An Analysis of the War of Attrition and the All-Pay Auction

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Abstract

We study the war of attrition and the all-pay auction when players' signals are affiliated and symmetrically distributed. We (a) find sufficient conditions for the existence of symmetric monotonic equilibrium bidding strategies; and (b) examine the performance of these auction forms in terms of the expected revenue accruing to the seller. Under our conditions the war of attrition raises greater expected revenue than all other other known sealed bid auction forms.

1 Introduction

Since the classic work of Vickrey [19], the ranking of various auction forms in terms of expected revenue has been the central question of auction theory (Milgrom [12] and Wilson [20] provide surveys). When the bidders are risk-neutral

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and their information about the value of the object is independently and identically distributed, the so-called "revenue equivalence principle" (see, for instance, Myerson [15]) provides a complete answer to the revenue ranking question. When the assumption of independence is relaxed, the answer is less well understood. Utilizing the assumption that the bidders' information is affiliated, Milgrom and Weber [13] develop the most comprehensive set of revenue ranking results to date; however, they restrict attention to "standard" auction forms in which only the winner is required to pay.

In this paper, we extend the analysis of Milgrom and Weber [13] to auctions with the property that losing bidders also pay positive amounts.[1] Specifically, we examine the performance of two alternative auction forms, the war of attrition and the all-pay auction. These auction forms share the common feature that all losing bidders pay exactly their bids and differ only in the amounts paid by the winning bidder. In the former, the winning bidder pays the second highest bid; whereas in the latter, the winner pays his own bid. As such, they are analogous to the standard second-price and first-price sealed bid auctions, respectively.[2] We identify circumstances in which the expected revenue from the auctions considered here exceeds that from the corresponding auctions in which only the winner pays. Thus, the war of attrition yields greater revenue than the second-price auction, and the all-pay auction yields greater revenue than the first-price auction. We also show that the war of attrition outperforms the all-pay auction and thus yields higher revenue than all known sealed bid auctions.

Our reasons for examining these alternative, "non-standard" auction forms are threefold: First, from a mechanism design standpoint, the restriction to allocation schemes which require only the person receiving the object to pay seems unwarranted. Second, although these forms may not be widely used in an auction setting, the underlying games are natural models of conflict in many situations. For instance, the war of attrition has been used as a model for conflict among animals and insects ([10], [3]) and the struggle for survival among firms ([8]), while the all-pay auction has been used to model the arms race ([17]) and rent-seeking activity, such as lobbying ([5], [4]). Our third reason for considering these forms

[1]Of course, in an auction with an entry fee each bidder would also pay a positive amount. However, the amount would be fixed by the seller and, unlike the auctions we consider, not depend on the bid itself.

[2]Indeed, the "war of attrition" is perhaps better described as a "second-price all-pay auction" and the "all-pay auction" as a "first-price all-pay auction." We have chosen, however, to retain existing terminology.

is that, as our results indicate, they raise greater revenue than the forms previously considered and hence are worthy of attention as auctions *per se*.[3]

This paper is organized as follows: Section 2 briefly describes the model used in the analysis. Since this is the same as the model in [13], we eschew a detailed description. As in [13], our major assumption is that bidders' signals are affiliated. In Section 3, we find sufficient conditions for the existence of a symmetric and increasing equilibrium in the war of attrition. Section 4 contains a parallel development for the all-pay auction. In both the war of attrition and the all-pay auction, we find that symmetric, increasing equilibria exist if bidders' signals are not "too affiliated." In Section 5, we develop revenue comparisons between these auctions and the standard forms. These comparisons form the basis for the results reported above. Section 6 studies circumstances under which the symmetric equilibria we consider are unique in the class of increasing equilibria. Technical results are collected in two appendices. Appendix A contains some useful results on affiliated random variables. Appendix B contains some results relevant for Section 6.

2 Preliminaries

We follow the model and notation of Milgrom and Weber [13] exactly. There is a single object to be auctioned and there are n bidders. Each bidder i receives a real valued *signal,* X_i, prior to the auction that affects the value of the object. Let $S = (S_1, S_2, ..., S_m)$ be other random variables that influence the value but are not observed by any bidder. The *value* of the object to bidder i is then:

$$V_i = u\left(S, X_i, \{X_j\}_{j \neq i}\right)$$

where u is non-negative, continuous, and increasing[4] in its variables.

The random variables $S_1, S_2, ..., S_m, X_1, X_2, ..., X_n$ have a joint density $f(S, X_1, X_2, ..., X_n)$, and the function f is symmetric in the bidders' signals. The random variables $S_1, S_2, ..., S_m, X_1, X_2, ..., X_n$ are assumed to be *affiliated*, and it is assumed that $E[V_i] < \infty$.

The reader should consult [13] for details.

[3] While it is known that in common value settings it is possible to extract nearly all the surplus ([7], [11]), the mechanisms that do this depend on the underlying distribution of signals.

[4] Throughout the paper, the term "increasing" is synonymous with "strictly increasing."

In what follows, the random variable $Y_1 = \max\{X_j\}_{j \neq 1}$. Let $f_{Y_1}(\cdot \mid x)$ denote the conditional density of Y_1 given that $X_1 = x$, and let $F_{Y_1}(\cdot \mid x)$ denote the corresponding cumulative distribution function. The variables X_1 and Y_1 are also affiliated; in Appendix A we derive some simple facts about the conditional distribution, $F_{Y_1}(\cdot \mid x)$, that are useful for our analysis.

Define $v(x, y) = E[V_1 \mid X_1 = x, Y_1 = y]$. Since X_1 and Y_1 are affiliated, $v(x, y)$ is a non-decreasing function of its arguments. As in [13], we assume that it is, in fact, increasing.

3 Equilibrium in the War of Attrition

We model the war of attrition as an auction in the following manner. Prior to the start of the auction each bidder, i, receives a signal, X_i, which gives him or her some information about the value of the object. Each bidder submits a sealed bid of b_i, and the payoffs are:

$$W_i = \begin{cases} V_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ -b_i & \text{if } b_i < \max_{j \neq i} b_j \\ \frac{1}{\#\{k : b_k = b_i\}} V_i - b_i & \text{if } b_i = \max_{j \neq i} b_j \end{cases}$$

where $i \neq j$. We have assumed that if $b_i = \max_{j \neq i} b_j$, the prize goes to each winning bidder with equal probability.

The analogy with the classic war of attrition model of conflict among two animals (Bishop, Canning and Smith [3]) should be clear. The derivation of a symmetric equilibrium with independent private values, that is, when $X_i = V_i$ and the X_i's are independently and identically distributed, is well known (see, for instance, [14]).

We begin with a heuristic derivation of the symmetric equilibrium strategy.

