, note that for "matching pennies" the sequence of play resulting from a CFP may be, for instance, $(H, H), (H, T), (T, T), (T, H)$ and is thus cyclic while the trajectory $(p(t), q(t))$ converges to the unique Nash equilibrium.

3 The Main Result

Our main result is that it is rare for CFP to converge cyclically to a mixed strategy equilibrium in which both players use more than two pure strategies.

Let Γ denote the set of all $I \times J$ games. Each game $G \in \Gamma$ can be associated with a point in the Euclidean space $R^{I \times J} \times R^{I \times J}$. A property P is said to hold *generically* in Γ if there is an open and dense subset Γ' of Γ in which the property holds. Similarly, a property is said to hold for generic initial conditions if there is an open and dense subset Z of $\Delta(I) \times \Delta(J)$ such that if $(p(1), q(1))$ belong to Z the property holds.

For generic games and initial conditions, if a CFP converges cyclically to (p^*, q^*) then $\# \, supp \, p^* = \# \, supp \, q^* \leq 2$.

The proof of Theorem 3 is somewhat involved and so we first present a brief outline of the argument.

3.1 An Outline of the Proof

Fictitious play (CFP) is a continuous time *non-linear* and *non-autonomous* dynamical system. The first step is to reformulate the system so that the problem reduces to the study of an associated (discrete) *linear* and *autonomous* system. Once this is done, standard tools can be brought to bear on the problem. In the second step, these tools are employed to analyze the linear difference equation system and obtain the main result.

The method we employ is to fix a particular (arbitrary) cycle of play, where each player uses at least three different pure strategies.[1] We then show that, for generic games, if this cycle is played the CFP does not converge. Since there are only countably many possible cycles, for generic games, cyclic convergence involves at most two pure strategies for each player.

Step 1: Reduction When the play is cyclic, a sequence of choices

$$(i_1, j_1), (i_2, j_2), (i_3, j_3), ..., (i_K, j_K)$$

are repeated over and over in the same order. K consecutive runs corresponding to the choices $(i_1, j_1), (i_2, j_2), (i_3, j_3), ..., (i_K, j_K)$ is a *round*. Thus, cyclic play consists of rounds $r = 1, 2, 3, ...$ A run corresponding to the choice (i_k, j_k) is referred to as a *k-run*. Let $n_k(r)$ denote the length of the k-run in round r, that is, $n_k(r)$ is the amount of time spent playing (i_k, j_k) in round r. Let $n(r) = (n_1(r), n_2(r), \ldots n_K(r))$.

We will argue that if the CFP is cyclic, then there exists a $K \times K$ matrix F such that for all r

$$n(r+1) = Fn(r) \tag{1}$$

Since CFP is completely determined by the associated system determining the run-lengths, the problem has been reduced to the study of a linear difference equation. This reduction also appears in Rosenmüller [12].

Step 2: Analysis of F The behavior of the discrete linear dynamical system (1) is determined by Eigen roots of F, and in the long run the evolution is determined by the dominant Eigen root. The crucial fact (Lemma 5) is that the product of the non-zero Eigen roots of F is one. Suppose each player uses at least three pure strategies in the cycle. We show that generically, not all Eigen roots of F can have absolute value equal to one. Thus, there exists an Eigen root λ of F such that $|\lambda| > 1$. Then, for generic initial conditions, the run-lengths increase exponentially, as in Shapley's [14] example, and CFP does not converge.

It is important to note that, in this proof, we fix a particular cycle (where each player uses at least three pure strategies) and show that for generic games, CFP does not converge along this cycle. But since there are only

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[1] As is well-known, if (p^*, q^*) is an equilibrium of a generic game then $\#\, supp\, p^* = \#\, supp\, q^*$.

countably many possible cycles, generically there is *no* cycle such that CFP will converge.

For *non-generic* classes of games (such as zero sum games) it may well happen that *all* non-zero Eigen roots of F have absolute value equal to one, which allows for convergence. We consider this issue in Section 8.

4 The Determination of the Run-Lengths

We start by assuming that CFP proceeds along a cycle

$$((i_1, j_1), (i_2, j_2), (i_3, j_3), ..., (i_K, j_K))$$

We will argue that there exists a $K \times K$ matrix F (which depends on the particular cycle and on the payoff matrices) such that for all r,

$$n(r+1) = Fn(r). \tag{2}$$

Let $(\alpha_1, \alpha_2, \ldots, \alpha_I)$ denote the I rows of A and let $(\beta_1, \beta_2, \ldots, \beta_J)$ denote the J columns of B. Let P^0 and Q^0 be vectors denoting the total amount of time each player has used each strategy prior to the start of round r. It is convenient to write $n = n(r)$ and $n' = n(r+1)$.

Define an $I \times K$ matrix P by:

$$P_{ik} = \begin{cases} 1 \text{ if } i_k = i \\ 0 \text{ if } i_k \neq i \end{cases} \tag{3}$$

and a $J \times K$ matrix Q by:

$$Q_{jk} = \begin{cases} 1 \text{ if } j_k = j \\ 0 \text{ if } j_k \neq j \end{cases} \tag{4}$$

Observe that $(Pn)_i$ is the amount of time player 1 played strategy i in round r and $(Qn)_j$ is the amount of time player 2 played strategy j in round r. Notice also that $\alpha_{i_k} Q = (a_{i_k j_1}, a_{i_k j_2}, ..., a_{i_k j_K})$ and $\beta_{j_k} P = (b_{i_1 j_k}, b_{i_2 j_k}, ..., b_{i_K j_k})$.

Let e_k denote the kth K-dimensional unit vector. It is convenient to define the $K \times K$ matrix:

$$E_k = (e_1, e_2, \ldots, e_k, 0, \ldots, 0) \tag{5}$$

whose first k columns are the first k unit vectors and the last $(K-k)$ columns are 0. By definition, $E_K = I$, the identity matrix. We also have $E_k n = (n_1, n_2, ..., n_k, 0, ..., 0)$.

Round r Equations Under CFP, when a player switches from one pure strategy to another he is precisely indifferent between these two strategies. Using this fact, we find that the players switch from (i_1, j_1) to (i_2, j_2) in round r when:

$$\alpha_{i_2} Q^0 + a_{i_2 j_1} n_1 = \alpha_{i_1} Q^0 + a_{i_1 j_1} n_1 \tag{6}$$

and

$$\beta_{j_2} P^0 + b_{i_1 j_2} n_1 = \beta_{j_1} P^0 + b_{i_1 j_1} n_1 \tag{7}$$

It is convenient to rewrite these as:

$$(\alpha_{i_2} - \alpha_{i_1}) Q E_1 n = -(\alpha_{i_2} - \alpha_{i_1}) Q^0 \tag{8}$$

and

$$(\beta_{j_2} - \beta_{j_1}) P E_1 n = -(\beta_{j_2} - \beta_{j_1}) P^0 \tag{9}$$

Typically, of course, only one of the two players will switch strategies in the transition from (i_1, j_1) to (i_2, j_2), and thus only one of equations (8) or (9) will be non-trivial. For instance, if only player 1 switches strategies, that is, if $i_2 \neq i_1$ but $j_2 = j_1$, then (9) is trivially satisfied and hence redundant.

In general, for $k = 1, 2, ..., K$, when the players switch from (i_k, j_k) to (i_{k+1}, j_{k+1}) we have:

$$\alpha_{i_{k+1}} Q^0 + \sum_{s=1}^{k} a_{i_{k+1} j_s} n_s = \alpha_{i_k} Q^0 + \sum_{s=1}^{k} a_{i_k j_s} n_s \tag{10}$$

and

$$\beta_{j_{k+1}} P^0 + \sum_{s=1}^{k} b_{i_s j_{k+1}} n_s = \beta_{j_k} P^0 + \sum_{s=1}^{k} b_{i_s j_k} n_s \tag{11}$$

which can be rewritten as:

$$(\alpha_{i_{k+1}} - \alpha_{i_k}) Q E_k n = -(\alpha_{i_{k+1}} - \alpha_{i_k}) Q^0 \tag{12}$$

and

$$(\beta_{j_{k+1}} - \beta_{j_k}) P E_k n = -(\beta_{j_{k+1}} - \beta_{j_k}) P^0 \tag{13}$$

where we always write $K + 1 \equiv 1$.

