Loss Aversion in a Multi-Period Model [1]

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Abstract

An individual faces a choice between streams of outcomes in several periods in the future. This paper examines an axiomatization of preference relations over these streams that includes preferences which do not satisfy temporal monotonicity, and which leads to a simple functional representation of these preferences. Motivated by the loss-aversion aspects of Tversky & Kahneman's prospect theory, the axioms lead to a representation that takes into account not only the utility of the per-period outcomes (instantaneous payoffs,) but also the differences between the utility of pairs of adjacent outcomes, and the direction of the differences (gains or losses). In this framework, loss aversion is defined and characterized.

Keywords: loss-aversion, utility, multi-period decisions.

1 Introduction

In many situations a decision-maker must choose an alternative out of a choice set, and the chosen element determines the outcome in several future periods. We call these situations multi-period problems. Examples of multi-period problems abound in many areas, such as investment, production, planning etc. Actually, single-period problems are the exception rather than the rule, since nearly every choice we make affects not only the present, but also the future.

In much of the economic literature, multi-period preferences are presented in the following manner. There exists an instantaneous utility function u on the per-period outcomes, which is independent of time, called the per-period or instantaneous payoff function, and a long-run function, U, such that each alternative $f = (f_1, f_2, \ldots)$ is assessed by $U(u(f_1), u(f_2), \ldots)$. In most of the cases in which U's functional form is specified, it is assumed to be a weighted average $\sum_i p_i u(f_i)$.

Savage (1954) defines an *act* as a function from the *states of the world* into the set of consequences. Interpreting the states of the world as points of time, an act is equivalent to a multi-period outcome, and the problems dealt with by Savage are equivalent to multi-period decision problems. In Savage, and also in Anscombe and Aumann (1963) (where acts are functions from states of the world into *lotteries* over consequences), axiomatic foundations are provided which lead to preferences represented by expected utility, where the relevant functional U is a weighted average of $\{u(f_t)\}_{t \in T}$, where f is an act[1], and T is the domain on which acts are defined.

While this form is mathematically convenient, there are many cases where its justification, particularly under the temporal interpretation of Savage's model, can be questioned. The inadequacy of the assumption of expected utility over periods of time has been shown in many experiments. Decision-makers remember (and are affected by) past results when evaluating present and future outcomes. This phenomenon is known as reference dependence. The human tendency to react primarily to changes was identified as early as the fifth Century BC, when the Greek philosopher Democritus wrote that

[1] To conform with the notation of Savage, we call multi-period outcomes acts.

> In reality we apprehend nothing for certain, but only as it changes according to the condition of our body and of the things that impinge upon or offer resistance to it. [Kirk and Raven (1957, p. 422)].

In more recent research, Kahneman and Thaler (1991) state that many studies support the proposition that comparison (of income) to one's past determines the standard of satisfaction with income.

Most of the literature on reference-dependence deals with the case of two periods – the reference point (in the past) and the present outcome. Utility functions incorporating reference-dependence usually include intertemporal changes in utility (differences between the utility level in the reference period and the level in a later period). Reference-dependent utility models can be found in Kahneman and Tversky (1979, 1984, 1991), Kahneman *et al* (1990, 1991) and Tversky and Griffin (1991). These references deal with reference points that are not necessarily time dependent, but cover the more general case where an alternative consists of a single reference point and one outcome. A number of these references, especially Kahneman *et al* (1991) and Kahneman and Tversky (1979 and 1991), are concerned with *loss-aversion*, a special case of reference dependence where decreases or negative changes in the level of utility (henceforth losses) are given harsher treatment than increases or positive changes (gains) of the same magnitude. In a related model, Benartzi and Thaler (1995) use loss-aversion to explain the equity premium paradox (the empirical fact that stocks have greatly outperformed bonds over the last century), by assuming a period length of one year, and that losses affect decisions twice as strongly as gains of the same magnitude.

Extending the number of periods in the model beyond two, Gilboa (1989) presented a *multi-period* reference-dependent model and found that under certain assumptions, multi-period decision rules are representable by a weighted average of the utility in each period, *and* the utility variation between each two consecutive periods. Our model is based on Gilboa's model, with a number of differences. The most fundamental of these is allowing preferences which do not satisfy temporal monotonicity.

Temporal monotonicity states that given two multi-period outcomes, if in every single period the outcome of the first act is preferred to the outcome of the second act, then overall the first act will be preferred to the second one. For many situations, temporal monotonicity is valid. However, it has been shown that there are situations where the differences between periods—gains or losses—are given greater weight in decision-making than the outcomes themselves.

The psychological literature contains many examples where non-monotonic preferences over series of outcomes have been found. Hsee and Abelson (1991) questioned subjects on their preferences over streams of (a) salaries, and (b) percentile rankings in class. They found in both cases that a significant majority of subjects preferred a rising outcome to a constant high outcome (the final outcome was the same in both cases), and more chose a constant low outcome than a falling outcome ending with the same low outcome[2]. In a study investigating factors that influence the relative weighting of positions (i.e. the actual values of outcomes) and velocity (i.e. the changes over time of outcomes), Hsee *et al* (1991) showed that the framing of the outcome in terms of changes caused the relative weight of velocity to loom large. Kahneman *et al* (1993) found violations of temporal monotonicity in experiments where subjects immersed their hands in painfully cold water. They found that adding some reduced pain after more intense pain was preferred to not adding the extra pain. In studies documented in Loewenstein and Prelec (1991), researchers found an appparently negative rate of time preference for choices among outcome sequences. Such preferences could violate temporal monotonicity. For example, Loewenstein and Sicherman (1991) found that a majority of museum visitors preferred increasing wage profiles over those that are flat or decline over time (holding total value constant). Pointing out that the flat and declining wage profiles could produce a *dominating* consumption stream through a suitable savings program did not have much impact on preference. Even in the atemporal setting of risk and uncertainty, Wakker (1993) showed that technically, Savage's axioms do not imply strict stochastic dominance. Instead, they usually involve violations of it, as demonstrated by Example 4 in Wakker's paper.

In the spirit of these findings, we present a model allowing non-monotonic

[2]The percentage of subjects indicating non-monotonic preferences was between 67% and 83%.

preferences—the importance of changes in the level of utility can be greater than that of the level of the instantaneous payoffs. We state a set of axioms which does not include monotonicity, and derive a functional representation of the decision-maker's decision rule from these axioms. This function is characterized by the way it takes into consideration interperiod changes of utility, dealing separately with gains and losses. This presentation enables the preferences to be presented almost solely through the interperiod differences, and enables identification of loss-aversion by comparison of the weights given to gains with those given to losses.

To avoid misunderstandings, the following should be emphasized: We are not suggesting that non-monotonic preferences are necessarily common or usual. Indeed, some decision makers will have monotonic preferences in any situation and in some situations every decision maker might have monotonic preferences. Our contribution is *allowing* non-monotonic preferences for some decision makers in some situations.

Section 2 presents the model and the main theorem. Section 3 contains the proof of the main theorem. In Section 4 we define and characterize loss aversion. Section 5 contains an extension of the model to an infinite horizon and a corresponding representation theorem, and we conclude with some remarks in Section 6.

2 The Model

We now describe the model, which follows the model in Gilboa (1989), and state the main theorem.