Suppose that players $j \neq 1$ follow the symmetric and increasing equilibrium strategy β. Suppose player 1 receives a signal, $X_1 = x$, and "bids" b. Then player

1's expected payoff is:

$$\begin{aligned}\Pi(b;x) &= \int_{-\infty}^{\beta^{-1}(b)} \left(E\left[V_1 \mid X_1 = x,\ Y_1 = y\right] - \beta(y)\right) f_{Y_1}(y \mid x)\, dy \\ &\quad - \left[1 - F_{Y_1}\left(\beta^{-1}(b) \mid x\right)\right] b \\ &= \int_{-\infty}^{\beta^{-1}(b)} (v(x,y) - \beta(y)) f_{Y_1}(y \mid x)\, dy \\ &\quad - \left[1 - F_{Y_1}\left(\beta^{-1}(b) \mid x\right)\right] b\end{aligned} \tag{1}$$

Maximizing (1) with respect to b yields the first order condition:

$$\begin{aligned}\tfrac{\partial \Pi}{\partial b} &= \left(v\left(x, \beta^{-1}(b)\right) - b\right) f_{Y_1}\left(\beta^{-1}(b) \mid x\right) \tfrac{1}{\beta'(\beta^{-1}(b))} \\ &\quad - \left[1 - F_{Y_1}\left(\beta^{-1}(b) \mid x\right)\right] + b f_{Y_1}\left(\beta^{-1}(b) \mid x\right) \tfrac{1}{\beta'(\beta^{-1}(b))} \\ &= v\left(x, \beta^{-1}(b)\right) f_{Y_1}\left(\beta^{-1}(b) \mid x\right) \tfrac{1}{\beta'(\beta^{-1}(b))} - \left[1 - F_{Y_1}\left(\beta^{-1}(b) \mid x\right)\right] \\ &= 0\end{aligned} \tag{2}$$

At a symmetric equilibrium, $b = \beta(x)$, and thus (2) becomes:

$$v(x,x) f_{Y_1}(x \mid x) \frac{1}{\beta'(x)} - \left[1 - F_{Y_1}(x \mid x)\right] = 0$$

Rearranging this yields:

$$\beta'(x) = \frac{v(x,x) f_{Y_1}(x \mid x)}{1 - F_{Y_1}(x \mid x)} \tag{3}$$

and thus:

$$\beta(x) = \int_{-\infty}^{x} \frac{v(t,t) f_{Y_1}(t \mid t)}{1 - F_{Y_1}(t \mid t)} dt \tag{4}$$

The derivation of (4) is only heuristic because (3) is merely a necessary condition, and the global optimality of $\beta(x)$ against β has not been established. Theorem 1 below provides a sufficient condition for β to be a symmetric equilibrium.

Definition 1 *The* hazard rate *of the distribution, $F_{Y_1}(\cdot \mid x)$, is defined as:*

$$\lambda(y \mid x) = \frac{f_{Y_1}(y \mid x)}{1 - F_{Y_1}(y \mid x)}.$$

With affiliation, the hazard rate, $\lambda(y \mid x)$, is a non-increasing function of x (Lemma 2 in Appendix A).

Definition 2 *Let $\varphi : \Re^2 \to \Re$ be defined by: $\varphi(x, y) = v(x, y)\lambda(y \mid x)$.*

Theorem 1 *Suppose that, for all y, $\varphi(\cdot, y)$ is an increasing function. A symmetric equilibrium in the war of attrition is given by the function β defined as:*

$$\beta(x) = \int_{-\infty}^{x} v(t, t)\lambda(t \mid t)dt. \tag{5}$$

Proof. Let $\bar{x}$ denote the supremum of the support of Y_1. If players $j \neq 1$ use the strategy β, then it is never profitable for player 1 to bid more than $\beta(\bar{x})$.

If player 1 bids an amount, $\beta(z)$, when the signal is x, his or her payoff is:

$$\begin{aligned}
\Pi(z; x) &= \int_{-\infty}^{z} \left(E\left[V_1 \mid X_1 = x,\ Y_1 = y\right] - \beta(y)\right) f_{Y_1}(y \mid x)\, dy \\
&\quad - \left[1 - F_{Y_1}(z \mid x)\right] \beta(z) \\
&= \int_{-\infty}^{z} (v(x, y) - \beta(y)) f_{Y_1}(y \mid x)\, dy \\
&\quad - \left[1 - F_{Y_1}(z \mid x)\right] \beta(z)
\end{aligned}$$

or:

$$\begin{aligned}
\Pi(z; x) &= \int_{-\infty}^{z} v(x, y) f_{Y_1}(y \mid x)\, dy - \int_{-\infty}^{z} \beta(y) f_{Y_1}(y \mid x)\, dy \\
&\quad - \left[1 - F_{Y_1}(z \mid x)\right] \beta(z) \\
&= \int_{-\infty}^{z} v(x, y) f_{Y_1}(y \mid x)\, dy - \beta(y) F_{Y_1}(y \mid x)\big|_{-\infty}^{z} + \int_{-\infty}^{z} \beta'(y) F_{Y_1}(y \mid x) dy \\
&\quad - \left[1 - F_{Y_1}(z \mid x)\right] \beta(z) \\
&= \int_{-\infty}^{z} v(x, y) f_{Y_1}(y \mid x)\, dy - \beta(z) F_{Y_1}(z \mid x) + \int_{-\infty}^{z} \beta'(y) F_{Y_1}(y \mid x) dy \\
&\quad - \left[1 - F_{Y_1}(z \mid x)\right] \beta(z) \\
&= \int_{-\infty}^{z} v(x, y) f_{Y_1}(y \mid x)\, dy + \int_{-\infty}^{z} \beta'(y) F_{Y_1}(y \mid x) dy - \beta(z)
\end{aligned}$$

Using the fact that:

$$\beta'(y) = v(y, y)\lambda(y \mid y)$$

we obtain:

$$\begin{aligned}
\Pi(z;x) &= \int_{-\infty}^{z} v(x,y) f_{Y_1}(y \mid x)\, dy - \beta(z) + \int_{-\infty}^{z} v(y,y)\lambda(y \mid y) F_{Y_1}(y \mid x) dy \\
&= \int_{-\infty}^{z} v(x,y) f_{Y_1}(y \mid x)\, dy - \int_{-\infty}^{z} \left[v(y,y)\lambda(y \mid y) - v(y,y)\lambda(y \mid y) F_{Y_1}(y \mid x) \right] dy \\
&= \int_{-\infty}^{z} v(x,y) f_{Y_1}(y \mid x)\, dy - \int_{-\infty}^{z} v(y,y) f_{Y_1}(y \mid x)\lambda(y \mid y) \left[\frac{1 - F_{Y_1}(y \mid x)}{f_{Y_1}(y \mid x)} \right] dy \\
&= \int_{-\infty}^{z} v(x,y) f_{Y_1}(y \mid x)\, dy - \int_{-\infty}^{z} v(y,y) f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)} \right] dy \\
&= \int_{-\infty}^{z} \left(v(x,y) - v(y,y) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)} \right] \right) f_{Y_1}(y \mid x) dy \\
&= \int_{-\infty}^{z} [\varphi(x,y) - \varphi(y,y)] \frac{f_{Y_1}(y \mid x)}{\lambda(y \mid x)} dy
\end{aligned}$$

Since, by assumption, $\varphi(\cdot, y)$ is an increasing function, for all $y < x$, $[\varphi(x,y) - \varphi(y,y)] > 0$, and, for all $y > x$, $[\varphi(x,y) - \varphi(y,y)] < 0$. Thus, $\Pi(z,x)$ is maximized by choosing $z = x$.

Finally, observe that the equilibrium payoff to a player who receives a signal of x is:

$$\Pi(x;x) = \int_{-\infty}^{x} [\varphi(x,y) - \varphi(y,y)] \frac{f_{Y_1}(y \mid x)}{\lambda(y \mid x)} dy \geq 0$$

and thus each player is willing to participate in the auction. □

Notice that $\varphi(x,y)$ is the product of $v(x,y)$ and $\lambda(y \mid x)$ where the former is increasing in x, while the latter is decreasing in x. As a result, $\varphi(\cdot, y)$ is increasing if the affiliation between X_1 and Y_1 is not so strong that it overwhelms the increase in the expected value of the object, $v(\cdot, y)$, resulting from a higher signal, x. Of course, the assumption is automatically satisfied if bidders' signals are independent.

The assumption that $\varphi(\cdot, y)$ is increasing can be replaced by the weaker condition that, for all x :

$$\begin{array}{c} \text{for all } y < x, [\varphi(x,y) - \varphi(y,y)] > 0 \\ \text{for all } y > x, [\varphi(x,y) - \varphi(y,y)] < 0. \end{array} \tag{6}$$

since this is all that is needed to ensure that $\Pi(z; x)$ is maximized at $z = x$.