Round $(r+1)$ Equations By the earlier arguments, for $k = 1, 2, ..., K$, when the players switch from (i_k, j_k) to (i_{k+1}, j_{k+1}) in round $r+1$ we have:

$$(\alpha_{i_{k+1}} - \alpha_{i_k})QE_k n' = -(\alpha_{i_{k+1}} - \alpha_{i_k})Qn - (\alpha_{i_{k+1}} - \alpha_{i_k})Q^0 \tag{14}$$

and

$$(\beta_{j_{k+1}} - \beta_{j_k})PE_k n' = -(\beta_{j_{k+1}} - \beta_{j_k})Pn - (\beta_{j_{k+1}} - \beta_{j_k})P^0 \tag{15}$$

The Basic Difference Equation By substituting (12) and (13) into (14) and (15) respectively, we obtain for $k = 1, 2, ..., K$:

$$(\alpha_{i_{k+1}} - \alpha_{i_k})QE_k n' = -(\alpha_{i_{k+1}} - \alpha_{i_k})Q(I - E_k)n \tag{16}$$

$$(\beta_{j_{k+1}} - \beta_{j_k})PE_k n' = -(\beta_{j_{k+1}} - \beta_{j_k})P(I - E_k)n \tag{17}$$

where

$$I - E_k = [0, 0, ..., e_{k+1}, e_{k+2}, ..., e_K] \,.$$

Assumption *The game (A, B) lies in an open and dense subset of Γ such that for all k, in the switch from (i_k, j_k) to (i_{k+1}, j_{k+1}) only one player switches strategies.*

Let $\sigma(k) \in \{1, 2\}$ denote the player who switches after k:

$$\sigma(k) = \begin{cases} 1 \text{ if } i_k \neq i_{k+1} \\ 2 \text{ if } j_k \neq j_{k+1} \end{cases} \tag{18}$$

For convenience, we usually assume player one is the first to switch in each round $(\sigma(1) = 1)$ and player 2 the last $(\sigma(K) = 2)$.

Consider the system of equations that results when out of equations (16) and (17), for each k, only the kth equation corresponding to the switching player $\sigma(k)$ is considered. This results in a system of equations of the form

$$Cn' = Dn \tag{19}$$

The kth row of C is

$$\begin{gathered}(\alpha_{i_{k+1}} - \alpha_{i_k})QE_k = \\ \left[a_{i_{k+1}j_1} - a_{i_k\,j_1}, a_{i_{k+1}j_2} - a_{i_k\,j_2}, ..., a_{i_{k+1}j_k} - a_{i_k\,j_k}, 0, 0, ..., 0\right]\end{gathered} \tag{20}$$

if $\sigma(k) = 1$, and is

$$\begin{gathered}(\beta_{j_{k+1}} - \beta_{j_k})PE_k = \\ \left[b_{i_1 j_{k+1}} - b_{i_1\, j_k}, b_{i_2 j_{k+1}} - b_{i_2\, j_k}, ..., b_{i_k j_{k+1}} - b_{i_k\, j_k}, 0, 0, ..., 0\right]\end{gathered} \tag{21}$$

if $\sigma(k) = 2$. Therefore, C is a *lower-triangular* matrix (that is, for all $k < l,\ c_{kl} = 0$).

The kth row of D is

$$\begin{gathered}-(\alpha_{i_{k+1}} - \alpha_{i_k})Q(I - E_k) = \\ -\left[0, 0, ..., 0, a_{i_{k+1} j_{k+1}} - a_{i_k\, j_{k+1}}, a_{i_{k+1} j_{k+2}} - a_{i_k\, j_{k+2}}, ..., a_{i_{k+1} j_K} - a_{i_k\, j_K}\right]\end{gathered} \tag{22}$$

if $\sigma(k) = 1$, and is

$$\begin{gathered}-(\beta_{j_{k+1}} - \beta_{j_k})P(I - E_k) = \\ -\left[0, 0, ..., 0, b_{i_{k+1} j_{k+1}} - b_{i_{k+1}\, j_k}, b_{i_{k+2} j_{k+1}} - b_{i_{k+2}\, j_k}, ..., b_{i_K j_{k+1}} - b_{i_K\, j_k}\right]\end{gathered} \tag{23}$$

if $\sigma(k) = 2$. Thus, D is a *strictly upper-triangular* matrix (that is, for all $k \geq l,\ d_{kl} = 0$).

The diagonal elements of C are all strictly positive (Monderer and Sella [10] call this the "better response property"). Thus, C is invertible and we can write

$$n' = C^{-1}Dn \tag{24}$$

This establishes that the run-lengths are determined by a first-order linear difference equation. The behavior of the difference equation (24) is determined by the Eigen roots of the matrix $C^{-1}D$.

5 The Eigen Roots of $C^{-1}D$

Notice from (23) that the first column of D is 0. Thus $C^{-1}D$ is singular and $\lambda_0 = 0$ is an Eigen root of $C^{-1}D$. Our first result concerns the non-zero roots of $C^{-1}D$.

For generic games, the product of the non-zero Eigen roots of $C^{-1}D$ is 1.

Proof. Write:

$$C = \begin{bmatrix} c_{11} & 0 \\ c & \overline{C} \end{bmatrix}$$

where $\overline{C}$ is the triangular $(K-1) \times (K-1)$ submatrix of C in the lower right corner and $c = [c_{21}, c_{31}, ..., c_{K1}]^T$. The diagonal elements of C are all strictly positive.

Similarly, write:

$$D = \begin{bmatrix} 0 & d \\ 0 & \overline{D} \end{bmatrix}$$

where $\overline{D}$ is the $(K-1) \times (K-1)$ submatrix of D in the lower right corner and $d = [d_{12}, d_{13}, ..., d_{1K}]$. Now:

$$C^{-1}D = \begin{bmatrix} 0 & \frac{1}{c_{11}}d \\ & \\ 0 & \overline{C}^{-1}\left(\overline{D} - \frac{1}{c_{11}}cd\right) \end{bmatrix} \tag{25}$$

Let I_k denote the $k \times k$ identity matrix. The characteristic polynomial of $C^{-1}D$ is:

$$\begin{aligned} 0 &= \det\left(\lambda I_K - C^{-1}D\right) \\ &= \det \begin{bmatrix} \lambda & -\frac{1}{c_{11}}d \\ & \\ 0 & \lambda I_{K-1} - \overline{C}^{-1}\left(\overline{D} - \frac{1}{c_{11}}cd\right) \end{bmatrix} \\ &= \lambda \times \det\left(\lambda I_{K-1} - \overline{C}^{-1}\left(\overline{D} - \frac{1}{c_{11}}cd\right)\right) \end{aligned} \tag{26}$$

using (25).

Claim:

$$\det\left(\overline{C}^{-1}\left(\overline{D} - \frac{1}{c_{11}}cd\right)\right) = 1 \tag{27}$$

or equivalently: $\det\left(\overline{D} - \frac{1}{c_{11}}cd\right) = \det \overline{C}$.

Proof of claim. Observe that

$$\overline{D} - \frac{1}{c_{11}}cd$$

$$= \begin{bmatrix} 0 & d_{23} & d_{24} & & d_{2K} \\ 0 & 0 & d_{34} & \cdots & d_{3K} \\ 0 & 0 & 0 & & d_{4K} \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \frac{1}{c_{11}} \begin{bmatrix} c_{21}d_{12} & c_{21}d_{13} & c_{21}d_{14} & & c_{21}d_{1K} \\ c_{31}d_{12} & c_{31}d_{13} & c_{31}d_{14} & \cdots & c_{31}d_{1K} \\ c_{41}d_{12} & c_{41}d_{13} & c_{41}d_{14} & & c_{41}d_{1K} \\ & \vdots & & \ddots & \\ c_{K1}d_{12} & c_{K1}d_{13} & c_{K1}d_{14} & \cdots & c_{K1}d_{1K} \end{bmatrix}$$

By repeated use of the rule that if a column of a matrix is the sum of two column vectors then the determinant is the sum of two determinants, we obtain:

$$\det\left(\overline{D} - \frac{1}{c_{11}}cd\right) = \frac{1}{c_{11}} \det \begin{bmatrix} -c_{21}d_{12} & d_{23} & d_{24} & & d_{2K} \\ -c_{31}d_{12} & 0 & d_{34} & \cdots & d_{3K} \\ -c_{41}d_{12} & 0 & 0 & & d_{4K} \\ & \vdots & & \ddots & \\ -c_{K1}d_{12} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Evaluating the determinant by expanding along the last row yields:

$$\det\left(\overline{D} - \frac{1}{c_{11}}cd\right) = (-1)^{K+1} \frac{1}{c_{11}} c_{K1} d_{12} d_{23} d_{34} \ldots d_{K-1,K}$$

But now recognize that from equations (20) to (23), for all k, $d_{k,k+1} = -c_{kk}$ and $c_{K1} = c_{KK}$. (Recall that if $\sigma(k) = 1$ then $j_k = j_{k+1}$ and if $\sigma(k) = 2$ then $i_k = i_{k+1}$.) Thus:

$$\begin{aligned} \det\left(\overline{D} - \frac{1}{c_{11}}cd\right) &= (-1)^{K+1} \frac{1}{c_{11}} c_{K1} (-c_{11})(-c_{22})(-c_{33}) \ldots (-c_{K-1,K-1}) \\ &= (-1)^{2K} c_{22}c_{33} \ldots c_{K-1,K-1}\, c_{KK} \\ &= \det \overline{C} \end{aligned} \tag{28}$$

This establishes the claim.