Let X be a non-empty set of *consequences* and let Y denote the set of finite-support distributions over X ("lotteries"), which are the per-period outcomes. Let $T = \{1, 2, \ldots, n\}$ be the (nonempty) set of points of time.[3] We define F, the set of *acts*, to be the set of functions from T to Y. The lottery given by the act f at period t is denoted by $f(t)$ or f_t. A binary (preference) relation $\succeq$ is given on F. Given $\succeq \ \subseteq F \times F$ we define $\succeq \ \subseteq Y \times Y$ by identifying a lottery y in Y with the constant act which as-

[3]Section 5 will extend the model to $T = \{1, 2, \ldots\}$.

sumes the value of y over all T. We denote this constant act by $\overline{y}$ when necessary to differentiate from the single-period outcome y. As usual, $\succ$ and $\sim$ are defined to be the asymmetric and symmetric parts of $\succeq$, respectively. Linear operations are performed on F pointwise.

Definition: Two acts f and g are *sequentially comonotonic (SC)* if for no two adjacent periods s, t it is true that $f_s \succ f_t$ and $g_t \succ g_s$.

The definition of SC differs from comonotonicity as defined in Schmeidler (1989) by using the inherent sequential ordering of time, and comparing only adjacent pairs of periods, rather than every pair of periods.

2.1 The Axioms

Our axioms are:

WO *Weak Ordering*: $\succeq$ is complete, i.e. for all $f, g \in F$, $f \succeq g$ or $g \succeq f$ (or both), and transitive.

SCI *Sequential Comonotonic Independence*: If for $f, g, h \in F$ both f, h and g, h are sequentially comonotonic and $\alpha \in (0,1)$, then

$$f \succeq g \iff \alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h. \tag{1}$$

In this weakening of Anscombe and Aumann's (1963) independence axiom we introduce two elements - the order of time periods and the separation of gains from losses. We do not require independence to hold for mixtures of non-SC acts. Thus, SCI is weaker than full-force independence. The SCI axiom is stronger than the axiom of comonotonic independence from Schmeidler (1989), which is used also by Gilboa (1989). Two other axioms whose strength is between independence and comonotonic independence are optimism independence and pessimism independence as defined in Wakker(1990). Neither of these is directly comparable in strength to SCI. Gilboa (1989) incorporates the sequentiality of time by a modification of Savage's sure-thing principle which he calls the variation-preserving sure-thing principle. Combining comonotonic independence and the variation-preserving sure-thing principle gives us the essence of SCI.

CONT *Continuity*: If $f, g, h \in F$ satisfy $f \succ g \succ h$, then there are $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.

CND *Constant Nondegeneracy*: There are $y, z \in Y$ such that $\overline{y} \succ \overline{z}$.

For one thing, this axiom implies that changes in utility levels are not the only element determining utility.

SUBST *Substitutability*: For any $f, g \in F$, if $\forall t \in T, f(t) \sim g(t)$ then $f \sim g$.

In our opinion, this axiom is the most questionable. It implies for instance that if one is indifferent between receiving a cup of coffee every day and receiving a cup of tea every day, then *any* stream of cups of coffee and tea is also indifferent to these two streams. However, this axiom is weaker than monotonicity, as assumed in the models of Schmeidler (1989) and Gilboa (1989), with Gilboa's model being equivalent in terms of dealing with periods of time.

2.2 The Main Theorem

Note: the notation $(x)^-$ designates $\min\{0, x\}$; the notation $(x)^+$ designates $\max\{0, x\}$.

Theorem 1 *$\succeq$ satisfies WO, SCI, CONT, CND and SUBST if and only if there exists an affine*[4] *utility function $u : Y \longrightarrow I\!R$ and functions $\delta^+, \delta^- : \{2, \ldots, n\} \longrightarrow I\!R$, such that*

$$U(f) = u(f_1) + \sum_{t=2}^{n} \left[\delta^+(t)\left(u(f_t) - u(f_{t-1})\right)^+ + \delta^-(t)\left(u(f_t) - u(f_{t-1})\right)^-\right]$$

represents $\succeq$ on F. Moreover, if the axioms are satisfied then u is unique up to a positive linear transformation and δ^+, δ^- are unique.

[4]An affine utility function on a mixture space satisfies $u(\alpha y_1 + (1-\alpha)y_2) = \alpha u(y_1) + (1-\alpha)u(y_2)$ for all $y_1, y_2 \in Y$ and $\alpha \in [0,1]$.

For brevity, we will abbreviate $\delta^+(t)$ by δ_t^+ and $\delta^-(t)$ by δ_t^-.

3 Proof of the main theorem

3.1 Overview of the proof

The proof of the main theorem consists of four parts. In Subsection 3.2 we derive the functions $u : Y \longrightarrow I\!R$ and $U : F \longrightarrow I\!R$. In Subsection 3.3 we show that U represents $\succeq$ on F. Subsection 3.4 extends the model so that for every act in F there exists a constant act with the same utility. We show that this extension is consistent with our model. In the final part of the proof, given in Subsection 3.5, we show the affinity of U over sequentially comonotonic acts and use this result to conclude the proof.

3.2 Deriving the utility functions

From the implication of the vN-M theorem (see Schmeidler (1989)), there exists a vN-M utility $u : Y \longrightarrow I\!R$ representing the preference relation $\succ$ induces on Y, which is affine, and unique up to a positive affine transformation.

From CND, there exist $y_+, y_- \in Y$ with $\overline{y_+} \succ \overline{y_-}$. Since u as defined above is given up to a positive affine transformation, choose u such that $u(y_+) = 1$ and $u(y_-) = -1$.

From the affinity of u, $u(\frac{1}{2}y_+ + \frac{1}{2}y_-) = 0$. Denote $y_0 = (\frac{1}{2}y_+ + \frac{1}{2}y_-)$.

For any $f \in F$, define

$$M_f = \{\alpha f + (1-\alpha)\overline{y} \mid y \in Y \text{ and } \alpha \in [0,1]\},$$

thus M_f is the convex hull of the union of f and the constant acts. It is easy to see that for any $f \in F$, any two acts in M_f are comonotonic.

Using the implication of the vN-M theorem again, there is an affine function $U_f : M_f \longrightarrow I\!R$, which represents $\succeq$ restricted to M_f. Since this function is given up to a positive affine transformation, we can choose U_f so that $U_f(y_+) = 1$ and $U_f(y_-) = -1$, and therefore $U_f(\overline{y}) = u(y)$ for all $y \in Y$.

Define $U : F \longrightarrow I\!R$ by $U(f) = U_f(f)$.

3.3 Representation of the preference relation

The following three lemmas prove that U represents $\succ$ on F. The rest of the proof will show that U has the functional form stated in the theorem.