As an example of a situation where the conditions of Theorem 1 are satisfied, suppose $n = 2$, and let X (resp. Y) be the random variable that denotes bidder 1's (resp. 2's) signal. Let:

$$f(x, y) = \frac{4}{5}(1 + xy) \text{ on } [0, 1] \times [0, 1] .$$

Then $f_{Y_1}(y \mid x) = \frac{2(1+xy)}{2+x}$, and $\lambda(y \mid x) = \frac{2(1+xy)}{2+x-2y-xy^2}$. If $v(x, y) = x$, then $\varphi(x, y) = \frac{2x(1+xy)}{2+x-2y-xy^2}$, and it may be verified that $\varphi(\cdot, y)$ is an increasing function.

Next, let *supp* f_X denote the common support of the bidders' signals, and let $\underline{x}$ and $\overline{x}$ denote the infimum and supremum of *supp* f_X, respectively. Of course, it may be that $\underline{x} = -\infty$ or $\overline{x} = \infty$ or both.

Two features of the equilibrium strategy, (5) deserve to be highlighted. First, a bidder receiving the lowest possible signal, $\underline{x}$, bids zero. This is true even if $v(\underline{x}, \underline{x})$ is strictly positive. Second, as a bidder's signal approaches $\overline{x}$, his bid becomes unbounded. Again, this holds even if the expected value of the object at $\overline{x}$, $v(\overline{x}, \overline{x})$ is finite.

Proposition 1 *Suppose that, for all* y, $\varphi(\cdot, y)$ *is an increasing function. Then* (i) $\lim_{x \to \underline{x}} \beta(x) = 0$; *and* (ii) $\lim_{x \to \overline{x}} \beta(x) = \infty$.

Proof. (i) follows directly from (5).

To verify (ii), choose z such that $v(z, z) > 0$. From (5), we can write:

$$\begin{aligned}
\beta(x) &= \int_{-\infty}^{x} v(y, y)\lambda(y \mid y)dy \\
&= \int_{-\infty}^{z} v(y, y)\lambda(y \mid y)dy + \int_{z}^{x} v(y, y)\lambda(y \mid y)dy \\
&\geq \int_{-\infty}^{z} v(y, y)\lambda(y \mid y)dy + \int_{z}^{x} v(z, y)\lambda(y \mid z)dy \\
&\geq \int_{-\infty}^{z} v(y, y)\lambda(y \mid y)dy + \int_{z}^{x} v(z, z)\lambda(y \mid z)dy
\end{aligned} \tag{7}$$

where the first inequality follows from the fact that $\varphi(\cdot, y) = v(\cdot, y)\lambda(y \mid \cdot)$ is increasing and the second from the fact that $v(z, \cdot)$ is increasing.

But now observe that for all y:

$$\lambda\left(y \mid z\right) = -\frac{d}{dy}\left(\ln\left[1 - F_{Y_1}(y \mid z)\right]\right)$$

and thus:

$$\int_z^x \lambda\left(y \mid z\right) dy = \ln\left(\frac{1 - F_{Y_1}(z \mid z)}{1 - F_{Y_1}(x \mid z)}\right) \tag{8}$$

Using (8) in (7) we obtain:

$$\beta(x) \geq \int_{-\infty}^{z} v(y,y)\lambda(y \mid y)dy + v(z,z)\ln\left(\frac{1 - F_{Y_1}(z \mid z)}{1 - F_{Y_1}(x \mid z)}\right)$$

As $x \to \overline{x}$, $F_{Y_1}(x \mid z) \to 1$. This completes the proof. $\square$

4 Equilibrium in the All-Pay Auction

In an all-pay auction each bidder submits a sealed bid of b_i and the payoffs are:

$$W_i = \begin{cases} V_i - b_i & \text{if } b_i > \max_{j\neq i} b_j \\ -b_i & \text{if } b_i < \max_{j\neq i} b_j \\ \frac{1}{\#\{k:b_k=b_i\}}V_i - b_i & \text{if } b_i = \max_{j\neq i} b_j \end{cases}$$

where $i{\neq}j$. As before, we have assumed that if $b_i = \max_{j\neq i} b_j$, the prize goes to each winning bidder with equal probability.

Once again it is useful to begin with a heuristic derivation.

Suppose players $j{\neq}1$ follow the symmetric (increasing) equilibrium strategy α. Suppose player 1 receives a signal, $X_1 = x$, and "bids" b. Then player 1's expected payoff is:

$$\begin{aligned} \Pi(b;x) &= \int_{-\infty}^{\alpha^{-1}(b)} E\left[V_1 \mid X_1 = x,\ Y_1 = y\right] f_{Y_1}\left(y \mid x\right) dy - b \\ &= \int_{-\infty}^{\alpha^{-1}(b)} v(x,y) f_{Y_1}(y \mid x)dy - b \end{aligned} \tag{9}$$

Maximizing (9) with respect to b yields the first order condition:

$$\begin{aligned} \tfrac{\partial \Pi}{\partial b} &= v(x,\alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b) \mid x)\tfrac{1}{\alpha'(\alpha^{-1}(b))} - 1 \\ &= 0 \end{aligned} \tag{10}$$

At a symmetric equilibrium, $\alpha(x) = b$ and thus (10) becomes:

$$\alpha'(x) = v(x,x)f_{Y_1}(x \mid x) \tag{11}$$

and thus:

$$\alpha(x) = \int_{-\infty}^{x} v(t,t)f_{Y_1}(t \mid t)\, dt$$

Once again, the derivation is heuristic since (11) is only a necessary condition.

Definition 3 *Let $\psi : \Re^2 \to \Re$ be defined by $\psi(x,y) = v(x,y)f_{Y_1}(y \mid x)$.*

Theorem 2 *Suppose that, for all y, $\psi(\cdot, y)$ is an increasing function. A symmetric equilibrium in the all-pay auction is given by the function α defined as:*

$$\alpha(x) = \int_{-\infty}^{x} v(t,t)f_{Y_1}(t \mid t)\, dt. \tag{12}$$

Proof. Let $\overline{x}$ denote the supremum of the support of Y_1. If players $j \neq 1$ use the strategy α, then clearly it cannot be a best response for player 1 to bid more than $\alpha(\overline{x})$.

If player 1 bids an amount $\alpha(z)$ when the signal is x, his or her payoff is:

$$\begin{aligned}
\Pi(z;x) &= \int_{-\infty}^{z} E\left[V_1 \mid X_1 = x,\ Y_1 = y\right] f_{Y_1}(y \mid x)\, dy - \alpha(z) \\
&= \int_{-\infty}^{z} v(x,y)f_{Y_1}(y \mid x)\, dy - \alpha(z) \\
&= \int_{-\infty}^{z} v(x,y)f_{Y_1}(y \mid x)\, dy - \int_{-\infty}^{z} v(y,y)f_{Y_1}(y \mid y)\, dy \\
&= \int_{-\infty}^{z} \left[\psi(x,y) - \psi(y,y)\right] dy
\end{aligned}$$

Since, by assumption, $\psi(\cdot, y)$ is an increasing function, for all $y < x$, $[\psi(x,y) - \psi(y,y)] > 0$, and, for all $y > x$, $[\psi(x,y) - \psi(y,y)] < 0$. Thus, $\Pi(z,x)$ is maximized by choosing $z = x$.