By (26), $\lambda \neq 0$ is an Eigen root of $C^{-1}D$ if and only if it is an Eigen root of $\overline{C}^{-1}\left(\overline{D} - \frac{1}{c_{11}}cd\right)$. Thus, the claim implies that the product of the non-zero roots of $C^{-1}D$ is one.

6 Unit Roots of $C^{-1}D$

Observe that the Eigen roots of $C^{-1}D$ are determined by the solutions to the equation:

$$\lambda x = C^{-1}Dx \tag{29}$$

where $x \neq 0$ is an Eigen vector, which are the same as the solutions to:

$$(\lambda C - D)x = 0 \tag{30}$$

Suppose player 1 plays strategy i_{k_0} in run k_0, then plays some other strategies (but not i_{k_0}) for a while, and then returns to playing $i_{k_1} = i_{k_0}$ in run k_1 in the same round. That is, $k_0 < k_1 - 1$, $i_{k_0} = i_{k_1}$, and $i_k \neq i_{k_0}$ for all k such that $k_0 < k < k_1$. If this happens we say that a *reversion* to strategy i_{k_0} occurs in run k_1. Suppose that there are ρ_i reversions for player i in each round. Let $\rho = \rho_1 + \rho_2$. Notice that there is always at least one reversion for each player, since $K + 1 \equiv 1$ by definition and hence $\rho \geq 2$.

Suppose the matrix $C^{-1}D$ has S distinct Eigen roots $\lambda_1, \lambda_2, ..., \lambda_S$ so that we can write the characteristic polynomial of $C^{-1}D$ in the form:

$$\left|\lambda I - C^{-1}D\right| = \prod_{s=1}^{S} (\lambda_s - \lambda)^{\alpha_s} = 0$$

The number α_s is called the *algebraic multiplicity* of the root λ_s.

For generic games, if each player uses at least three different pure strategies in the cycle, the algebraic multiplicity of the unit root of $C^{-1}D$ is ρ.

Proof. The proof is by induction on ρ, the number of reversions in the cycle of play.

Initial Step ($\rho = 2$) In this step we show that, generically, if $\rho = 2$ then the algebraic multiplicity of the unit root is two. It is useful to initially consider the *simple cycle* where the players alternate in switching strategies. That is, for all k, $\sigma(k) \neq \sigma(k+1)$.

We may assume without loss of generality that $\sigma(k) = 1$ if and only if k is odd, and perhaps after a relabeling of strategies, write the cycle as

$$\begin{aligned} & \langle (i_1, j_1), (i_2, j_2), (i_3, j_3), (i_4, j_4), ..., (i_K, j_K) \rangle \\ = \; & \langle (1,1), (2,1), (2,2), (3,2), ..., (\kappa, \kappa), (1, \kappa) \rangle \end{aligned} \tag{31}$$

where by assumption the number of pure strategies used by each player is $\kappa = \frac{K}{2} > 2$. In this case, $(i_k, j_k) = \left(\frac{k+1}{2}, \frac{k+1}{2}\right)$ if k is odd, $(i_k, j_k) = \left(\frac{k}{2} + 1, \frac{k}{2}\right)$

if k is even. Call such a cycle *simple*. For a simple cycle we have:

$$C =$$

$$\begin{bmatrix}
a_{21}-a_{11} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
b_{12}-b_{11} & b_{22}-b_{21} & 0 & & 0 & & 0 & 0 \\
a_{31}-a_{21} & a_{31}-a_{21} & a_{32}-a_{22} & & 0 & & 0 & 0 \\
b_{13}-b_{12} & b_{23}-b_{22} & b_{23}-b_{22} & & 0 & & 0 & 0 \\
\vdots & & & \ddots & \vdots & \ddots & & \vdots \\
b_{1\kappa}-b_{1\kappa-1} & b_{2\kappa}-b_{2\kappa-1} & b_{2\kappa}-b_{2\kappa-1} & & b_{i_k\kappa}-b_{i_k\kappa-1} & & 0 & 0 \\
a_{11}-a_{\kappa1} & a_{11}-a_{\kappa1} & a_{12}-a_{\kappa2} & & a_{1j_k}-a_{\kappa j_k} & & a_{1\kappa}-a_{\kappa\kappa} & 0 \\
b_{11}-b_{1\kappa} & b_{21}-b_{2\kappa} & b_{21}-b_{2\kappa} & \cdots & b_{i_k1}-b_{i_k\kappa} & \cdots & b_{\kappa1}-b_{\kappa\kappa} & b_{11}-b_{1\kappa}
\end{bmatrix}$$

and

$$-D =$$

$$\begin{bmatrix}
0 & a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2j_k}-a_{1\,j_k} & \cdots & a_{2\kappa}-a_{1\kappa} & a_{2\kappa}-a_{1\kappa} \\
0 & 0 & b_{22}-b_{21} & & b_{i_k2}-b_{i_k1} & & b_{\kappa2}-b_{\kappa1} & b_{12}-b_{11} \\
0 & 0 & 0 & & a_{3j_k}-a_{2j_k} & & a_{3\kappa}-a_{2\kappa} & a_{3\kappa}-a_{2\kappa} \\
0 & 0 & 0 & & b_{i_k3}-b_{i_k2} & & b_{\kappa3}-b_{\kappa2} & b_{13}-b_{12} \\
\vdots & & & \ddots & \vdots & \ddots & & \vdots \\
0 & 0 & 0 & & 0 & & b_{\kappa\kappa}-b_{\kappa\kappa-1} & b_{1\kappa}-b_{1\kappa-1} \\
0 & 0 & 0 & & 0 & & 0 & a_{1\kappa}-a_{\kappa\kappa} \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{bmatrix}$$

Thus,

$$[\lambda C - D] =$$

$$\begin{bmatrix}
\lambda(a_{21}-a_{11}) & a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2j_k}-a_{1\,j_k} & \cdots & a_{2\kappa}-a_{1\kappa} \\
\lambda(b_{12}-b_{11}) & \lambda(b_{22}-b_{21}) & b_{22}-b_{21} & & b_{i_k2}-b_{i_k1} & & b_{12}-b_{11} \\
\lambda(a_{31}-a_{21}) & \lambda(a_{31}-a_{21}) & \lambda(a_{32}-a_{22}) & & a_{3j_k}-a_{2j_k} & & a_{3\kappa}-a_{2\kappa} \\
\lambda(b_{13}-b_{12}) & \lambda(b_{23}-b_{22}) & \lambda(b_{23}-b_{22}) & & b_{i_k3}-b_{i_k2} & & b_{13}-b_{12} \\
\vdots & & & \ddots & \vdots & \ddots & \vdots \\
\lambda(b_{1\kappa}-b_{1\kappa-1}) & \lambda(b_{2\kappa}-b_{2\kappa-1}) & \lambda(b_{2\kappa}-b_{2\kappa-1}) & & \lambda(b_{i_k\kappa}-b_{i_k\kappa-1}) & & b_{1\kappa}-b_{1\kappa-1} \\
\lambda(a_{11}-a_{\kappa1}) & \lambda(a_{11}-a_{\kappa1}) & \lambda(a_{12}-a_{\kappa2}) & & \lambda(a_{1j_k}-a_{\kappa j_k}) & & a_{1\kappa}-a_{\kappa\kappa} \\
\lambda(b_{11}-b_{1\kappa}) & \lambda(b_{21}-b_{2\kappa}) & \lambda(b_{21}-b_{2\kappa}) & \cdots & \lambda(b_{i_k1}-b_{i_k\kappa}) & \cdots & \lambda(b_{11}-b_{1\kappa})
\end{bmatrix}$$

Observe that for this matrix:

$$\begin{aligned} & \text{row } 1+\text{row } 3+\text{row } 5+...+\text{row } K-1 \\ = \quad & \left(0,\left(1-\lambda\right)\left(a_{21}-a_{11}\right),\left(1-\lambda\right)\left(a_{22}-a_{12}\right),...,\left(1-\lambda\right)\left(a_{i_kj_k}-a_{1j_k}\right),..,0\right) \end{aligned}$$

and

$$\begin{aligned} & \text{row } 2+\text{row } 4+\text{row } 6+...+\text{row } K \\ = \quad & \left(0,0,\left(1-\lambda\right)\left(b_{22}-b_{21}\right),...,\left(1-\lambda\right)\left(b_{i_kj_k}-b_{i_k1}\right),...,\left(1-\lambda\right)\left(b_{1\kappa}-b_{11}\right)\right) \end{aligned}$$