Lemma 1 *If $f \in M_g$ then $U_g(f) = U_f(f) = U(f)$.*

1. If $f \sim \overline{y_+}$ or $f \sim \overline{y_0}$ then the proof is trivial.

2. If $\overline{y_+} \succ f \succ \overline{y_0}$ then from CONT $\exists \alpha \in (0,1)$ such that[5]

$$\alpha \cdot \overline{y_+} + (1-\alpha) \cdot \overline{y_0} \sim f.$$

Therefore, since U_g represents $\succeq$ on M_g,

$$U_g(\alpha \cdot \overline{y_+} + (1-\alpha) \cdot \overline{y_0}) = U_g(f).$$

Affinity of U_g allows us to conclude that

$$\alpha \cdot U_g(\overline{y_+}) + (1-\alpha) \cdot U_g(\overline{y_0}) = U_g(f),$$

therefore,

$$U_g(f) = \alpha.$$

Similarly, since U_f represents $\succeq$ on M_f and is affine, we derive that $U_f(f) = \alpha$, and therefore

$$\alpha = U_g(f) = U_f(f) = U(f).$$

3. If $f \succ \overline{y_+}$ then from CONT, $\exists \alpha \in (0,1)$ such that

$$\alpha \cdot f + (1-\alpha) \cdot \overline{y_0} \sim \overline{y_+}.$$

Using the same method as in Step 2 we derive that

$$\frac{1}{\alpha} = U_g(f) = U_f(f) = U(f).$$

[5] The existence of such an α is implied by CONT, as $\alpha_0 = inf\{\alpha \in (0,1) \mid \alpha \cdot \overline{y_+} + (1-\alpha) \cdot \overline{y_0} \succ f\}$ exists, and if $\alpha_0 \cdot \overline{y_+} + (1-\alpha_0) \cdot \overline{y_0} \succ f$, we have a contradiction to α_0 being the infimum as defined.

4. If $f \prec \overline{y_0}$ then from CONT, $\exists \alpha \in (0,1)$ such that

$$\alpha \cdot \overline{y_+} + (1-\alpha) \cdot f \sim \overline{y_0},$$

and similarly to Step 2 we derive that

$$\frac{-\alpha}{1-\alpha} = U_g(f) = U_f(f) = U(f).$$

■(Lemma 1)

Lemma 2 *If $f, g \in F$ are sequentially comonotonic and $1 \geq \alpha > \beta \geq 0$, then $f \succ g \iff \alpha \cdot f + (1-\alpha) \cdot g \succ \beta \cdot f + (1-\beta) \cdot g$.*

Proof: The following five statements are equivalent:

$$\alpha \cdot f + (1-\alpha) \cdot g \succ \beta \cdot f + (1-\beta) \cdot g \tag{2}$$

$$\beta \cdot f + (1-\beta) \cdot \left[\frac{\alpha-\beta}{1-\beta} \cdot f + \frac{1-\alpha}{1-\beta} \cdot g\right] \succ \beta \cdot f + (1-\beta) \cdot g \tag{3}$$

$$\frac{\alpha-\beta}{1-\beta} \cdot f + \frac{1-\alpha}{1-\beta} \cdot g \succ g \tag{4}$$

$$\frac{\alpha-\beta}{1-\beta} \cdot f + \frac{1-\alpha}{1-\beta} \cdot g \succ \frac{\alpha-\beta}{1-\beta} \cdot g + \frac{1-\alpha}{1-\beta} \cdot g \tag{5}$$

$$f \succ g \tag{6}$$

(2) and (3) are equivalent since $\alpha > \beta$, (3) and (4) are equivalent from SCI, and the same holds for (5) and (6).

■(Lemma 2)

Corollary 1 *If $f \succ g$ and $\alpha \cdot f + (1-\alpha) \cdot g \succ \beta \cdot f + (1-\beta) \cdot g$ and $\alpha, \beta \in [0,1]$ then $\alpha > \beta$.*

Lemma 3 *U represents $\succ$ on F, i.e.*

$$\forall f, g \in F,\ f \succ g \iff U(f) > U(g).$$

Proof:

1. $\forall f \in F$,

$$f \succ \overline{y_+} \iff U(f) > 1,$$

$$f \prec \overline{y_0} \iff U(f) < 0,$$

and

$$\overline{y_0} \preceq f \preceq \overline{y_+} \iff 0 \leq U(f) \leq 1.$$

2. If $U(f)$ and $U(g)$ are on different sides of 0 or 1, then transitivity and Step 1 complete the proof. If they are on the same side of 0 and 1, there are three cases to be dealt with.

3. If $\overline{y_+} \succeq f \succeq \overline{y_0}$ and $\overline{y_+} \succeq g \succeq \overline{y_0}$, then from CONT there exist $\alpha, \beta \in [0,1]$ such that

$$\begin{aligned} f &\sim \alpha \cdot \overline{y_+} + (1-\alpha) \cdot \overline{y_0} \\ g &\sim \beta \cdot \overline{y_+} + (1-\beta) \cdot \overline{y_0} \end{aligned}$$

and from the affinity of U_f and U_g, $U(f) = \alpha$ and $U(g) = \beta$. Therefore, $U(f) > U(g) \iff \alpha > \beta \iff f \succ g$, where the last equivalence is from Lemma 2 and Corollary 1 ($\overline{y_+}$ and $\overline{y_0}$ play the parts of f and g respectively in the lemma and the corollary).

4. If $f \succeq \overline{y_+}$, $g \succeq \overline{y_+}$, then from CONT there exist $\alpha, \beta \in [0,1]$ such that

$$\overline{y_+} \sim \alpha \cdot f + (1-\alpha) \cdot \overline{y_0} \quad (7)$$

$$\overline{y_+} \sim \beta \cdot g + (1-\beta) \cdot \overline{y_0} \quad (8)$$

and from the affinity of U_f and U_g, $U(f) = \frac{1}{\alpha}$ and $U(g) = \frac{1}{\beta}$. Therefore

$$U(f) > U(g)$$

$\iff$

$$\beta > \alpha$$

$\iff$ (from Lemma 2, Corollary 1, (7) and (8))

$$\alpha \cdot g + (1-\alpha) \cdot \overline{y_0} \prec \beta \cdot g + (1-\beta) \cdot \overline{y_0} \sim \alpha \cdot f + (1-\alpha) \cdot \overline{y_0}$$

$\iff$ (from the transitity of $\succeq$)

$$\alpha \cdot f + (1-\alpha) \cdot \overline{y_0} \succ \alpha \cdot g + (1-\alpha) \cdot \overline{y_0}$$

$\iff$ (from SCI)

$$f \succ g$$

5. The case where $f \preceq \overline{y_0}$ and $g \preceq \overline{y_0}$ is analogous to Step 4. ■(Lemma 3)

The next lemma shows the affinity of U over any act combined with a constant act.

Lemma 4 $\forall f \in F$, $\forall y \in Y$, $\forall \alpha \in [0,1]$,

$$U(\alpha \cdot f + (1-\alpha) \cdot \overline{y}) = \alpha \cdot U(f) + (1-\alpha) \cdot u(y).$$

Proof:

$$\begin{aligned} & U(\alpha \cdot f + (1-\alpha) \cdot \overline{y}) = \\ &= U_f(\alpha \cdot f + (1-\alpha) \cdot \overline{y}) = && (9) \\ &= \alpha \cdot U_f(f) + (1-\alpha) \cdot U_f(\overline{y}) = && (10) \\ &= \alpha \cdot U(f) + (1-\alpha) \cdot u(y) && (11) \end{aligned}$$

where (9) is from Lemma 1, (10) is from affinity of U_f and (11) is from the defnition of U and U_f.