Finally, observe that the equilibrium payoff to a player who receives a signal of x is:

$$\Pi(x,x) = \int_{-\infty}^{x} \left[\psi(x,y) - \psi(y,y)\right] dy \geq 0 \tag{13}$$

and thus each player is willing to participate in the auction. $\square$

It is useful to compare some qualitative features of the equilibrium strategy (12) in the all-pay auction to the equilibrium strategy (5) of the war of attrition. It is still the case that a bidder receiving the lowest possible signal, $\underline{x}$, bids zero. This is true even if $v(\underline{x}, \underline{x})$ is strictly positive. However, in an all-pay auction, as a bidder's signal approaches $\overline{x}$, his bid is bounded if the expected value of the object at $\overline{x}$, $v(\overline{x}, \overline{x})$ is finite. This is in contrast to the unboundedness of the bids in the war of attrition.

Proposition 2 *Suppose that, for all* y, $\psi(\cdot, y)$ *is an increasing function. Then* (i) $\lim_{x\to\underline{x}} \alpha(x) = 0$; *and* (ii) $\lim_{x\to\overline{x}} \alpha(x) \leq \lim_{x\to\overline{x}} v(x, x)$.

Proof. (i) follows immediately from (12).

To verify (ii) notice that from (13):

$$\alpha(x) \leq \int_{-\infty}^{x} v(x, y) f_{Y_1}(y \mid x)\, dy \leq v(x, x) \int_{-\infty}^{x} f_{Y_1}(y \mid x)\, dy$$

and as $x \to \overline{x}$, the right hand side tends to $\lim_{x\to\overline{x}} v(x, x)$. $\square$

5 Revenue Comparisons

In this section we examine the performance of the war of attrition and the all-pay auction in terms of the expected revenue accruing to the seller. As a benchmark, recall from Milgrom and Weber [13] that, with affiliation, the expected revenue from a second-price auction is greater than the expected revenue from a first-price auction.

5.1 War of Attrition versus Second-Price Auction

Our first result is that, under the condition that $\varphi(x, y)$ is increasing in x, the revenue from the war of attrition is greater than that from a second-price auction.

Theorem 3 *Suppose* $\varphi(\cdot, y)$ *is increasing. Then the expected revenue from the war of attrition is greater than or equal to the expected revenue from a second-price auction.*

Proof. In a second-price auction, the equilibrium bid by a bidder who receives a signal of x is $v(x, x)$ (see [13], pages 1100-1101), and thus the expected payment by such a bidder is:

$$e_2(x) = \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid x) dy \tag{14}$$

In a war of attrition, the expected payment in equilibrium by a bidder who receives a signal of x is:

$$\begin{aligned}
e_W(x) &= \int_{-\infty}^{x} \beta(y) f_{Y_1}(y \mid x) dy + \left[1 - F_{Y_1}(x \mid x)\right] \beta(x) \\
&= \beta(x) F_{Y_1}(x \mid x) - \int_{-\infty}^{x} \beta'(y) F_{Y_1}(y \mid x) dy + \left[1 - F_{Y_1}(x \mid x)\right] \beta(x) \\
&= \beta(x) - \int_{-\infty}^{x} \beta'(y) F_{Y_1}(y \mid x) dy \\
&= \int_{-\infty}^{x} \left[\frac{v(y, y) f_{Y_1}(y \mid y)}{1 - F_{Y_1}(y \mid y)} - \frac{v(y, y) f_{Y_1}(y \mid y)}{1 - F_{Y_1}(y \mid y)} F_{Y_1}(y \mid x)\right] dy \\
&= \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid x) \left[\frac{f_{Y_1}(y \mid y)}{1 - F_{Y_1}(y \mid y)}\right] \left[\frac{1 - F_{Y_1}(y \mid x)}{f_{Y_1}(y \mid x)}\right] dy \\
&= \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)}\right] dy.
\end{aligned} \tag{15}$$

For $y \leq x$, Lemma 2 in Appendix A implies that $\lambda(y \mid y) \geq \lambda(y \mid x)$ and thus:

$$\begin{aligned}
e_W(x) &= \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)}\right] dy \\
&\geq \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid x) dy \\
&= e_2(x)
\end{aligned} \tag{16}$$

using (14). $\square$

5.2 All-Pay Auction versus First-Price Auction

Our second result is that, under the condition that $\psi(x, y)$ is increasing in x, the revenue from the all-pay auction is greater than that from a first-price auction.[5]

Theorem 4 *Suppose $\psi(\cdot, y)$ is increasing. Then the expected revenue from the all-pay auction is greater than or equal to the expected revenue from a first-price sealed bid auction.*

Proof. Let $b^*(x)$ denote the equilibrium bid of a bidder who receives a signal of x (see [13], page 1107). Then the expected payment is:

$$\begin{aligned} e_1(x) &= F_{Y_1}(x \mid x) b^*(x) \\ &= F_{Y_1}(x \mid x) \int_{-\infty}^{x} v(y, y) dL(y \mid x) \end{aligned} \tag{17}$$

where $L(y \mid x) = \exp\left(-\int_y^x \frac{f_{Y_1}(t|t)}{F_{Y_1}(t|t)} dt\right)$. (17) can be rewritten as:

$$e_1(x) = \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid y) \left[\frac{F_{Y_1}(x \mid x)}{F_{Y_1}(y \mid y)}\right] \exp\left(-\int_y^x \frac{f_{Y_1}(t \mid t)}{F_{Y_1}(t \mid t)} dt\right) dy. \tag{18}$$

From Lemma 4 in Appendix A, for all $y < x$:

$$\exp\left(-\int_y^x \frac{f_{Y_1}(t \mid t)}{F_{Y_1}(t \mid t)} dt\right) \leq \left[\frac{F_{Y_1}(y \mid y)}{F_{Y_1}(x \mid x)}\right]$$

Thus, we can write:

$$\begin{aligned} e_1(x) &= \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid y) \left[\frac{F_{Y_1}(x \mid x)}{F_{Y_1}(y \mid y)}\right] \exp\left(-\int_y^x \frac{f_{Y_1}(t \mid t)}{F_{Y_1}(t \mid t)} dt\right) dy \\ &\leq \int_{-\infty}^{x} v(y, y) f_{Y_1}(y \mid y) dy \\ &= \alpha(x) \\ &= e_A(x). \ \square \end{aligned}$$

[5] Amann and Leininger [2] have independently provided an alternative proof of this result.

5.3 War of Attrition versus All-Pay Auction

Our next result compares the expected revenue from the war of attrition to that from an all-pay auction.

We first establish that the sufficient condition identified in Section 3 implies that there is an increasing equilibrium in the all-pay auction also.

Proposition 3 *Suppose that $\varphi(\cdot, y)$ is an increasing function of x. Then $\psi(\cdot, y)$ is an increasing function of x.*

Proof. Let $x < x'$. Then since $\varphi(\cdot, y)$ is an increasing function, we have that:

$$v(x, y)\left[\frac{f_{Y_1}(y \mid x)}{1 - F_{Y_1}(y \mid x)}\right] < v(x', y)\left[\frac{f_{Y_1}(y \mid x')}{1 - F_{Y_1}(y \mid x')}\right] \tag{19}$$

By Lemma 3 in Appendix A, $F_{Y_1}(y \mid x) \geq F_{Y_1}(y \mid x')$ and thus:

$$\frac{1}{1 - F_{Y_1}(y \mid x)} \geq \frac{1}{1 - F_{Y_1}(y \mid x')}$$

Now, from (19), we can immediately infer that:

$$v(x, y) f_{Y_1}(y \mid x) < v(x', y) f_{Y_1}(y \mid x')$$

which completes the proof. $\square$

We now show that if $\varphi(\cdot, y)$ is increasing, the war of attrition outperforms the all-pay auction.[6]

Theorem 5 *Suppose $\varphi(\cdot, y)$ is increasing. Then the expected revenue from the war of attrition is greater than or equal to the expected revenue from an all-pay auction.*

[6] Bulow and Klemperer [6] have constructed a very interesting example in which the all-pay auction extracts *all* the surplus from the bidders. In the example, $\psi(\cdot, y)$ is increasing but $\varphi(\cdot, y)$ is not.