Therefore, we can add the odd-numbered rows to row 1 and the even-numbered rows to row 2 to obtain

$$|\lambda C-D| =$$

$$\begin{vmatrix}
0 & \begin{array}{c}\left(1-\lambda\right)\times \\ \left(a_{21}-a_{11}\right)\end{array} & \cdots & \begin{array}{c}\left(1-\lambda\right)\times \\ \left(a_{i_kj_k}-a_{1j_k}\right)\end{array} & \cdots & \begin{array}{c}\left(1-\lambda\right)\times \\ \left(a_{\kappa\kappa}-a_{1\kappa}\right)\end{array} & 0 \\
0 & 0 & & \begin{array}{c}\left(1-\lambda\right)\times \\ \left(b_{i_kj_k}-b_{i_k1}\right)\end{array} & & \begin{array}{c}\left(1-\lambda\right)\times \\ \left(b_{\kappa\kappa}-b_{\kappa1}\right)\end{array} & \begin{array}{c}\left(1-\lambda\right)\times \\ \left(b_{1\kappa}-b_{11}\right)\end{array} \\
\lambda\left(a_{31}-a_{21}\right) & \lambda\left(a_{31}-a_{21}\right) & & a_{3j_k}-a_{2j_k} & & a_{3\kappa}-a_{2\kappa} & a_{3\kappa}-a_{2\kappa} \\
\lambda\left(b_{13}-b_{12}\right) & \lambda\left(b_{23}-b_{22}\right) & \cdots & b_{i_k3}-b_{i_k2} & \cdots & b_{\kappa3}-b_{\kappa2} & b_{13}-b_{12} \\
\vdots & \vdots & \ddots & \vdots & \ddots & & \vdots \\
\lambda\left(b_{1\kappa}-b_{1\kappa-1}\right) & \lambda\left(b_{2\kappa}-b_{2\kappa-1}\right) & \cdots & \lambda\left(b_{i_k\kappa}-b_{i_k\kappa-1}\right) & \cdots & b_{\kappa\kappa}-b_{\kappa\kappa-1} & b_{1\kappa}-b_{1\kappa-1} \\
\lambda\left(a_{11}-a_{\kappa1}\right) & \lambda\left(a_{11}-a_{\kappa1}\right) & & \lambda\left(a_{1j_k}-a_{\kappa j_k}\right) & & \lambda\left(a_{1\kappa}-a_{\kappa\kappa}\right) & a_{1\kappa}-a_{\kappa\kappa} \\
\lambda\left(b_{11}-b_{1\kappa}\right) & \lambda\left(b_{21}-b_{2\kappa}\right) & \cdots & \lambda\left(b_{i_k1}-b_{i_k\kappa}\right) & \cdots & \lambda\left(b_{\kappa1}-b_{\kappa\kappa}\right) & \lambda\left(b_{11}-b_{1\kappa}\right)
\end{vmatrix}$$

$$= \lambda\left(1-\lambda\right)^2 \times |L(\lambda)|$$

where

$$|L(\lambda)| \equiv$$

$$\begin{vmatrix}
0 & a_{21}-a_{11} & \cdots & a_{i_kj_k}-a_{1j_k} & \cdots & a_{\kappa\kappa}-a_{1\kappa} & 0 \\
0 & 0 & & b_{i_kj_k}-b_{i_k1} & & b_{\kappa\kappa}-b_{\kappa1} & b_{1\kappa}-b_{11} \\
a_{31}-a_{21} & \lambda\left(a_{31}-a_{21}\right) & & a_{3j_k}-a_{2j_k} & & a_{3\kappa}-a_{2\kappa} & a_{3\kappa}-a_{2\kappa} \\
b_{13}-b_{12} & \lambda\left(b_{23}-b_{22}\right) & & b_{i_k3}-b_{i_k2} & & b_{\kappa3}-b_{\kappa2} & b_{13}-b_{12} \\
\vdots & & \ddots & \vdots & \ddots & & \vdots \\
b_{1\kappa}-b_{1\kappa-1} & \lambda\left(b_{2\kappa}-b_{2\kappa-1}\right) & & \lambda\left(b_{i_k\kappa}-b_{i_k\kappa-1}\right) & & b_{\kappa\kappa}-b_{\kappa\kappa-1} & b_{1\kappa}-b_{1\kappa-1} \\
a_{11}-a_{\kappa1} & \lambda\left(a_{11}-a_{\kappa1}\right) & \cdots & \lambda\left(a_{1j_k}-a_{\kappa j_k}\right) & \cdots & \lambda\left(a_{1\kappa}-a_{\kappa\kappa}\right) & a_{1\kappa}-a_{\kappa\kappa} \\
b_{11}-b_{1\kappa} & \lambda\left(b_{21}-b_{2\kappa}\right) & & \lambda\left(b_{i_k1}-b_{i_k\kappa}\right) & & \lambda\left(b_{\kappa1}-b_{\kappa\kappa}\right) & \lambda\left(b_{11}-b_{1\kappa}\right)
\end{vmatrix}$$

Our claim is that for generic games

$$|L(1)| \neq 0$$

and this implies that there does not exist a third unit root. Since $|L(1)|$ is a polynomial function of the payoffs (a_{ij}, b_{ij}), $|L(1)| = 0$ on an open set in $R^{\kappa^2} \times R^{\kappa^2}$ if and only if $|L(1)| = 0$ identically on $R^{\kappa^2} \times R^{\kappa^2}$. But this is false since if A and B are defined by

$$a_{ij} = \begin{cases} 0 & \text{if } i = j = 1 \\ 1 & \text{if } i = j > 1 \\ 0 & \text{if } i \neq j \end{cases} \tag{32}$$

$$b_{ij} = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{33}$$

then we obtain

$$|L(1)| = \begin{vmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \end{vmatrix} \tag{34}$$

$$= -\tfrac{1}{2}(\kappa - 2)(\kappa - 1) \neq 0$$

recalling that $\kappa > 2$. (See Appendix A for the explicit evaluation of this determinant.)

So far we have only considered the simple cycle (31). However, the analysis for more complicated switching patterns is similar. A cycle where players switch several times consecutively results, after some rearrangement of rows and columns, in a matrix similar to $L(1)$. One then checks that a determinant similar to (34) is not zero. The somewhat laborious details can be found in Appendix B.

We have thus established that for generic games, when $\rho = 2$, the algebraic multiplicity of the unit root is 2.

The Induction Step Suppose there exists $\overline{\rho} \geq 2$ such that the statement of Lemma 6 is true for any cycle such that $\rho_1 + \rho_2 = \overline{\rho}$. Now consider an arbitrary cycle c of length K in which the number of reversals is $\rho_1 + \rho_2 = \overline{\rho} + 1$. Since $\overline{\rho} + 1 \geq 3$, there is a player, say player 1, who switches "back" to some strategy during the cycle. We may, for simplicity, suppose that this occurs in run $K - 1$. Moreover, we can relabel the strategies so that the cycle c is of the form:

$$c = \left\langle \underset{1}{11},\ \ldots,\ \underset{k_0}{\kappa j_{k_0}}\ ,\ \ldots,\ \underset{K-2}{(\kappa-1)\,\kappa},\ \underset{K-1}{\kappa\kappa},\ \underset{K}{\kappa 1} \right\rangle$$

The numbers under the strategy labels are the runs; thus player 1 uses strategy $\kappa \neq 1$ in run k_0, then plays some other strategies, and then switches back ("reverts") to playing κ again in run $K-1$.[2]

Consider the matrix $C - D$ that corresponds to the cycle c.

Add column $K-2$ of $C-D$ to column K and subtract column $K-1$ from column K. Add row $K-2$ to row K. Call the resulting matrix $\widehat{X}$.

For $k \neq K$, the kth element in the Kth column of $\widehat{X}$ is

$$\left(a_{i_{k+1}\kappa} - a_{i_k\kappa}\right) - \left(a_{i_{k+1}\kappa} - a_{i_k\kappa}\right) + \left(a_{i_{k+1}1} - a_{i_k1}\right) = \left(a_{i_{k+1}1} - a_{i_k1}\right) \qquad (35)$$

if $\sigma(k) = 1$, and

$$\left(b_{(\kappa-1)j_{k+1}} - b_{(\kappa-1)j_k}\right) - \left(b_{1j_{k+1}} - b_{1j_k}\right) + \left(b_{1j_{k+1}} - b_{1j_k}\right) = \left(b_{(\kappa-1)j_{k+1}} - b_{(\kappa-1)j_k}\right) \qquad (36)$$

if $\sigma(k) = 2$.

Similarly, for $k \neq K$, the kth element in the Kth row of $\widehat{X}$ is

$$(a_{1j_k} - a_{\kappa j_k}) + (a_{\kappa j_k} - a_{\kappa-1,j_k}) = a_{1j_k} - a_{\kappa-1,j_k} \qquad (37)$$

Finally, delete row $K-2$ and column $K-1$ from $\widehat{X}$. Call the new matrix $\overline{X}$.