■(Lemma 4)

3.4 Extension of the model

The following definitions extend Y, F, and U. The reason that this extension is needed for the proof, while it was not required in Schmeidler (1989) or Gilboa (1989), is the lack of the monotonicity axiom. If we have any act f that has various outcomes, the monotonicity axiom postulates that f is preferred to an act with a constant outcome equal to f's worst outcome, and is less preferred than an act with a constant outcome equal to f's best outcome. Therefore, from continuity, there exists a constant act $\overline{y}$ such that the decision-maker is indifferent between f and $\overline{y}$. Without the monotonicity axiom, for some act f there may be no constant act that is indifferent to f. The extension that we use allows us to bypass this problem by extending the possible utilities of constant acts to all the real numbers. The extension is used only for the proofs, which give us the results we require on the original domain.

1. $\hat{Y} = Y \cup I\!R$ (assuming without loss of generality that $Y \cap I\!R = \emptyset$).

2. $\hat{u} : \hat{Y} \longrightarrow I\!R$ by

$$\hat{u}(\hat{y}) = \begin{cases} u(\hat{y}) & \text{if } \hat{y} \in Y \\ \hat{y} & \text{if } \hat{y} \in I\!R \end{cases}$$

3. $\hat{F} = \hat{Y}^T$,

4. For $\hat{f} \in \hat{F}$, define $\tau : \hat{F} \longrightarrow I\!R$ by

$$\tau(\hat{f}) = \sup_{t \in T} |\hat{u}(\hat{f}_t)|$$

   and $\eta : \hat{F} \longrightarrow F$ by

$$\eta(\hat{f})(t) = \begin{cases} y_0 & \text{if } \tau(\hat{f}) = 0 \\ \frac{\hat{u}(\hat{f}_t)}{\tau(\hat{f})} \cdot y_+ + \left(1 - \frac{\hat{u}(\hat{f}_t)}{\tau(\hat{f})}\right) \cdot y_0 & \text{if } \hat{u}(\hat{f}_t) \geq 0 \text{ and } \tau(\hat{f}) > 0 \\ \frac{|\hat{u}(\hat{f}_t)|}{\tau(\hat{f})} \cdot y_- + \left(1 - \frac{|\hat{u}(\hat{f}_t)|}{\tau(\hat{f})}\right) \cdot y_0 & \text{if } \hat{u}(\hat{f}_t) < 0 \end{cases}$$

5. $\hat{U}(\hat{f}) = \tau(\hat{f}) \cdot U(\eta(\hat{f}))$.

These definitions ensure that $\hat{U}$ is well-defined on $\hat{F}$, which includes all the n-tuples of real numbers and/or elements of Y.

The next lemma shows that $\hat{U}$ coincides with U on F.

Lemma 5 $\forall f \in F$, $U(f) = \hat{U}(f)$.

Proof:

1. If $\tau(f) \geq 1$ then

$$\eta(f) \sim \frac{1}{\tau(f)} \cdot f + \frac{\tau(f) - 1}{\tau(f)} \cdot \overline{y_0}$$

(from the definition of τ and η and SUBST). Then, from Lemma 4,

$$U(\eta(f)) = \frac{1}{\tau(f)} \cdot U(f),$$

and from the definition of $\hat{U}$,

$$U(f) = \tau(f) \cdot U(\eta(f)) = \hat{U}(f).$$

2. If $0 < \tau(f) < 1$ then

$$\tau(f) \cdot \eta(f) + (1 - \tau(f)) \cdot \overline{y_0} \sim f$$

from the definition of τ and η. Then, from Lemma 4 and the definition of $\hat{U}$,

$$U(f) = \tau(f) \cdot U(\eta(f)) = \hat{U}(f).$$

3. If $\tau(f) = 0$ then both $U(f)$ and $\hat{U}(f)$ are equal to zero, and the result is immediate.

■(Lemma 5)

We now extend the definition of $\succ$ to $\hat{F}$ by the following definition:

$$\hat{f} \hat{\succ} \hat{g} \text{ if and only if } \hat{U}(\hat{f}) > \hat{U}(\hat{g})$$

Thus, since $\hat{\succ}$ coincides with $\succ$ on F, $\hat{U}$ represents $\hat{\succ}$ on $\hat{F}$, and $\hat{U}$ restricted to F represents $\succ$.

We now define multiplication of elements of $\hat{F}$ by a scalar, and addition of pairs of elements of $\hat{F}$:

$$\text{For } \alpha \geq 0 \text{ and } \hat{f} \in \hat{F},\ (\alpha \otimes \hat{f})(t) = \alpha \cdot \hat{u}(\hat{f}_t) \ \ \forall t \in T,$$

$$\text{For } \hat{f}, \hat{g} \in \hat{F},\ (\hat{f} \oplus \hat{g})(t) = \hat{u}(\hat{f}_t) + \hat{u}(\hat{g}_t) \ \ \forall t \in T.$$

Where there is no possibility of confusion, we use $\cdot$ instead of $\otimes$ and $+$ instead of $\oplus$.

The following lemma states that τ and η are homogeneous (of degree one and zero respectively).

Lemma 6 $\forall \alpha \geq 0,\ \forall \hat{f} \in \hat{F}$

1. $\tau(\alpha \otimes \hat{f}) = \alpha \cdot \tau(\hat{f})$,
2. $\eta(\alpha \otimes \hat{f}) = \eta(\hat{f})$.

Proof:
The proof of (1) follows from the fact that $\hat{u}$ is affine, and therefore, $\forall t \in T$, $\hat{u}(\alpha \otimes \hat{f})(t) = \alpha \cdot \hat{u}(\hat{f}(t))$. The proof of (2) follows from the definition of η, and from (1). ■(Lemma 6)

Corollary 2 $\hat{U}(\alpha \otimes \hat{f}) = \alpha \cdot \hat{U}(\hat{f})$

We now extend the notions of SC and SCI to the extended domain.
Definition: $\hat{f}$ and $\hat{g}$ are *sequentially comonotonic* ($\widehat{SC}$), if $\eta(\hat{f})$ and $\eta(\hat{g})$ are sequentially comonotonic.

The axiom SCI is extended in the following way:

$\widehat{SCI}$: If for $\hat{f}, \hat{g}, \hat{h} \in \hat{F}$ both $\hat{f}, \hat{h}$ and $\hat{g}, \hat{h}$ are $\widehat{SC}$, and $\alpha \in (0,1)$, then

$$\hat{f} \hat{\succ} \hat{g} \iff (\alpha \otimes \hat{f}) \oplus ((1-\alpha) \otimes \hat{h}) \hat{\succ} (\alpha \otimes \hat{g}) \oplus ((1-\alpha) \otimes \hat{h})$$

The following lemma shows that SCI in our original model implies the analogous characteristic $\widehat{SCI}$ in the extended model.

Lemma 7 *SCI on $F \Longrightarrow \widehat{SCI}$ on $\hat{F}$.*

Proof: Assume $\hat{f}, \hat{h}$ and $\hat{g}, \hat{h}$ are both $\widehat{SC}$, $\hat{f}, \hat{g}, \hat{h} \in \hat{F}$, and $\alpha \in (0,1)$.

Define $\mu = \max\{\tau(\hat{f}), \tau(\hat{g}), \tau(\hat{h})\}$.

If $\mu = 0$, then the proof is immediate. Therefore, assume $\mu > 0$.