Proof. In an all-pay auction, the expected payment in equilibrium by a bidder who receives a signal of x is:

$$e_A(x) = \int_{-\infty}^{x} v(y,y) f_{Y_1}(y \mid y) dy \tag{20}$$

From (15) in a war of attrition, the expected payment in equilibrium is:

$$\begin{aligned}
e_W(x) &= \int_{-\infty}^{x} v(y,y) f_{Y_1}(y \mid x) \left[\frac{\lambda(y \mid y)}{\lambda(y \mid x)}\right] dy \\
&= \int_{-\infty}^{x} v(y,y) f_{Y_1}(y \mid x) \left[\frac{f_{Y_1}(y \mid y)}{1 - F_{Y_1}(y \mid y)}\right] \left[\frac{1 - F_{Y_1}(y \mid x)}{f_{Y_1}(y \mid x)}\right] dy \\
&= \int_{-\infty}^{x} v(y,y) f_{Y_1}(y \mid y) \left[\frac{1 - F_{Y_1}(y \mid x)}{1 - F_{Y_1}(y \mid y)}\right] dy \\
&\geq \int_{-\infty}^{x} v(y,y) f_{Y_1}(y \mid y) dy \\
&= e_A(x).
\end{aligned}$$

since, by Lemma 3 in Appendix A, for all $y < x$, $F_{Y_1}(y \mid x) \leq F_{Y_1}(y \mid y)$ and thus $\left[\frac{1-F_{Y_1}(y|x)}{1-F_{Y_1}(y|y)}\right] \geq 1$. □

Analogous to the result that the equilibrium bids in the second-price auction exceed those in the first-price auction, we have the result that the equilibrium bids in the war of attrition exceed those in the all-pay auction. To see this, observe that:

$$\begin{aligned}
\beta(x) &= \int_{-\infty}^{x} v(t,t) \lambda(t \mid t) dt \\
&= \int_{-\infty}^{x} v(t,t) \left[\frac{f_{Y_1}(t \mid t)}{1 - F_{Y_1}(t \mid t)}\right] dt \\
&\geq \int_{-\infty}^{x} v(t,t) f_{Y_1}(t \mid t) dt \\
&= \alpha(x).
\end{aligned}$$

5.4 All-Pay Auction versus Second-Price Auction

It remains to compare the expected revenues from the all-pay auction to that from the second-price auction. We now show by means of two examples that no general ranking of the two auctions is possible.

Let:

$$f(x, y) = \frac{4}{5}(1 + xy) \text{ on } [0, 1] \times [0, 1] .$$

The density function f satisfies the affiliation inequality.

If $v(x, y) = x$, $\psi(x, y)$ is an increasing function of x so that the conditions of Theorem 2 are satisfied. Routine calculations show that, for all $x > 0$, $e_A(x) > e_2(x)$, and thus, in this case, the expected revenue from an all-pay auction is *greater* than that from a second-price auction.

On the other hand, if $v(x, y) = x^4$, we have that for $x^* \simeq 0.707$:

$$\begin{aligned} e_A(x) > e_2(x) \quad &\text{if } x < x^* \\ e_A(x) < e_2(x) \quad &\text{if } x > x^* \end{aligned}$$

and furthermore:

$$e_A = \int_0^1 e_A(x) f_X(x)\, dx < \int_0^1 e_2(x) f_X(x)\, dx = e_2$$

so that, in this case, the expected revenue from an all-pay auction is *less* than that from a second-price auction.

5.5 Summary of Revenue Comparisons

The relationships between the expected revenues from the various auction forms may be summarized as follows:

$$\begin{array}{ccccc} & & e_2 & & \\ & \leq & & \leq_\varphi & \\ e_1 & & \vee\wedge & & e_W \\ & \leq_\psi & & \leq_\varphi & \\ & & e_A & & \end{array}$$

where e_M is the expected revenue from the auction mechanism "M" and the symbols " $\leq_\varphi$ " and " $\leq_\psi$ " mean, respectively, that the inequality holds when φ or ψ is increasing. The result that $e_1 \leq e_2$ is, of course, due to Milgrom and Weber [13].

6 On the Uniqueness of Equilibrium

In this section, we examine whether the equilibrium in the war of attrition which we previously analyzed, is unique in the class of increasing equilibrium strategies. We then examine the same question for the all-pay auction. For both the war of attrition and the all-pay auction we provide sufficient conditions to guarantee the uniqueness of the respective equilibria of Theorems 1 and 2.

Let $supp\, f_X \subseteq \Re$ denote the common support of the bidders' signals, and let $\underline{x} \in \Re$ be the infimum of $supp\, f_X$ and $\overline{x}$ the supremum of $supp\, f_X$. We allow for the possibility that $\overline{x} = \infty$; thus, we assume that either (i) $supp\, f_X = [\underline{x}, \infty)$ or (ii) $supp\, f_X = [\underline{x}, \overline{x}]$.

Throughout this section, we assume that there are only two bidders. Bidder 1 receives a signal of x and bidder 2 of y. The value of the object to bidder 1 is then $v(x, y)$. It is assumed that v is continuously differentiable in its arguments.

The symmetric density f of the signals is assumed to be everywhere positive on its support and continuously differentiable on the interior of the support. Finally, we assume that the strategies employed by each bidder, β_1 and β_2 are strictly increasing.

6.1 The War of Attrition

It is well known that the war of attrition is plagued with a multiplicity of equilibria (see Riley [18] and Nalebuff and Riley [16]). Thus it is not surprising that we need strong conditions to guarantee uniqueness. Our main result is:

Theorem 6 *Suppose* $\varphi(\cdot, y)$ *is increasing and* $v(\underline{x}, \underline{x}) > 0$. *Then the symmetric equilibrium* (β, β) *from Theorem 1 is the unique increasing equilibrium of the two bidder war of attrition.*

Proof. It is routine to verify that if β_1 and β_2 form an equilibrium, then for all x and y :

$$\begin{aligned} \beta_2'\left(\beta_2^{-1}(\beta_1(x))\right) &= v\left(x, \beta_2^{-1}(\beta_1(x))\right)\lambda\left(\beta_2^{-1}(\beta_1(x))|x\right) \\ \beta_1'\left(\beta_1^{-1}(\beta_2(y))\right) &= v\left(y, \beta_1^{-1}(\beta_2(y))\right)\lambda\left(\beta_1^{-1}(\beta_2(y))|y\right) \end{aligned} \tag{21}$$

By Lemmas 7 and 8 in Appendix B, we know that the equilibrium bidding strategies must be continuous and have a common range.

Define a function $y : supp\, f_X \to supp\, f_X$ by:

$$y(x) = \beta_2^{-1}\left(\beta_1(x)\right). \tag{22}$$

Given a signal x for player 1, $y(x)$ is the unique signal for player 2 which induces him to bid $\beta_1(x)$ also.

Using (22) in (21) results in:

$$\begin{aligned} \beta_2'\left(y\left(x\right)\right) &= v\left(x, y\left(x\right)\right)\lambda\left(y\left(x\right)|x\right) \\ \beta_1'\left(x\right) &= v\left(y\left(x\right), x\right)\lambda\left(x|y\left(x\right)\right) \end{aligned} \tag{23}$$

Differentiating (22) with respect to x yields:

$$y'(x) = \frac{\beta_1'(x)}{\beta_2'(y(x))}. \tag{24}$$

Substituting from (23) into (24) yields the ordinary differential equation:

$$y' = \frac{v\left(y, x\right)\lambda(x|y)}{v\left(x, y\right)\lambda(y|x)} \tag{25}$$

We may write this differential equation as $y'(x) = g\left(x, y\right)$.