Now consider the $K-1$ cycle

$$\overline{c} = \left\langle \underset{1}{11},\ \ldots,\ \underset{k_0}{\kappa j_{k_0}}\ ,\ \ldots,\ \underset{K-2}{(\kappa-1)\,\kappa},\ \underset{K-1}{(\kappa-1)\,1} \right\rangle \qquad (38)$$

which is the same as c except that the sequence $((\kappa-1)\kappa,\ \kappa\kappa,\ \kappa 1)$ at the end has been replaced by the shorter sequence $((\kappa-1)\kappa,\ (\kappa-1)1)$. In $\overline{c}$ player 1 makes a direct transition from strategy $(\kappa-1)$ to strategy 1. Otherwise the cycle is as before.

---

[2] For ease of notation, we are assuming that the players alternate in switching in the last three runs of the cycle (that is, player 1 switches from $\kappa-1$ to κ, then player 2 switches from κ to 1, and finally player 1 switches from κ to 1), but it should be clear that results do not depend on this.

Note that the number of reversions in $\bar{c}$ is $\overline{\rho}$. Let $\overline{C} - \overline{D}$ denote the matrix corresponding to the cycle $\bar{c}$.

We now claim that $\overline{X} = \overline{C} - \overline{D}$. Indeed, the elements in (35) and (36) are the entries in the last column of $\overline{X}$, and they correspond to a run where player 1 uses $\kappa - 1$ and player 2 uses 1. This is precisely the last run in the cycle $\bar{c}$. Similarly, the last row of the matrix $\overline{X}$ corresponds to a switch from strategy $\kappa - 1$ to strategy 1 by player 1. The other rows and columns have not been disturbed, and hence $\overline{X} = \overline{C} - \overline{D}$.

We have shown that we can write

$$|C - D| = \begin{vmatrix} \overline{C} - \overline{D} & \gamma \\ \alpha & \beta \end{vmatrix} \tag{39}$$

since the operations we have performed on $C - D$ do not affect the determinant. In (39), $\overline{C} - \overline{D}$ is the matrix resulting from the smaller cycle $\bar{c}$, $\begin{bmatrix} \gamma \\ \beta \end{bmatrix}$ is the $(K-1)$th column of $\widehat{X}$, and $\begin{bmatrix} \alpha & \beta \end{bmatrix}$ is the $(K-2)$th row of $\widehat{X}$. Thus in particular, $\beta = a_{\kappa\kappa} - a_{\kappa-1,\kappa}$.

In the form (39), γ is a linear combination of the columns of $\overline{C} - \overline{D}$, and α is a linear combination of the rows. That is, there exist δ and η such that:

$$\begin{aligned} \alpha + \delta\left(\overline{C} - \overline{D}\right) &= 0 \\ \gamma + \left(\overline{C} - \overline{D}\right)\eta &= 0 \end{aligned} \tag{40}$$

More precisely, we can assume (without loss of generality) that $\sigma(k_0) = 1$. Then η is defined by:

$$\eta_k = \begin{cases} -1 & k = k_0 \\ -1 & \text{if } k_0 < k \leq K-2, \sigma(k-1) = 2 \text{ and } \sigma(k) = 1 \\ 1 & \text{if } k_0 < k \leq K-2, \sigma(k-1) = 1 \text{ and } \sigma(k) = 2 \\ 0 & \text{otherwise} \end{cases}$$

and δ is defined by:

$$\delta_k = \begin{cases} 1 & \text{if } k_0 \leq k < K-2 \text{ and } \sigma(k) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It can then be checked that (40) holds.

Now consider the matrix $T(\lambda)$ that results when the row and column operations described above are performed on $\lambda C - D$ (rather than on $C - D$). To be precise, we have

$$[\lambda C - D] =$$

$$\begin{bmatrix} \lambda(a_{21} - a_{11}) & \cdots & a_{2j_k} - a_{1\,j_k} & \cdots & a_{2\kappa} - a_{1\kappa} & a_{2\kappa} - a_{1\kappa} & a_{21} - a_{11} \\ \vdots & \ddots & & \ddots & & & \vdots \\ \lambda(a_{\kappa 1} - a_{k-1,1}) & & \lambda(a_{\kappa j_k} - a_{\kappa-1\,j_k}) & & \lambda(a_{\kappa\kappa} - a_{\kappa-1\kappa}) & a_{\kappa\kappa} - a_{\kappa-1\kappa} & a_{\kappa 1} - a_{\kappa-11} \\ \lambda(b_{11} - b_{1\kappa}) & & \lambda(b_{i_k 1} - b_{i_k \kappa}) & & \lambda(b_{\kappa-1,1} - b_{\kappa-1,\kappa}) & \lambda(b_{\kappa 1} - b_{\kappa\kappa}) & b_{\kappa 1} - b_{\kappa\kappa} \\ \lambda(a_{11} - a_{\kappa 1}) & \cdots & \lambda(a_{1j_k} - a_{\kappa j_k}) & \cdots & \lambda(a_{1\kappa} - a_{\kappa\kappa}) & \lambda(a_{1\kappa} - a_{\kappa\kappa}) & \lambda(a_{11} - a_{\kappa 1}) \end{bmatrix}$$

The operations are: add column $K-2$ to column K and subtract column $K-1$ from column K. Add row $K-2$ to row K. Finally interchange row $K-2$ and row K, and column $K-1$ and column K. The result is

$$T(\lambda) =$$

$$\begin{bmatrix} \lambda(a_{21} - a_{11}) & \cdots & a_{2j_k} - a_{1\,j_k} & \cdots & a_{21} - a_{11} & a_{2\kappa} - a_{1\kappa} \\ \vdots & \ddots & & \ddots & \vdots & \vdots \\ \lambda(a_{11} - a_{\kappa-1,1}) & & \lambda(a_{1j_k} - a_{\kappa-1j_k}) & & \begin{array}{c} \lambda(a_{11} - a_{\kappa-1\kappa}) + \\ (1-\lambda)(a_{\kappa 1} - a_{\kappa\kappa}) \end{array} & \begin{array}{c} \lambda(a_{1\kappa} - a_{\kappa\kappa}) + \\ a_{\kappa\kappa} - a_{\kappa-1\kappa} \end{array} \\ \lambda(b_{11} - b_{1\kappa}) & & \lambda(b_{i_k 1} - b_{i_k\kappa}) & & \begin{array}{c} (1-\lambda)(b_{\kappa 1} - b_{\kappa\kappa}) + \\ \lambda(b_{\kappa-1,1} - b_{\kappa-1,\kappa}) \end{array} & \lambda(b_{\kappa 1} - b_{\kappa\kappa}) \\ \lambda(a_{\kappa 1} - a_{k-1,1}) & \cdots & \lambda(a_{\kappa j_k} - a_{\kappa-1\,j_k}) & \cdots & \begin{array}{c} a_{\kappa 1} - a_{\kappa-1,1} + \\ (\lambda-1)(a_{\kappa\kappa} - a_{\kappa-1\kappa}) \end{array} & a_{\kappa\kappa} - a_{\kappa-1\kappa} \end{bmatrix}$$

Observe that $|T(\lambda)| = |\lambda C - D|$. From (39) it follows that:

$$T(\lambda) = \begin{bmatrix} \overline{L}_1(\lambda) & \gamma(\lambda) \\ \alpha(\lambda) & \beta(\lambda) \end{bmatrix} \tag{41}$$

where $\overline{L}_1(\lambda)$ is a $(K-1) \times (K-1)$ matrix with the property that $\overline{L}_1(1) = \overline{C} - \overline{D}$.

Now add $\delta \left[\begin{array}{cc} \overline{L}_1(\lambda) & \gamma(\lambda) \end{array} \right]$ to the last row of $T(\lambda)$, and add $\left[\begin{array}{c} \overline{L}_1(\lambda) \\ \alpha(\lambda) \end{array} \right] \eta$ to the last column of $T(\lambda)$. By (40), this results in a matrix:

$$T_1(\lambda) = \left[\begin{array}{cc} \overline{L}_1(\lambda) & (1-\lambda)\gamma_0 \\ (1-\lambda)\alpha_0 & (1-\lambda)\beta_0 \end{array} \right] \tag{42}$$

where again we have that $|T_1(\lambda)| = |\lambda C - D|$.

Consider the form of β_0. After adding $\delta \left[\begin{array}{cc} \overline{L}_1(\lambda) & \gamma(\lambda) \end{array} \right]$ to the last row of $T(\lambda)$, the last row is:

$$(0, ..., 0, ..., (1-\lambda)\,(a_{i_k j_k} - a_{\kappa\, j_k}), ..., (1-\lambda)\,(a_{\kappa-1\kappa} - a_{\kappa\kappa}), (1-\lambda)\,(a_{\kappa-1\kappa} - a_{\kappa\kappa}), 0)$$

where we observe that the jth entry is zero if $j \leq k_0$. Multiplying this row by η, the result is

$$(1-\lambda)\beta_0 = (1-\lambda) \sum \eta_k (a_{i_k j_k} - a_{\kappa\, j_k})$$

Observe that for generic games, $\beta_0 \neq 0$.