Assume

$$(\alpha \otimes \hat{f}) \oplus ((1-\alpha) \otimes \hat{h}) \hat{\succ} (\alpha \otimes \hat{g}) \oplus ((1-\alpha) \otimes \hat{h}). \tag{12}$$

Therefore, from the definitions,

$$\hat{U}(\alpha \cdot \tau(\hat{f}) \cdot \eta(\hat{f}) + (1-\alpha) \cdot \tau(\hat{h}) \cdot \eta(\hat{h})) >$$

$$\hat{U}(\alpha \cdot \tau(\hat{g}) \cdot \eta(\hat{g}) + (1-\alpha) \cdot \tau(\hat{h}) \cdot \eta(\hat{h})). \tag{13}$$

Define $f' = \frac{\tau(\hat{f})}{\mu} \cdot \eta(\hat{f}) + (1 - \frac{\tau(\hat{f})}{\mu}) \cdot \hat{y}_0$, and g', h' similarly.

From Lemma 6,

$$\mu \cdot U(\alpha \cdot f' + (1-\alpha) \cdot h') > \mu \cdot U(\alpha \cdot g' + (1-\alpha) \cdot h').$$

Note that $f'(s) \succ f'(t) \iff \hat{f}(s) \succ \hat{f}(t)$ for all $s, t \in T$, and similarly for $g', \hat{g}$ and $h', \hat{h}$. Thus, both f' and h' are SC, and g' and h' are SC. Therefore from SCI and Lemma 5,

$$\mu \cdot \hat{U}\left(\frac{\tau(\hat{f})}{\mu} \cdot \eta(\hat{f})\right) > \mu \cdot \hat{U}\left(\frac{\tau(\hat{g})}{\mu} \cdot \eta(\hat{g})\right).$$

Therefore, from Corollary 2, $\hat{U}(\hat{f}) > \hat{U}(\hat{g})$, and thus $\hat{f} \hat{\succ} \hat{g}$.

The other direction is proved similarly. ■(Lemma 7)

Lemma 8 *If $\hat{f}, \hat{g} \in \hat{F}$ and $\forall t \in T$, $\overline{\hat{f}_t} \hat{\sim} \overline{\hat{g}_t}$ then $\hat{f} \hat{\sim} \hat{g}$.*

This is the analogy of SUBST for $\hat{\succ}$.

Proof:

$$\overline{\hat{f}_t} \hat{\sim} \overline{\hat{g}_t} \quad \forall t \in T \Longrightarrow$$

$$\hat{u}(\hat{f}_t) = \hat{u}(\hat{g}_t) \quad \forall t \in T \Longrightarrow$$

$$\tau(\hat{f}) = \tau(\hat{g}) \text{ and } \eta(\hat{f}) = \eta(\hat{g}) \Longrightarrow$$

$$\hat{U}(\hat{f}) = \hat{U}(\hat{g}) \Longrightarrow$$

$$\hat{f} \hat{\sim} \hat{g}.$$

■(Lemma 8)

Lemma 9 shows that the utility of mixtures in F is preserved in $\hat{F}$.

Lemma 9 *If $f, g \in F, \alpha \in [0,1]$ then $\hat{U}((\alpha \otimes f) \oplus ((1-\alpha) \otimes g)) = U(\alpha \cdot f + (1-\alpha) \cdot g)$.*

Proof:

1. Starting from the right hand side, we have that

$$\begin{aligned} & U(\alpha \cdot f + (1-\alpha) \cdot g) = \\ &= \hat{U}\left(\alpha \cdot f + (1-\alpha) \cdot g\right) = \\ &= \hat{U}\left(\alpha \cdot f_1 + (1-\alpha) \cdot g_1, \ldots\right) = \\ &= \hat{U}\left(\hat{u}(\alpha \cdot f_1 + (1-\alpha) \cdot g_1), \ldots\right) = \quad (14) \\ &= \hat{U}\left(\alpha \cdot \hat{u}(f_1) + (1-\alpha) \cdot \hat{u}(g_1), \ldots\right) \quad (15) \end{aligned}$$

   Where (14) is from Lemma 8, and (15) is from the affinity of $\hat{u}$.

2. Starting from the left hand side, we have that

$$\begin{aligned}
&\hat{U}((\alpha \otimes f) \oplus ((1-\alpha) \otimes g)) = \\
&= \hat{U}\left((\alpha \cdot \hat{u}(f_1), \ldots) \oplus ((1-\alpha) \cdot \hat{u}(g_1), \ldots)\right) = \qquad (16) \\
&= \hat{U}\left(\alpha \cdot \hat{u}(f_1) + (1-\alpha) \cdot \hat{u}(g_1), \ldots\right) \qquad (17)
\end{aligned}$$

Where (16) is from the definition of $\otimes$ and (17) is from the definition of $\oplus$.

■(Lemma 9)

3.5 Affinity of U on sequentially comonotonic acts

Lemma 10 *For $f, g \in F$, $\alpha \in [0,1]$, if f and g are SC, then $U(\alpha \cdot f + (1-\alpha) \cdot g) = \alpha \cdot U(f) + (1-\alpha) \cdot U(g)$*

Proof:

$$\begin{aligned}
&\alpha \cdot f + (1-\alpha) \cdot g \hat{\sim} \\
&\hat{\sim} \quad (\alpha \otimes f) \oplus ((1-\alpha) \otimes g) \hat{\sim} \qquad (18) \\
&\hat{\sim} \quad (\alpha \otimes \overline{U(f)}) \oplus ((1-\alpha) \otimes \overline{U(g)}) \hat{\sim} \qquad (19) \\
&\hat{\sim} \quad \overline{\alpha \cdot U(f) + (1-\alpha) \cdot U(g)} \qquad (20)
\end{aligned}$$

Where (18) is from Lemma 9, (19) is from two applications of $\widehat{SCI}$ and (20) is from the definition of $\oplus$ and $\otimes$.

Take U of the first and $\hat{U}$ of the last expressions, and the proof is completed.

■(Lemma 10)

Lemma 11 *If $\hat{f}, \hat{g}$ are $\widehat{SC}$, then $\hat{U}(\hat{f} \oplus \hat{g}) = \hat{U}(\hat{f}) + \hat{U}(\hat{g})$.*

Proof:

$$
\begin{aligned}
&\hat{U}(\hat{f} \oplus \hat{g}) = \\
&= \hat{U}(\hat{u}(\hat{f}_1) + \hat{u}(\hat{g}_1), \ldots) = && (21)\\
&= \hat{U}(\tau(\hat{f}) \cdot \hat{u}(\eta(\hat{f})(1)) + \tau(\hat{g}) \cdot \hat{u}(\eta(\hat{g})1)), \ldots) = && (22)\\
&= \hat{U}\left((\tau(\hat{f})+\tau(\hat{g}))\cdot\left[\frac{\tau(\hat{f})}{\tau(\hat{f})+\tau(\hat{g})}\cdot\hat{u}(\eta(\hat{f})(1))+\frac{\tau(\hat{g})}{\tau(\hat{f})+\tau(\hat{g})}\cdot\hat{u}(\eta(\hat{g})(1))\right],\ldots\right) = \\
&= (\tau(\hat{f}) + \tau(\hat{g})) \cdot U\left(\frac{\tau(\hat{f})}{\tau(\hat{f})+\tau(\hat{g})}\cdot\eta(\hat{f})+\frac{\tau(\hat{g})}{\tau(\hat{f})+\tau(\hat{g})}\cdot\eta(\hat{g})\right) = && (23)\\
&= \tau(\hat{f}) \cdot U(\eta(\hat{f})) + \tau(\hat{g}) \cdot U(\eta(\hat{g})) = && (24)\\
&= \hat{U}(\hat{f}) + \hat{U}(\hat{g}) && (25)
\end{aligned}
$$

Where (21) is from the definition of $\oplus$, (22) is from the definition of τ and η, (23) is from Corollary 2 and Lemma 5, (24) is from Lemma 10 and (25) is from the definition of $\hat{U}$.