From Lemma 9, we get the boundary condition, $y(\underline{x}) = \underline{x}$, and since $f(\underline{x}|\underline{x}) > 0$, $\lambda(\underline{x}|\underline{x}) > 0$. By assumption $v(\underline{x}, \underline{x}) > 0$. Thus (25) is well defined at the boundary point $\underline{x}$. (Indeed $g(\underline{x}, \underline{x}) = 1$.)

Since both $v(\cdot,\cdot)$ and $f(\cdot,\cdot)$ are continuously differentiable in all their arguments, g is Lipschitz in y.

Thus, by Theorem 1 in Chapter 15 of Hirsch and Smale [9], we know that the solution to the differential equation (25) is locally unique. By standard techniques the local solution may be extended (see Hirsch and Smale [9] Chapter 8, Section 5).

Since $y(x) = x$ solves the differential equation (25), it is the unique solution and hence the symmetric equilibrium (β, β) is the unique equilibrium. □

To illustrate the workings of Theorem 6 it is useful to consider an example due to Riley [18].

Suppose that $v(x, y) = x$, and X and Y are independently and identically distributed on $[0, \infty)$ with the density function $f_X(x) = e^{-x}$. Then $\lambda(x) = 1$ for all x, and the differential equation (25) becomes:

$$y' = \frac{y}{x} \tag{26}$$

together with the boundary condition that $y(0) = 0$.

Observe that for all $k > 0$, $y(x) = kx$ is a solution to (26) and $y(0) = 0$. We have a continuum of solutions because the right hand side of (26) is not continuously differentiable in x (or even defined) at the point $(0, 0)$, and thus, the conditions that guarantee a unique solution are not satisfied. For each $k > 0$, the following increasing strategies form an asymmetric equilibrium:

$$\begin{aligned} \beta_1(x) &= \tfrac{k}{2}x^2 \\ \beta_2(y) &= \tfrac{1}{2k}y^2 \end{aligned}$$

In terms of Theorem 6 the example fails to satisfy the condition that $v(\underline{x}, \underline{x}) > 0$.

Consider a modification of the example so that, for $\underline{x} > 0$, X and Y are independently and identically distributed on $[\underline{x}, \infty)$ with the density function $f_X(x) = e^{-(x-\underline{x})}$. Again $\lambda(x) = 1$ for all x and the differential equation (25) remains:

$$y' = \frac{y}{x} \tag{27}$$

together with a new boundary condition $y(\underline{x}) = \underline{x}$.

Now, however, $y(x) = x$ is the unique solution to (27) , and thus the symmetric equilibrium is the unique equilibrium with increasing strategies.

Consequently, we have that if $\underline{x} > 0$, there is a unique equilibrium in the above example and multiplicity arises only when $\underline{x} = 0$.

Nalebuff and Riley [16] have shown that in the war of attrition there is an abundance of equilibria that are non-decreasing. In particular, these have the character that one player concedes to the other by bidding 0 when his signal is in some interval $[\underline{x}, x^*]$. Of course, such equilibria are inefficient in the sense that the auction may be won by the bidder who attaches a lower value to the object.

6.2 The All-Pay Auction

In this subsection we provide sufficient conditions for the symmetric equilibrium of Section 4 to be the unique equilibrium of the two bidder all-pay auction. Instead of providing detailed proofs which are very similar to the proofs in the case of the war of attrition, we indicate the arguments informally.

Suppose α_1 and α_2 are equilibrium strategies in the all-pay auction and that these are both increasing.

First, by following arguments similar to those in Lemmas 6, 7, and 8 in Appendix B, we can establish that α_1 and α_2 are both continuous and have identical ranges.

Next, the first order conditions for an all-pay auction imply that for all x and y:

$$\begin{aligned} \alpha_2'\left(\alpha_2^{-1}(\alpha_1(x))\right) &= v\left(x, \alpha_2^{-1}(\alpha_1(x))\right) f\left(\alpha_2^{-1}(\alpha_1(x))|x\right) \\ \alpha_1'\left(\alpha_1^{-1}(\alpha_2(y))\right) &= v\left(y, \alpha_1^{-1}(\alpha_2(y))\right) f\left(\alpha_1^{-1}(\alpha_2(y))|y\right) \end{aligned} \tag{28}$$

As in the proof of Theorem 6 define:

$$y(x) = \alpha_2^{-1}(\alpha_1(x)) \tag{29}$$

And as before this results in a differential equation:

$$y' = \frac{v(y,x) f(x|y)}{v(x,y) f(y|x)} \tag{30}$$

which is analogous to (25).

Applying the uniqueness result once again (Theorem 1 in Chapter 15 in [9]) we obtain:

Theorem 7 *Suppose that* $\psi(\cdot, y)$ *is increasing and either* (i) $v(\underline{x}, \underline{x}) > 0$ *or that* (ii) *supp* $f_X = [\underline{x}, \overline{x}]$. *Then the symmetric equilibrium* (α, α) *from Theorem 2 is the unique increasing equilibrium of the two bidder all-pay auction.*[7]

[7] Amann and Leininger [1] show that there is a unique increasing equilibrium in the case when X and Y are independently, but not necessarily symmetrically, distributed on $[0, 1]$. Theorem 7 specializes this to the symmetric case but allows for affiliation and, if (i) holds, an infinite support.

Proof. In case (i), the boundary condition $y(\underline{x}) = \underline{x}$ may be used to show that (30) has a unique solution. In case (ii), the boundary condition $y(\overline{x}) = \overline{x}$ may be used. Recall that $f(\underline{x}|\underline{x})$ and $f(\overline{x}|\overline{x})$ are both assumed to be positive. □

The reader may wonder why condition (ii) in the above Theorem does not suffice for the conclusion of Theorem 6. The reason is that if $supp\, f_X = [\underline{x}, \overline{x}]$ then it must be that $\lambda(\overline{x}|\overline{x}) = \infty$ and so the right hand side of (25) is not defined at $y = x = \overline{x}$.

7 Conclusion

We have identified conditions under which the war of attrition and the all-pay auction generate higher expected revenue than their standard counterparts: the second-price and first-price sealed bid auctions. These conditions guarantee that the equilibrium strategies are increasing in the signals received by the bidders; and implicitly require that the affiliation between bidders' signals is not too strong.

If bidders' signals are strongly affiliated the possibility of non-increasing equilibria is not implausible. Suppose there is an increasing equilibrium. In such a situation, bidder 1, say, who receives a higher signal would consider it very likely that other bidders also received higher signals. Hence, the reduction in his conditional probability of winning the auction would overwhelm the increase in his expected value from receiving the higher signal. This would induce the bidder to lower rather than raise his bid because his prospects of losing the auction have increased, and, in these circumstances, he would be required to pay his bid. This destroys the increasing equilibrium. Under the presumption that there is a symmetric, pure-strategy equilibrium, such an equilibrium must be non-increasing. The characterization of symmetric equilibrium strategies in the two auctions when there is no symmetric increasing equilibrium remains an open problem.

8 Appendix A: Affiliation

In this Appendix we collect some miscellaneous results on affiliated random variables that are used to derive the results of this paper.

Definition 4 *Suppose the random variables X and Y have a joint density $f : \Re^2 \to \Re$. X and Y are said to be* affiliated *if for all $x' \geq x$ and $y' \geq y$,*

$$f(x', y)f(x, y') \leq f(x, y)f(x', y').$$

Let X and Y have a joint density of f and let $F_Y(\cdot \mid x)$ denote the conditional distribution of Y given $X = x$.