We now investigate the unit roots of the matrix $C^{-1}D$. Since

$$\left| \lambda I - C^{-1} D \right| = \left| C^{-1} \right| |\lambda C - D|$$

we can equally well investigate the roots of the polynomial $|\lambda C - D| = 0$. We can write:

$$\begin{aligned} |\lambda C - D| &= |T_1(\lambda)| \\ &= (1-\lambda)\,|L_1(\lambda)| \end{aligned} \tag{43}$$

where

$$L_1(\lambda) = \left[\begin{array}{cc} \overline{L}_1(\lambda) & (1-\lambda)\gamma_0 \\ \alpha_0 & \beta_0 \end{array} \right]$$

Hence there is at least one unit root. If there is another one, then $|L_1(1)| = 0$ and there exists a vector $y = (\bar{y}, y_K) \neq 0$ such that

$$L_1(1) y = \left[\begin{array}{cc} \overline{L}_1(1) & 0 \\ \alpha_0 & \beta_0 \end{array} \right] y = 0 \tag{44}$$

Since $\overline{L}_1(1) = \overline{C} - \overline{D}$, this implies that

$$\begin{aligned} \left(\overline{C} - \overline{D} \right) \bar{y} &= 0 \\ \alpha_0 \bar{y} + y_K \beta_0 &= 0 \end{aligned} \tag{45}$$

If $\bar{y} = 0$, then $y_K \neq 0$. Since $\beta_0 \neq 0$, this contradicts (45). Thus, $\bar{y} \neq 0$. Suppose without loss of generality that $\bar{y}_j = 1$ for some $j < K$.

Since $L_1(1)y = 0$, we have

$$L_1(\lambda)y = (1-\lambda)v$$

for some vector v. Replace the jth column of $L_1(\lambda)$ by $(1-\lambda)v$ and call the resulting matrix $T_2(\lambda)$. Let

$$L_2(\lambda) = \begin{bmatrix} \overline{L}_2(\lambda) & (1-\lambda)\gamma_0 \\ \alpha(\lambda) & \beta_0 \end{bmatrix}$$

be the matrix that obtains when the jth column of $L_1(\lambda)$ is replaced by v. Then

$$|\lambda C - D| = (1-\lambda)\,|L_1(\lambda)| = (1-\lambda)\,|T_2(\lambda)| = (1-\lambda)^2\,|L_2(\lambda)| \qquad (46)$$

Suppose that there is a third unit root. Then $|L_2(1)| = 0$ and there is $z = (\bar{z}, z_K) \neq 0$ such that

$$L_2(1)z = \begin{bmatrix} \overline{L}_2(1) & 0 \\ \alpha(1) & \beta_0 \end{bmatrix} z = 0 \qquad (47)$$

Again $\beta_0 \neq 0$ implies that $\bar{z} \neq 0$. Therefore, as in (46), we can use z to show that

$$|\lambda C - D| = (1-\lambda)^3\,|L_3(\lambda)|$$

for some matrix $L_3(\lambda)$. This procedure can be repeated until $|L_k(1)| \neq 0$ for some k.

Now we note that one unit root resulted directly from (43). After that each step of the procedure corresponds not only to a unit root of $|\lambda C - D| = 0$, but clearly also to a unit root of $\left|\lambda\overline{C} - \overline{D}\right| = 0$, where the matrix $[\lambda\overline{C} - \overline{D}]$ comes from the smaller cycle $\bar{c}$. By the induction hypothesis, $\overline{C}^{\,-1}\overline{D}$ has exactly $\overline{\rho}$ unit roots. Therefore, we can repeat the argument precisely $\overline{\rho}$ times. Therefore,

$$|\lambda C - D| = (1-\lambda)^{\overline{\rho}+1}\,|L_{\overline{\rho}}(\lambda)|$$

and $|L_{\overline{\rho}}(1)| \neq 0$. This completes the induction step and the proof of Lemma 6.

7 Non-convergence

We can now complete the proof of Theorem 3.

Proof of Theorem 3 Let the cycle be such that there are ρ reversions. As a result of Lemma 6 we know that for generic games, $\lambda_0 = 0$ is an Eigen root of $C^{-1}D$ and there are exactly ρ unit roots:

$$\lambda_1 = \lambda_2 = \ldots = \lambda_\rho = 1$$

There are $K - \rho - 1$ remaining Eigen roots. Since $K = 2\kappa + \rho - 2$, the number of remaining roots is odd. Since the number of complex roots is always even, there is an odd number of *real* roots. Not all of these can equal -1 since from Lemma 5 we have,

$$\lambda_{\rho+1} \times \lambda_{\rho+2} \times \ldots \times \lambda_{K-1} = 1$$

This implies that there is a non-zero real root, say $\lambda_{\rho+1}$, such that $\lambda_{\rho+1} \neq 1$ or -1 and hence $|\lambda_{\rho+1}| \neq 1$. Since the product of all non-zero roots is one, there exists a root, λ_s such that $|\lambda_s| > 1$. Let λ_S be the dominant root, that is, the root with the largest absolute value. If λ_S is either negative or complex, the cycle cannot persist since run-lengths would become negative. Thus λ_S must be real and positive and hence $\lambda_S > 1$.

The final step of the argument is similar to that used by Shapley [14].

Consider the (arbitrary) vectors P^0 and Q^0 that describe the "initial conditions" before the cycle begins. Let $n(0)$ be the vector of run lengths in the initial round that the cycle is played. We know from (12) and (13) that:

$$(\alpha_{i_{k+1}} - \alpha_{i_k})QE_k n(0) = -(\alpha_{i_{k+1}} - \alpha_{i_k})Q^0$$

and

$$(\beta_{j_{k+1}} - \beta_{j_k})PE_k n(0) = -(\beta_{j_{k+1}} - \beta_{j_k})P^0$$

which can be rewritten as:

$$n(0) = \left(I - C^{-1}D\right) m\left(0\right)$$

where $m\left(0\right) \in R^K$ is a vector determined by the initial conditions P^0 and Q^0.

Thus:

$$n(r) = \left(C^{-1}D\right)^r \left(I - C^{-1}D\right) m\left(0\right) = \left(I - C^{-1}D\right) \left(C^{-1}D\right)^r m\left(0\right)$$

Define $m\left(r\right) \equiv \left(C^{-1}D\right)^r m\left(0\right)$. By the Jordan decomposition theorem (see Hirsch and Smale [5]) we can write:

$$m\left(r\right) = c_1 m_1\left(r\right) + c_2 m_2\left(r\right) + ... + c_S m_S\left(r\right)$$

where each $m_s\left(r\right)$ corresponds to a different Eigen roots λ_s of $C^{-1}D$ and is of the form:

$$m_s\left(r\right) = \sum_{i=0}^{k_s} \frac{r^i}{i!} \lambda_s^{r-i} x_{si}$$

where each x_{si} is a generalized Eigen vector ($x_{si} \in \ker\left(\lambda_s I - C^{-1}D\right)^i$).

Then $n\left(r\right) = \left(I - C^{-1}D\right) m\left(r\right)$ is also a sum of similar terms, one of which is:

$$c_S \lambda_S^r \left(I - C^{-1}D\right) x_{S0} \neq 0$$

where as above, λ_S is the dominant root.

Since $\lambda_S > 1$ the run-lengths grow exponentially. Hence, as in Shapley [14], for generic initial conditions, CFP cannot converge.

This completes the proof of Theorem 3.

8 Discussion

The class of 2 × 2 games forms an exception to our main result: there is open set of 2 × 2 games with a unique equilibrium in mixed strategies for which every CFP is convergent. However, it can be shown that every 2 × 2 game with a unique equilibrium in mixed strategies has the same best-response correspondence as a zero-sum game. Since CFP depends only on the best-response correspondence, the 2 × 2 exception is a consequence of this equivalence.

We now present an example in order to demonstrate that there exist (non-generic) non-zero sum games and initial conditions for which CFP can converge to a mixed strategy equilibrium in which more than two strategies are used.

"Rock-Paper-Scissors":

	R	P	S
R	$0,\ 0$	$-1,\ \alpha$	$\alpha, -1$
P	$\alpha, -1$	$0,\ 0$	$-1,\ \alpha$
S	$-1,\ \alpha$	$\alpha, -1$	$0,\ 0$

There exists a unique equilibrium which is completely mixed: each player assigns equal probabilities to each of the three pure strategies.

The game is symmetric and hence non-generic. Consider a CFP process with an initial condition satisfying $p(1) = q(1)$. Notice that the initial condition is also non-generic.