■(Lemma 11)

Corollary 3 *If $a_1, \ldots, a_k \geq 0$, $\hat{f}_1, \ldots, \hat{f}_k \in \hat{F}$ and are pairwise $\widehat{SC}$, then*

$$\hat{U}\left(\sum_{i=1}^{k} a_i \hat{f}_i\right) = \sum_{i=1}^{k} a_i \hat{U}(\hat{f}_i).$$

Define, for $0 \leq t \leq n$, $\overset{\nearrow}{x}_t \in F$ by

$$\overset{\nearrow}{x}_t(s) = \begin{cases} y_0 & \text{if } s \leq t \\ y_+ & \text{if } s > t \end{cases}$$

and $\overset{\searrow}{x}_t \in F$ by

$$\overset{\searrow}{x}_t(s) = \begin{cases} y_+ & \text{if } s \leq t \\ y_0 & \text{if } s > t \end{cases}$$

Define, for $1 \leq t \leq n$, $\delta_t^+ = U(\overset{\nearrow}{x}_{t-1})$, and $\delta_t^- = 1 - U(\overset{\searrow}{x}_{t-1})$. Note that $\delta_1^+ = u(\overline{y_+}) = 1$ and $\delta_1^- = 1 - u(\overline{y_0}) = 1$.

Lemma 12 $\forall \hat{f} \in \hat{F}$,

$$\hat{U}(\hat{f}) = \hat{u}(f_1) + \sum_{t=2}^{n} \left[\delta_t^+ (\hat{u}(\hat{f}_t) - \hat{u}(\hat{f}_{t-1}))^+ + \delta_t^- (\hat{u}(\hat{f}_t) - \hat{u}(\hat{f}_{t-1}))^- \right]$$

Proof: Take $v \in \hat{F}$ such that $v_t = \hat{u}(\hat{f}_t)\ \forall t \in T$. (i.e. $v \in I\!R^T$ and $v \sim \hat{f}$ from Lemma 8). Define $v_0 = 0$.

Define $T_d = \{t \in T | \hat{u}(\hat{f}_t) < \hat{u}(\hat{f}_{t-1})\}$ and $T_i = T \setminus T_d$. T_d is the set of periods where there was a decrease in utility from the previous period, and T_i is the set where there was an increase (or no change).

From this definition, and algebraic manipulation, we derive that

$$v + \sum_{t \in T_d} \overset{\nearrow}{x_0} (v_{t-1} - v_t) = \sum_{t \in T_i} \overset{\nearrow}{x_{t-1}} (v_t - v_{t-1}) + \sum_{t \in T_d} \overset{\searrow}{x_{t-1}} (v_{t-1} - v_t).$$

The acts $\overset{\nearrow}{x_0}$, $\overset{\nearrow}{x_{t-1}}\ \forall t \in T_i$, $\overset{\searrow}{x_{t-1}}\ \forall t \in T_d$ and v are pairwise SC, therefore (since if $t \in T_d$ then $v_{t-1} - v_t \geq 0$ and if $t \in T_i$ then $v_t - v_{t-1} \geq 0$) from Corollary 3,

$$\hat{U}(v) + \sum_{t \in T_d} U(\overset{\nearrow}{x_0})(v_{t-1} - v_t) =$$

$$\sum_{t \in T_i} (v_t - v_{t-1}) U(\overset{\nearrow}{x_{t-1}}) + \sum_{t \in T_d} (v_{t-1} - v_t) U(\overset{\searrow}{x_{t-1}}). \tag{26}$$

Therefore,

$$\begin{aligned} \hat{U}(v) &= \sum_{t \in T_i} \delta_t^+ (v_t - v_{t-1}) + \\ &\quad \sum_{t \in T_d} \delta_t^- (v_t - v_{t-1}) + = \qquad (27) \\ &= \sum_{T} \left[\delta_t^+ (v_t - v_{t-1})^+ + \delta_t^- (v_t - v_{t-1})^- \right]. \qquad (28) \end{aligned}$$

Since $v_0 = 0$ and $\delta_1^+ = \delta_1^- = 1$, this implies that

$$\hat{U}(v) = \sum_{t=1}^{n} \left[\delta_t^+ (v_t - v_{t-1})^+ + \delta_t^- (v_t - v_{t-1})^- \right],$$

which is what we need, since $v \sim \hat{f}$, and therefore $\hat{U}(v) = \hat{U}(\hat{f}) = U(\hat{f})$. ■(Lemma 12)

Proof of the main theorem: The only if direction is immediate from Lemma 5 and Lemma 12. Uniqueness of δ_t^+, δ_t^- is immediate, since the values of δ_t^+ and δ_t^- are uniquely determined from the values of $U(\overset{\nearrow}{x_t})$ and $U(\overset{\searrow}{x_t})$, which are well defined. For the other direction, if a functional U represents $\succeq$ on F and satisfies the other conditions, it is immediate that the axioms are all satisfied.

■(Theorem 1)

4 Definition and Characterization of Loss Aversion

The aspect of loss-aversion we refer to here is the tendency of decision makers to be more sensitive to reductions in their level of well-being than to increases. In experiments referred to in Tversky and Kahneman (1991), this tendency has been empirically measured, and on average the importance assigned to decreases is about double that given to increases.

We will use an additional axiom on $\succeq$ in the proof of the characterization of loss-aversion to be given in this section, named constant-tail monotonicity. This axiom, which is much weaker than the monotonicity axioms of Gilboa (1989) and Schmeidler (1989), implies that if from a certain period onwards the outcome is constant, the decision-maker gains utility from increasing this constant outcome, and loses utility when this outcome is decreased (according to his preferences over constant acts). Formally,

CTM *Constant-Tail Monotonicity*: For all $f, g \in F$, for all $i \in T$, if

1. $f(t) = g(t) \ \ \forall t < i$,

2. $f(s) = f(t)$ and $g(s) = g(t) \ \ \forall s, t \geq i$,

and

3. $f(n) \succ g(n)$

then $f \succ g$.

Note that if the axioms of the main theorem hold, then CTM implies that for all $t \in T$, $\delta_t^+ > 0$ and $\delta_t^- > 0$, since from CTM, increasing the tail of an act that is constant from period t onwards must increase the utility of the act, both when there was an increase in period t and when there was a decrease in that period.

Now we will give the axiom of loss-aversion, derived from Gilboa's definition of variation aversion, which, together with our axioms, will be used to characterize loss-aversion in our utility functional.