Lemma 1 *Suppose X and Y are affiliated. Then $F_Y(y \mid x)/f_Y(y \mid x)$ is non-increasing in x.*

Proof. See Milgrom and Weber [13] (page 1107). □

Definition 5 *The* hazard rate *of the distribution $F_Y(\cdot \mid x)$ is defined as:*

$$\lambda(y \mid x) = \frac{f_Y(y \mid x)}{1 - F_Y(y \mid x)}.$$

Lemma 2 *Suppose X and Y are affiliated. Then $\lambda(y \mid x)$ is non-increasing in x.*

Proof. Suppose $x' \geq x$. By the affiliation inequality, for any $t \geq y$ we have:

$$f(x, t)\, f(x', y) \quad \leq \quad f(x, y)\, f(x', t)$$

which can be rearranged as:

$$\frac{f(x, t)}{f(x, y)} \leq \frac{f(x', t)}{f(x', y)}$$

or:

$$\frac{f(t \mid x) f(x)}{f(y \mid x) f(x)} \leq \frac{f(t \mid x') f(x')}{f(y \mid x') f(x')}$$

which is the same as:

$$\frac{f_Y(t \mid x)}{f_Y(y \mid x)} \leq \frac{f_Y(t \mid x')}{f_Y(y \mid x')}$$

Integrating with respect to t over the range $[y, \infty)$ yields:

$$\frac{1 - F_Y(y \mid x)}{f_Y(y \mid x)} \leq \frac{1 - F_Y(y \mid x')}{f_Y(y \mid x')}$$

which is the same as:

$$\lambda(y \mid x) \geq \lambda(y \mid x').$$

□

Lemma 3 *Suppose X and Y are affiliated. Then $F_Y(y \mid x)$ is non-increasing in x.*

Proof. Notice that:

$$\frac{F_Y(y \mid x)}{1 - F_Y(y \mid x)} = \left[\frac{F_Y(y \mid x)}{f_Y(y \mid x)}\right] \times \left[\frac{f_Y(y \mid x)}{1 - F_Y(y \mid x)}\right]$$

is a non-increasing function of x since both terms in the brackets are positive and non-increasing by Lemmas 1 and 2. Thus $F_Y(y \mid x)$ is also non-increasing in x. □

Lemma 4 *Suppose X and Y are affiliated. Then for all $y < x$:*

$$\exp\left(-\int_y^x \frac{f_Y(t \mid t)}{F_Y(t \mid t)} dt\right) \leq \left[\frac{F_Y(y \mid y)}{F_Y(x \mid x)}\right]$$

Proof. Note that:

$$\begin{aligned} -\int_y^x \frac{f_Y(t \mid t)}{F_Y(t \mid t)} dt &\leq -\int_y^x \frac{f_Y(t \mid y)}{F_Y(t \mid y)} dt \\ &= \ln F_Y(y \mid y) - \ln F_Y(x \mid y) \\ &\leq \ln F_Y(y \mid y) - \ln F_Y(x \mid x) \end{aligned}$$

where the first inequality follows from the fact that $[f_Y(t \mid \cdot)/F_Y(t \mid \cdot)]$ is a non-decreasing function; (Lemma 1) and the second inequality follow from the fact that $F_Y(x \mid \cdot)$ is non-increasing. (Lemma 3). □

9 Appendix B: Lemmas for Section 6

In this appendix we establish that if (β_1, β_2) is an equilibrium of the two bidder war of attrition and both β_1 and β_2 are increasing, then *Range* β_1 = *Range* β_2. This allows the construction of the function $y : supp\, f_X \to supp\, f_X$ used in the proof of Theorem 6.

Lemma 5 *Suppose $\varphi(\cdot, y)$ is increasing. Then*[8]

$$\lim_{x \to \bar{x}} \beta_1(x) = \lim_{y \to \bar{x}} \beta_2(y) = \infty.$$

[8]Nalebuff and Riley [16] provide a similar result for the case when X and Y are independent. We have adapted their proof to the affiliated case.

Proof. Suppose not, then $\lim_{y\to\bar{x}} \beta_2(y) \to b^* < \infty$. To rule this out, consider a subinterval $[\beta_2(y'), \beta_2(y'')]$ of $(0, b^*)$ with associated signals y satisfying $y' < y < y''$. From the necessary conditions for an equilibrium, for all $y \in (y', y'')$, we have

$$\begin{aligned}
1 &= \frac{v\left(y,\beta_1^{-1}(\beta_2(y))\right)\lambda\left(\beta_1^{-1}(\beta_2(y))|y\right)}{\beta_1'\left(\beta_1^{-1}(\beta_2(y))\right)} \\
&> \frac{v\left(y',\beta_1^{-1}(\beta_2(y))\right)\lambda\left(\beta_1^{-1}(\beta_2(y))|y'\right)}{\beta_1'\left(\beta_1^{-1}(\beta_2(y))\right)} \\
&> \frac{v\left(y',\beta_1^{-1}(\beta_2(y'))\right)\lambda\left(\beta_1^{-1}(\beta_2(y))|y'\right)}{\beta_1'\left(\beta_1^{-1}(\beta_2(y))\right)}
\end{aligned}$$

where the first inequality follows from the assumption that $\varphi\left(\cdot, \beta_1^{-1}(\beta_2(y))\right) = v\left(\cdot, \beta_1^{-1}(\beta_2(y))\right)\lambda\left(\beta_1^{-1}(\beta_2(y))|\cdot\right)$ is increasing. The second follows from the assumption that $v(y', \cdot)$, β_1 and β_2 are all increasing.

Integrating with respect to $\beta_2(y)$ over the range $[\beta_2(y'), \beta_2(y'')]$ yields

$$\beta_2(y'') - \beta_2(y') > v\left(y', \beta_1^{-1}(\beta_2(y'))\right) \ln\left[\frac{1 - F\left(\beta_1^{-1}(\beta_2(y'))|y'\right)}{1 - F\left(\beta_1^{-1}(\beta_2(y''))|y'\right)}\right]$$

where we have used the identity $\lambda(y \mid z) \equiv -\frac{d}{dy}\ln[1 - F(y \mid z)]$.

Now observe that, as $y'' \to \bar{x}$, $\beta_2(y'') \to b^*$ and $1 - F\left(\beta_1^{-1}(\beta_2(y''))|y'\right) \to 0$. Furthermore, $1 - F\left(\beta_1^{-1}(\beta_2(y'))|y'\right) > 0$. Thus, the right hand side of the above expression becomes unbounded.

The proof for β_1 is symmetric. □

Let *Range* $\beta_i = \{\beta_i(y) : z \in supp\, f_X\}$, and $cl[Range\ \beta_i]$ be its closure.

Lemma 6 $cl[Range\ \beta_1] = cl[Range\ \beta_2]$

Proof. We argue that $cl[Range\ \beta_1] \subseteq cl[Range\ \beta_2]$. Suppose not, that is, there exists a $b \in cl[Range\ \beta_1]$ such that $b \notin cl[Range\ \beta_2]$.

Case 1. $b > 0$

Then there exists $\epsilon > 0$ such that $(b - \epsilon, b + \epsilon) \notin cl[Range\ \beta_2]$.

Define $y^* = \sup\{y : \beta_2(y) < b\}$.

We assume without loss of generality that $b \in Range\ \beta_1$. This is because if $b \in cl[Range\ \beta_1]$, there exists a sequence $b^n \to b$ such that $b^n \in Range\ \beta_1$, and, for large n, there exists $\epsilon > 0$, $(b^n - \epsilon, b^n + \epsilon) \notin cl[Range\ \beta_2]$.

Suppose $b = \beta_1(x)$. If player 1 bids b after receiving a signal of x, his expected payoff is:

$$\Pi_1(b; x) = \int_{\underline{x}}^{y^*} (v(x, y) - \beta_2(y)) f(y|x) dy - [1 - F(y^* \mid x)]\, b \tag{31}$$

Alternatively, if player 1 bids $b - \epsilon$ after a signal of x his payoff is:

$$\Pi_1(b - \epsilon; x) = \int_{\underline{x}}^{y^*} (v(x, y) - \beta_2(y)) f(y|x) dy - [1 - F(y^* \mid x)]\,(b - \epsilon) \tag{32}$$

since player 2 never bids in the interval $(b - \epsilon, b)$.