It can be shown that if $\alpha \geq 1$, any such CFP *converges* in a cyclical manner; both players play the same pure strategy at all times and follow the cycle: $(R, R) \rightarrow (P, P) \rightarrow (S, S)$.

For this cyclical CFP we have:

$$C^{-1}D = \begin{bmatrix} 0 & -\alpha^{-1} & 1+\alpha^{-1} \\ 0 & -\left(\alpha^{-1}+\alpha^{-2}\right) & 1+\alpha^{-1}+\alpha^{-2} \\ 0 & -\left(\alpha^{-1}+\alpha^{-2}+\alpha^{-3}\right) & 1+\alpha^{-1}+\alpha^{-2}+\alpha^{-3} \end{bmatrix}$$

and the Eigen roots of this matrix are: $0, 1$ and α^{-3}. For $\alpha \geq 1$, the largest root is, therefore, 1. This implies that the run-lengths are, in the limit, constant and the CFP converges.

Notice that if $\alpha > 1$, the product of the non-zero roots is *less* than 1; and thus the conclusion of Lemma 5 does not hold. This is because in the cycle given above, both players switch simultaneously. Recall that the assumption that the players did not switch simultaneously played an important role in the proof of Lemma 5.

Thus we have a family of games in which CFP converges cyclically to a mixed strategy equilibrium with more than two strategies; of course, the class of games is non-generic, as are the initial conditions.

9 Appendix A

Consider the determinant of the form

$$D = \begin{vmatrix}
0 & 0 & \delta_1 & 0 & \delta_2 & 0 & \delta_3 & \cdots & 0 & \delta_{\kappa-1} & 0 \\
0 & 0 & -\delta_1 & 0 & -\delta_2 & 0 & -\delta_3 & & 0 & -\delta_{\kappa-1} & \delta_0 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\
\vdots & & & & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & -1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1
\end{vmatrix} \tag{48}$$

where each δ_j is either zero or one.

$D = \sum_{j=1}^{\kappa-1} j\delta_j - \sum_{j=1}^{\kappa-1}(\kappa-1)\delta_j$.

Proof. Multiply the even numbered columns by (-1), add row 1 to the row 2, and finally expand by the first column (which has only one non-zero entry). The result is:

$$D = \left| \begin{array}{cccccccccc}
0 & \delta_1 & 0 & \delta_2 & 0 & \delta_3 & \cdots & 0 & \delta_{\kappa-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & -1 \\
0 & -1 & 1 & 1 & -1 & 0 & & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 & & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\
\vdots & & & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1
\end{array} \right|$$

To column 1, add all the remaining columns. To column 2, add the all the columns except column 1, etc. The result is:

$$D = \left| \begin{array}{cccccccccc}
\sum_{j=1}^{\kappa-1} \delta_j & \sum_{j=1}^{\kappa-1} \delta_j & \sum_{j=2}^{\kappa-1} \delta_j & \sum_{j=2}^{\kappa-1} \delta_j & \cdots & \delta_{\kappa-1} + \delta_{\kappa-2} & \delta_{\kappa-1} + \delta_{\kappa-2} & \delta_{\kappa-1} & \delta_{\kappa-1} & 0 \\
-1 & -1 & -1 & -1 & & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & \ddots & & & & & \vdots \\
0 & 0 & 0 & 0 & & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1
\end{array} \right|$$

Add the second last column to the fourth last, the fourth last to the sixth last etc. to get

$$D = \begin{vmatrix} \sum_{j=1}^{\kappa-1}\delta_j & \sum_{j=1}^{\kappa-1} j\delta_j & \sum_{j=2}^{\kappa-1}\delta_j & \sum_{j=2}^{\kappa-1}(j-1)\delta_j & \cdots & 2\delta_{\kappa-1}+\delta_{\kappa-2} & \delta_{\kappa-1} & \delta_{\kappa-1} & 0 \\ -1 & -(\kappa-1) & -1 & -(\kappa-2) & & -2 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{vmatrix}$$

Thus,

$$D = \begin{vmatrix} \sum_{j=1}^{\kappa-1}\delta_j & \sum_{j=1}^{\kappa-1} j\delta_j \\ \\ -1 & -(\kappa-1) \end{vmatrix} = \sum_{j=1}^{\kappa-1} j\delta_j - \sum_{j=1}^{\kappa-1}(\kappa-1)\delta_j \qquad (49)$$

Since $\delta_j \in \{0,1\}$ by assumption, we have:

$D = 0$ if and only if $\delta_1 = \delta_2 = \ldots = \delta_{\kappa-2} = 0$.

Finally, observe that $|L(1)|$ from (34) is of the form (48) with each $\delta_j = 1$. Thus,

$$|L(1)| = \sum_{j=1}^{\kappa-1} j - \sum_{j=1}^{\kappa-1}(\kappa-1) = \frac{\kappa(\kappa-1)}{2} - (\kappa-1)^2 = -\frac{1}{2}(\kappa-1)(\kappa-2)$$

10 Appendix B

In this appendix we consider the case when $\rho_1 + \rho_2 = 2$ but there is at least one k such that $\sigma(k) = \sigma(k+1)$. That is, at least one player switches twice in succession. We claim that the algebraic multiplicity of the unit root of $C^{-1}D$ is still 2.

Consider first a cycle

$$\langle 11, 21, 31, 32, 33, 43, ..., \kappa\kappa, 1\kappa \rangle$$

where player 1 switches twice in succession (from 1 to 2 to 3), after which player 2 also switches twice in succession (from 1 to 2 to 3). For this cycle, we have

$$[\lambda C - D] =$$

$$\begin{bmatrix}
\lambda\,(a_{21} - a_{11}) & a_{21} - a_{11} & a_{21} - a_{11} & a_{22} - a_{12} & \cdots & a_{2\kappa} - a_{1\kappa} \\
\lambda\,(a_{31} - a_{21}) & \lambda\,(a_{31} - a_{21}) & a_{31} - a_{21} & a_{32} - a_{22} & & a_{3\kappa} - a_{2\kappa} \\
\lambda\,(b_{12} - b_{11}) & \lambda\,(b_{22} - b_{21}) & \lambda\,(b_{32} - b_{31}) & b_{32} - b_{31} & & b_{12} - b_{11} \\
\lambda\,(b_{13} - b_{12}) & \lambda\,(b_{23} - b_{22}) & \lambda\,(b_{33} - b_{32}) & \lambda\,(b_{33} - b_{32}) & & b_{13} - b_{12} \\
\lambda(a_{41} - a_{31}) & \lambda(a_{41} - a_{31}) & \lambda(a_{41} - a_{31}) & \lambda(a_{42} - a_{32}) & & \\
\vdots & & & \vdots & \ddots & \vdots \\
\lambda\,(b_{1\kappa} - b_{1\kappa-1}) & \lambda\,(b_{2\kappa} - b_{2\kappa-1}) & \lambda\,(b_{3\kappa} - b_{3\kappa-1}) & \lambda\,(b_{3\kappa} - b_{3\kappa-1}) & & b_{1\kappa} - b_{1\kappa-1} \\
\lambda\,(a_{11} - a_{\kappa 1}) & \lambda\,(a_{11} - a_{\kappa 1}) & \lambda\,(a_{11} - a_{\kappa 1}) & \lambda\,(a_{12} - a_{\kappa 2}) & & a_{1\kappa} - a_{\kappa\kappa} \\
\lambda\,(b_{11} - b_{1\kappa}) & \lambda\,(b_{21} - b_{2\kappa}) & \lambda\,(b_{31} - b_{3\kappa}) & \lambda\,(b_{31} - b_{3\kappa}) & \cdots & \lambda\,(b_{11} - b_{1\kappa})
\end{bmatrix}$$

If column 3 of this matrix is replaced by (column 2− column 3+column 4), and rows 2 and 3 are interchanged, we obtain the matrix:

$$\begin{bmatrix}
\lambda\,(a_{21} - a_{11}) & a_{21} - a_{11} & a_{22} - a_{12} & \cdots & a_{2\kappa} - a_{1\kappa} \\
\lambda\,(b_{12} - b_{11}) & \lambda\,(b_{22} - b_{21}) & \lambda\,(b_{22} - b_{21}) + (1 - \lambda)\,(b_{32} - b_{31}) & & b_{12} - b_{11} \\
\lambda\,(a_{31} - a_{21}) & \lambda\,(a_{31} - a_{21}) & (\lambda - 1)\,(a_{31} - a_{21}) + a_{32} - a_{22} & & a_{3\kappa} - a_{2\kappa} \\
\lambda\,(b_{13} - b_{12}) & \lambda\,(b_{23} - b_{22}) & \lambda\,(b_{23} - b_{22}) & & b_{13} - b_{12} \\
\vdots & & \vdots & \ddots & \vdots \\
\lambda\,(b_{1\kappa} - b_{1\kappa-1}) & \lambda\,(b_{2\kappa} - b_{2\kappa-1}) & \lambda\,(b_{2\kappa} - b_{2\kappa-1}) & & b_{1\kappa} - b_{1\kappa-1} \\
\lambda\,(a_{11} - a_{\kappa 1}) & \lambda\,(a_{11} - a_{\kappa 1}) & \lambda\,(a_{12} - a_{\kappa 2}) & & a_{1\kappa} - a_{\kappa\kappa} \\
\lambda\,(b_{11} - b_{1\kappa}) & \lambda\,(b_{21} - b_{2\kappa}) & \lambda\,(b_{21} - b_{2\kappa}) & \cdots & \lambda\,(b_{11} - b_{1\kappa})
\end{bmatrix}$$