LA *Loss Aversion*: For all $f^1, f^2, g^1, g^2 \in F$ and $1 < i < n$, if

1. $f^2(i) = g^2(i) \succ f^1(i) = g^1(i)$
2. $f^1(j) = f^2(j)$ and $g^1(j) = g^2(j)$ for all $j \neq i$
3. $f^1 \sim g^1$
4. $g^1(i) \succeq g^1(i+1)$ and $g^1(i-1) \succeq g^2(i)$
   or
   $g^2(i) \preceq g^1(i+1)$ and $g^1(i-1) \preceq g^1(i)$

then $g^2 \succeq f^2$.

That is, if the decision-maker is indifferent between f^1 and g^1, and we improve both of them on period i to the level $f^2(i) = g^2(i)$, but f has increased total losses, while g's total losses remain constant, then the modified f^1 (i.e. f^2) is (weakly) less preferred than the modified g^1 (which is g^2).

Theorem 2 *If $\succeq$ satisfies the axioms of Theorem 1 and CTM, then $\succeq$ satisfies LA if and only if $\delta_t^- \geq \delta_t^+ > 0 \ \ \forall t \in T$.*

Proof: We start with the only-if direction. Assume loss-aversion holds and take any $i \in \{2, \ldots, n-1\}$. We construct f^1, f^2, g^1, g^2 that fulfill the requirements of the definition of loss-aversion and this enables us to conclude that $\delta_t^- \geq \delta_t^+$. Define $\hat{f}, \hat{g}$ as follows:

$$\hat{f}(t) = \begin{cases} 0 & \text{if } t < i \\ \frac{1}{2} & \text{if } t = i \\ -1 & \text{if } t > i \end{cases}$$

$$\hat{g}(t) = \begin{cases} 1 & \text{if } t < i \\ \frac{1}{2} & \text{if } t = i \\ -1 & \text{if } t > i \end{cases}$$

We now modify the acts to make the decision-maker indifferent between them, without changing their relative structure. From CTM, we can decrease $\hat{f}$ or $\hat{g}$ from period $i+1$ to period n, and thus reduce its utility (without changing the structure). If either $\hat{f}$ or $\hat{g}$ is preferred to the other (in $\hat{F}$), decrease the utility of the preferred act until they have the same utility. Since $\hat{Y}$ is not bounded, from the note to CTM this is possible. Denote the resulting acts $\hat{f}'$ and $\hat{g}'$. Our next step is to normalize these two acts to get f^1 and g^1 in F. Denote $\mu = \max\{\tau(\hat{f}'), \tau(\hat{g}')\}$. Define

$$f^1 = \frac{\tau(\hat{f}')}{\mu} \cdot \eta(\hat{f}') + \left(1 - \frac{\tau(\hat{f}')}{\mu}\right) \cdot y_0$$

$$g^1 = \frac{\tau(\hat{g}')}{\mu} \cdot \eta(\hat{g}') + \left(1 - \frac{\tau(\hat{g}')}{\mu}\right) \cdot y_0$$

Now define $f^2(i) = g^2(i) = \frac{1}{2}g^1(i-1) + \frac{1}{2}g^1(i)$, and the four requirements of the definition of loss-aversion are fulfilled. Thus, since we assumed that $\succeq$ satisfies loss-aversion, we can conclude that $g^2 \succeq f^2$. Therefore, $U(g^2) \geq U(f^2)$ and $U(g^1) = U(f^1)$, which means that $U(g^2) - U(g^1) \geq U(f^2) - U(f^1)$, thus $(u(g^2(i)) - u(g^1(i)) \cdot \delta_i^- \geq (u(g^2(i)) - u(g^1(i)) \cdot \delta_i^+$ $\Longrightarrow \delta_i^- \geq \delta_i^+$, since $g^2(i) \succ g^1(i)$.

This covers $i = 1$ to $i = n-1$. To prove the relation for n, take $i = n-1$, and build the functions such that g is increasing at period i (the second option of item 4 in the definition of loss-aversion), and proceed similarly, to conclude that $\delta_n^- \geq \delta_n^+$. The other direction of the proof is direct. ■ (Theorem 2)

5 Extension to an Infinite Horizon

For this extension, we assume that T is the set of natural numbers, i.e. $T = \{1, 2, \ldots\}$. We denote by F the set of *bounded* functions from T to Y, and retain all the other aspects of the model. To extend our representation theorem for the infinite case, we need another continuity axiom.

Definition: Let $\{f^n\}_{n\geq 1} \subset F$ and $f \in F$. We say that $\{f^n\}_{n\geq 1}$ *monotonically converges* to f, and denote this by $f^n \longrightarrow f$, if the following conditions hold:

1. $\forall t \in T \;\; \exists n_t \geq 1$ such that $\forall n \geq n_t \;\; f^n(t) = f(t)$.

2. Either $f^n(t) \succeq f^{n+1}(t) \succeq f(t)$ for all $t \in T$ and all $n \geq 1$, or $f(t) \succeq f^{n+1}(t) \succeq f^n(t)$ for all $t \in T$ and all $n \geq 1$.

Our additional axiom is the following.

TC *Time Continuity*: suppose that $\{f^n\}$ monotonically converges to f and that $f \succ g$ ($g \succ f$). Then $\exists n_0 \geq 1$ such that $f^n \succ g$ ($g \succ f^n$) for all $n \geq n_0$.

Among other things, TC implies that if one goes far enough, the tail of any act carries negligible weight in evaluating the act.

Most of the results given in the finite case carry over to the infinite case with T defined as above. Specifically, the proof of the main theorem is valid, up to Corollary 3.

Define $d : F \longrightarrow I\!R^T$ (the sequence of differences in utility) by $d_t(f) = u(f_t) - u(f_{t-1})$, where u is derived as in the finite case. Define

$$U(f) = u(f_1) + \sum_{t=2}^{\infty} \left[\delta_t^+ (d_t(f))^+ + \delta_t^- (d_t(f))^-\right], \tag{29}$$

where δ^+ and δ^- are defined as in Section 3.5. This formulation is analogous to the formulation of Theorem 1, with n replaced by ∞.

Theorem 3 *The preference relation $\succeq$ on F satisfies the axioms of Theorem 1 and TC if and only if*

1. *There exists an affine function* $u : Y \longrightarrow I\!R$

2. *There exist unique functions* $\delta^+, \delta^- : \{2, 3, \ldots\} \longrightarrow I\!R$ *such that* $\sum_{t=2}^{\infty} |\delta_t^+ - \delta_t^-|$ *and* $\sum_{t=2}^{\infty} |\delta_{t+1}^+ - \delta_t^-|$ *are finite, and* $lim_{t\to\infty}\delta_t^+ = lim_{t\to\infty}\delta_t^- = 0$.

3. $U(f)$ *as defined by (29) represents* $\succeq$ *on* F.

Define $\delta_1^+ = \delta_1^- = 1$. We present a series of lemmas which we use to prove the theorem.

Lemma 13 *If* $U(f)$ *converges and is finite for all* $f \in F$, *then*

1. $\sum_{t=1}^{\infty} |\delta_t^+ - \delta_t^-| < \infty$.

2. $\sum_{t=1}^{\infty} |\delta_t^- - \delta_{t+1}^+| < \infty$.