From (31) and (32)

$$\Pi_1(b - \epsilon; x) - \Pi_1(b; x) = [1 - F(y^* \mid x)]\, \epsilon \tag{33}$$

By Lemma 5, $y^* < \overline{x}, F(y^* \mid x) < 1$. Now, from (33), it follows that $\Pi_1(b - \epsilon; x) > \Pi_1(b; x) = \Pi_1(\beta_1(x); x)$, contradicting the fact that β_1 and β_2 form an equilibrium.

Thus, we have established that $cl[Range\ \beta_1] \cap \Re_{++} = cl[Range\ \beta_1] \cap \Re_{++}$.

Case 2. $b = 0$, that is, $0 \in cl[Range\ \beta_1]$ and $0 \notin cl[Range\ \beta_2]$.

Once again we can assume that $0 \in Range\ \beta_1$. If not, then there exists a sequence $b^n \to 0$ such that $b^n \in Range\ \beta_1$ and $b^n > 0$. By *Case* 1, $b^n \in cl[Range\ \beta_2]$ and hence $0 \in cl[Range\ \beta_2]$.

Since $0 \notin cl[Range\ \beta_2]$, $\beta_2(\underline{x}) > 0$. By *Case* 1, $\beta_2(\underline{x}) \in cl[Range\ \beta_1]$. We claim that, for all $b' < \beta_2(\underline{x})$ such that $b' > 0$, $b' \notin cl[Range\ \beta_1]$. This follows from the fact that since β_2 is increasing, $b' \notin cl[Range\ \beta_2]$ and, hence, by *Case* 1, $b' \notin cl[Range\ \beta_1]$.

Thus, if player 2 were to bid a b' satisfying $0 < b' < \beta_2(\underline{x})$, when the signal is $\underline{x}$, he would win in exactly the same circumstances as with a bid of $\beta_2(\underline{x})$, and his expected payoff would be higher. Hence, $\Pi_2(\beta_2(\underline{x}); \underline{x}) < \Pi_2(b'; \underline{x})$ which is a contradiction. □

Lemma 7 *β_1 and β_2 are continuous in x and y, respectively.*

Proof. Suppose β_1 is discontinuous at x.

Case 1. Suppose $b' \equiv \lim_{\epsilon \to 0} \beta_1(x-\epsilon) < \beta_1(x)$.

Define $y^* = \sup\{y : \beta_2(y) < \beta_1(x)\}$.

Player 1's expected payoff when bidding $\beta_1(x)$ with a signal of x is:

$$\Pi_1\left(\beta_1(x); x\right) = \int_{\underline{x}}^{y^*} (v(x,y) - \beta_2(y)) f(y|x) dy - \left[1 - F(y^*|x)\right] \beta_1(x)$$

Suppose player 1 deviates by bidding b' instead. By Lemma 6, the closures of the ranges of the two bidding functions are equal, thus, a bid of b' wins whenever $y \leq y^*$. Hence, player 1's expected payoff becomes:

$$\Pi_1\left(b'; x\right) = \int_{\underline{x}}^{y^*} (v(x,y) - \beta_2(y)) f(y|x) dy - \left[1 - F(y^*|x)\right] b'$$

Again, from Lemma 5, $y^* < \overline{x}$ and thus, $F(y^*|x) < 1$. But now we have $\Pi_1\left(b'; x\right) > \Pi_1\left(\beta_1(x); x\right)$, which contradicts the fact that β_1 and β_2 constitute an equilibrium.

Case 2. Suppose $\beta_1(x) < \lim_{\epsilon \to 0} \beta_1(x+\epsilon) \equiv b''$.

Define $y^* = \sup\{y : \beta_2(y) < \beta_1(x)\}$ and $y^{**} = \sup\{y : \beta_2(y) < b''\}$. We claim that $y^* = y^{**}$. Clearly $y^* \leq y^{**}$, and if $y^* < y^{**}$, then there is a y such that $\beta_1(x) < \beta_2(y) < b''$. But this contradicts Lemma 6.

For small ϵ, define $y(\epsilon) = \sup\{y : \beta_2(y) < \beta_1(x+\epsilon)\}$. It may be verified that:

$$\lim_{\epsilon \to 0} y(\epsilon) = \lim_{\epsilon \to 0} \left[\sup\{y : \beta_2(y) < \beta_1(x+\epsilon)\}\right] = y^{**} = y^* \tag{34}$$

If player 1 bids $\beta_1(x+\epsilon)$ when his signal is $x+\epsilon$, his payoff is:

$$\begin{aligned}\Pi_1\left(\beta_1(x+\epsilon); x+\epsilon\right) &= \int_{\underline{x}}^{y(\epsilon)} (v(x+\epsilon,y) - \beta_2(y)) f(y|x+\epsilon) dy \\ &\quad - \left[1 - F(y(\epsilon)|x+\epsilon)\right] \beta_1(x+\epsilon)\end{aligned}$$

Suppose player 1 deviates by bidding $\beta_1(x)$ when the signal is $x+\epsilon$. By Lemma 6, the closures of the ranges of the two bidding functions are equal; thus, player 1's bid wins whenever $y \leq y^*$. Hence, player 1's expected payoff becomes:

$$\begin{aligned}\Pi_1\left(\beta_1(x); x+\epsilon\right) &= \int_{\underline{x}}^{y^*} (v(x+\epsilon,y) - \beta_2(y)) f(y|x+\epsilon) dy \\ &\quad - \left[1 - F(y^*|x+\epsilon)\right] \beta_1(x)\end{aligned}$$

The difference between the two is:

$$\begin{aligned}[\Pi_1(\beta_1(x);x+\epsilon) &- \Pi_1(\beta_1(x+\epsilon);x+\epsilon)] \\ &= -\int_{y^*}^{y(\epsilon)} (v(x+\epsilon,y)-\beta_2(y))f(y|x+\epsilon)dy \\ &\quad + \left[1-F(y(\epsilon)|x+\epsilon)\right]\beta_1(x+\epsilon) \\ &\quad - \left[1-F(y^*|x+\epsilon)\right]\beta_1(x)\end{aligned}$$

Since $\lim_{\epsilon\to 0} y(\epsilon) = y^*$, taking limits results in:

$$\begin{aligned}\lim_{\epsilon\to 0}[\Pi_1(\beta_1(x);x+\epsilon) &- \Pi_1(\beta_1(x+\epsilon);x+\epsilon)] \\ &= \left[1-F(y^*|x)\right]\left(b''-\beta_1(x)\right) \\ &> 0\end{aligned}$$

Again, from Lemma 5, $y^* < \overline{x}$ and thus, $F(y^*|x) < 1$. Hence, for some small ϵ, $\Pi_1(\beta_1(x);x+\epsilon) > \Pi_1(\beta_1(x+\epsilon);x+\epsilon)$, which again contradicts the fact that β_1 and β_2 constitute an equilibrium. □

From Lemmas 6 and 7, the following is immediate.

Lemma 8 *Range* β_1 = *Range* β_2.

Lemma 9 $\beta_1(\underline{x}) = \beta_2(\underline{x}) = 0$.

Proof. The fact that $\beta_1(\underline{x}) = \beta_2(\underline{x})$ follows from Lemma 8.

If $\beta_1(\underline{x}) = \beta_2(\underline{x}) > 0$, then player 1's expected payoff when his signal is $\underline{x}$ is negative since he always loses and pays $\beta_1(\underline{x})$. But then, a bid of 0 when his signal is $\underline{x}$ is a profitable deviation. □

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