For this matrix:

$$\begin{aligned} & \text{row } 1+\text{row } 3+\text{row } 5+\ldots+\text{row } K-1 \\ = \quad & (0,(1-\lambda)(a_{21}-a_{11}),(1-\lambda)(a_{32}-a_{12}-(a_{31}-a_{21})),\ldots,(1-\lambda)(a_{i_kj_k}-a_{1j_k}),..,0) \end{aligned}$$

and

$$\begin{aligned} & \text{row } 2+\text{row } 4+\text{row } 6+\ldots+\text{row } K \\ = \quad & (0,0,(1-\lambda)(b_{32}-b_{31}),\ldots,(1-\lambda)(b_{i_kj_k}-b_{i_k1}),\ldots,(1-\lambda)(b_{1\kappa}-b_{11})) \end{aligned}$$

Therefore, we can add the odd-numbered rows to row 1 and the even-numbered rows to row 2 to obtain

$$|\lambda C-D|=\lambda(1-\lambda)^2\times$$

$$\left|\begin{array}{cccccc} 0 & a_{21}-a_{11} & \begin{array}{c}[(a_{32}-a_{12})\\ -(a_{31}-a_{21})]\end{array} & \cdots & a_{\kappa\kappa}-a_{1\kappa} & 0 \\ 0 & 0 & b_{32}-b_{31} & & b_{\kappa\kappa}-b_{\kappa 1} & b_{1\kappa}-b_{11} \\ a_{31}-a_{21} & \lambda(a_{31}-a_{21}) & \begin{array}{c}[(\lambda-1)(a_{31}-a_{21})\\ +(a_{32}-a_{22})]\end{array} & & a_{3\kappa}-a_{2\kappa} & a_{3\kappa}-a_{2\kappa} \\ b_{13}-b_{12} & \lambda(b_{23}-b_{22}) & \lambda(b_{23}-b_{22}) & & b_{\kappa 3}-b_{\kappa 2} & b_{13}-b_{12} \\ \vdots & & \vdots & \ddots & & \vdots \\ b_{1\kappa}-b_{1\kappa-1} & \lambda(b_{2\kappa}-b_{2\kappa-1}) & \lambda(b_{2\kappa}-b_{2\kappa-1}) & & b_{\kappa\kappa}-b_{\kappa\kappa-1} & b_{1\kappa}-b_{1\kappa-1} \\ a_{11}-a_{\kappa 1} & \lambda(a_{11}-a_{\kappa 1}) & \lambda(a_{12}-a_{\kappa 2}) & & \lambda(a_{1\kappa}-a_{\kappa\kappa}) & a_{1\kappa}-a_{\kappa\kappa} \\ b_{11}-b_{1\kappa} & \lambda(b_{21}-b_{2\kappa}) & \lambda(b_{21}-b_{2\kappa}) & \cdots & \lambda(b_{\kappa 1}-b_{\kappa\kappa}) & \lambda(b_{11}-b_{1\kappa}) \end{array}\right| \tag{50}$$

$$=\lambda(1-\lambda)^2\times|M(\lambda)|$$

If we use the payoff matrix given by (32) and (33), i.e.

$$a_{ij}=\begin{cases} 0 & \text{if } i=j=1 \\ 1 & \text{if } i=j>1 \\ 0 & \text{if } i\neq j \end{cases}$$

$$b_{ij}=\begin{cases} -1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

then

$$|M(1)| = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \end{vmatrix} \tag{51}$$

Thus, $|M(1)|$ is of the form (48), and by the corollary to Lemma 9, $|M(1)| \neq 0$. Thus, again the algebraic multiplicity of the unit root is 2 (for generic games).

Similarly, if there are several places in the cycle where a player switches twice in a row, we can reduce the matrix $|\lambda C - D|$ to the form (48). This procedure is the same as the one that resulted in (50). Each time we "reduce" a sequential switch, in the first two rows one of the δ_j will change from one to zero, just as δ_1 became zero in (51). However, because sometime between run 2 and run $K-1$ there *must* exist two consecutive runs where first player 2 switches and then player 1 switches, not *all* of $\delta_1, \delta_2, ..., \delta_{\kappa-2}$ will be zero. For example, if the cycle is

$$\langle 11, 21, 31, 32, 33, 43, 53, 54, ...\rangle \tag{52}$$

then the "reduction" will simulate the cycle $\langle 11, 21, 22, 32, 33, 43, 44, 54, ...\rangle$. Then, in columns 3 and 7, the δ_1 and δ_3 will change from 1 to zero, but in column 5, $\delta_2 = 1$. By the corollary to Lemma 9, the determinant is still non-zero. Thus, if during the cycle any number of times players switch twice in a row, the algebraic multiplicity of the unit root is still 2 (for generic games).

If some player switches three or more consecutive times, the procedure is the same. By rearranging rows and columns, we can obtain a matrix of the form (48) which is non-singular. Note that there must exist a run k^*, $1 < k^* < \kappa$, such that (i) $\sigma(k^*) = 2$ and $\sigma(k^*+1) = 1$, and (ii) $i_{k^*} \equiv i^* \neq 1$ and $j_{k^*} \equiv j^* \neq 1$. This is because player one cannot switch from strategy 1 to 2 to ... back to 1 again consecutively, but player 2 must make some switch in-between, and a similarly player 1 cannot make consecutive switches from 1 to 2 to ... back to 1. (In the case of (52) take $k^* = 5$, whence $i^* = j^* = 3$).

It will suffice to illustrate this with the cycle

$$\langle 11, 21, 31, 41, 42, 43, ...(i^*, j^*-1), (i^*, j^*), (i^*+1, j^*), ..., 1\kappa\rangle$$

Here player 1 switches thrice in succession (from 1 to 2 to 3 to 4) and then player 2 switches three times. In this case, we operate on the matrix $\lambda C - D$ as follows. If we replace column 4 by (column 3 − column 4 + column 5) and interchange rows 3 and 4, we obtain a matrix $M_1(\lambda)$ which, for $\lambda = 1$, is identical to the matrix $|C - D|$ corresponding to the cycle

$$\langle 11, 21, 31, 32, 42, 43, ..., (i^*, j^*-1), (i^*, j^*), (i^*+1, j^*), ..., \kappa\kappa, 1\kappa\rangle$$

Next, in $M_1(\lambda)$ replace column 3 by (column 2− column 3+ column 4) and interchange rows 2 and 3. The result is a matrix $M_2(\lambda)$ which, for $\lambda = 1$, is identical to the matrix $|C - D|$ corresponding to the cycle

$$\langle 11, 21, 22, 32, 42, 43, ..., (i^*, j^*-1), (i^*, j^*), (i^*+1, j^*), ..., \kappa\kappa, 1\kappa\rangle$$

Finally, in $M_2(\lambda)$ replace column 5 by (column 4− column 5+ column 6) and interchange rows 4 and 5. The result is a matrix $M_3(\lambda)$ which, for $\lambda = 1$, is identical to the matrix $|C - D|$ corresponding to the cycle

$$\langle 11, 21, 22, 32, 33, 43, ..., (i^*, j^*-1), (i^*, j^*), (i^*+1, j^*), ..., \kappa\kappa, 1\kappa\rangle$$

It is clear how to proceed this way, to obtain a matrix $M(\lambda)$ which, when $\lambda = 1$, corresponds to the matrix $\lambda C - D$ for the simple cycle where $\sigma(k) \neq \sigma(k+1)$. In fact, for any λ, columns k^*-1, k^* and k^*+1 of $M(\lambda)$ will be identical to columns k^*-1, k^*, k^*+1 of the matrix $\lambda C - D$ for the simple cycle. This is because none of the operations will have affected these columns. Refer to the case (52), where columns 4, 5 and 6 remained unchanged.

If we then, in the matrix $M(1)$ sum the rows corresponding to player 1's (resp. player 2's) switches to obtain a matrix of the form (48), at least one of the δ_j for $j < \kappa - 1$ will be non-zero (viz. in the column k^*, since this was true for the matrix corresponding to the simple cycle). By the corollary to Lemma 9, $|M(1)| \neq 0$.

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