Proof: Define $T_1 = \{t \in T |\ \delta_t^+ \geq \delta_t^-\}$ and $T_2 = T \setminus T_1$. Define $f^1, f^2, f^3, f^4 \in F$ as follows (regard 0 as synonymous with y_0, 1 with y_+ and -1 with y_-):

$$f^1(1) = \begin{cases} 1 & \text{if } 1 \in T_1 \\ 0 & \text{if } 1 \in T_2 \end{cases}$$

and for $t \geq 2$

$$f^1(t) = \begin{cases} f^1(t-1) & \text{if } t \in T_2 \\ 1 - f^1(t-1) & \text{if } t \in T_1 \end{cases}$$

$$f^2(1) = \begin{cases} -1 & \text{if } 1 \in T_1 \\ 0 & \text{if } 1 \in T_2 \end{cases}$$

and for $t \geq 2$

$$f^2(t) = \begin{cases} f^2(t-1) & \text{if } t \in T_2 \\ -1 - f^2(t-1) & \text{if } t \in T_1 \end{cases}$$

Define f^3 as f^1, and f^4 as f^2, exchanging T_1 and T_2. Now

$$U(f^1) + U(f^2) = \sum_{t=1}^{\infty} (\delta_t^+ - \delta_t^-)^+$$

and

$$U(f^3) + U(f^4) = \sum_{t=1}^{\infty} (\delta_t^+ - \delta_t^-)^-$$

thus

$$U(f^1) + U(f^2) - (U(f^3) + U(f^4)) = \sum_{t=1}^{\infty} |\delta_t^+ - \delta_t^-|$$

and since all four terms on the LHS of the equation converge and are finite, so does the RHS. We now prove that $\sum_{t=1}^{\infty} |\delta_t^- - \delta_{t+1}^+| < \infty$ in a similar manner. Define $T_3 = \{t \in T |\ \delta_t^- \geq \delta_{t+1}^+\}$ and $T_4 = T \setminus T_3$. Define $f^5, f^6, f^7, f^8 \in F$ as follows:

$$f^5(1) = \begin{cases} -1 & \text{if } 1 \in T_3 \\ 0 & \text{if } 1 \in T_4 \end{cases}$$

and for $t \geq 2$

$$f^5(t) = \begin{cases} -1 - f^5(t-1) & \text{if } t \in T_3 \text{ or } [t-1 \in T_3 \text{ and } f(t-1) = -1] \\ f^5(t-1) & \text{otherwise} \end{cases}$$

$$f^6(1) = 0$$

and for $t \geq 2$

$$f^6(t) = \begin{cases} -1 - f^6(t-1) & \text{if } [t \in T_3 \text{ and } t-1 \in T_3] \\ & \text{or } [t-1 \in T_3 \text{ and } t-2 \in T_3 \text{ and } f(t-1) = -1] \\ f^6(t-1) & \text{otherwise} \end{cases}$$

Define f^7 as f^5, and f^8 as f^6, exchanging T_3 and T_4.

Now

$$U(f^5) + U(f^6) = \sum_{t=1}^{\infty} (\delta_t^- - \delta_{t+1}^+)^+$$

and

$$U(f^7) + U(f^8) = \sum_{t=1}^{\infty} (\delta_t^- - \delta_{t+1}^+)^-$$

thus

$$U(f^5) + U(f^6) - (U(f^7) + U(f^8)) = \sum_{t=1}^{\infty} |\delta_t^- - \delta_{t+1}^+|$$

and since all four terms on the LHS of the equation converge and are finite, the same is true for the RHS. ■(Lemma 13)

Lemma 14 *If* $\sum_{t=1}^{\infty} |\delta_t^+ - \delta_t^-|$ *and* $\sum_{t=1}^{\infty} |\delta_t^- - \delta_{t+1}^+|$ *are finite, and* $lim_{t\to\infty}\delta_t^+ = lim_{t\to\infty}\delta_t^- = 0$*, then for all* $f \in F$*,* $U(f)$ *converges and is finite.*

Proof: Take $f \in F$. Since $\{u(f_t)\}_{t\geq 1}$ is bounded, denote $M = \sup_{t\geq 1} |u(f(t))|$. Define $\{\gamma_t\}_{t\geq 1}$, $\{e_t\}_{t\geq 1}$ and $\{E_t\}_{t\geq 1}$ as follows:

$$\gamma_t = \begin{cases} \delta^+_{\frac{t+1}{2}} & \text{if t is odd} \\ \delta^-_{\frac{t}{2}} & \text{if t is even} \end{cases}$$

$$e_t = \begin{cases} (d_{\frac{t+1}{2}}(f))^+ & \text{if t is odd} \\ (d_{\frac{t}{2}}(f))^- & \text{if t is even} \end{cases}$$

and $E_t = \sum_{n=1}^{t} e_n$, $E_0 = 0$. Thus, $U(f) = \sum_{t=1}^{\infty} e_t\gamma_t$. Abel's partial summation formula states that

$$\forall n \geq 1, \quad \sum_{t=1}^{n} e_t\gamma_t = E_n\gamma_{n+1} - \sum_{t=1}^{n} E_t(\gamma_{n+1} - \gamma_n)$$

therefore if both terms on the RHS converge and are finite, the same is true for the LHS. The sequence $\{E_n\}$ is bounded, since $\max_{n\geq 1}|E_n| \leq 2M$. The sequence $\{\gamma_n\}$ tends to zero from the assumptions of the lemma, and $\sum_{t=1}^{\infty} E_t(\gamma_{t+1} - \gamma_t) \leq 2M(\sum_{t=1}^{\infty}|\delta_t^+ - \delta_t^-| + \sum_{t=1}^{\infty}|\delta_t^- - \delta_{t+1}^+|)$, with the RHS of this inequality convergent and finite from the assumption of the lemma. ■(Lemma 14)

We now use the extension of the model as defined in Section 3.4, so that any act will be indifferent to some constant act in the extended model. The next lemma will extend TC to $\hat{F}$.

Lemma 15 *If $\succeq$ satisfies TC on F then $\hat{\succeq}$ satisfies TC on $\hat{F}$.*

Proof: For brevity, if $h \in F$ and $0 \leq \alpha \leq 1$, let $\alpha \cdot h$ denote $\alpha \cdot h + (1-\alpha) \cdot y_0$.

Assume that $f, g \in \hat{F}$, $\{f^n\}_{n\geq 1} \longrightarrow f$ and $f \hat{\succ} g$. Therefore, if $\tau(f) > 0$ then

$$\left\{\frac{\tau(f^n)}{\tau(f)} \cdot \eta(f^n)\right\}_{n\geq 1} \longrightarrow \eta(f), \tag{30}$$

where the convergence is in F. There are three cases to consider.

1. If $\tau(f) \geq \tau(g)$, then $\eta(f) \succ \frac{\tau(g)}{\tau(f)} \cdot \eta(g)$ and from TC and (30), for some $n_0 \geq 1$, we have that for all $n \geq n_0$,

$$\frac{\tau(f^n)}{\tau(f)} \cdot \eta(f^n) \succ \frac{\tau(g)}{\tau(f)} \cdot \eta(g),$$

   and thus, for all $n \geq n_0$, $\tau(f^n)U(\eta(f^n)) > \tau(g)U(\eta(g))$, and therefore $f^n \hat{\succ} g$.

2. If $\tau(g) > \tau(f) > 0$, then $\frac{\tau(f)}{\tau(g)} \cdot \eta(f) \succ \eta(g)$. From (30) we can infer that

$$\left\{\frac{\tau(f)}{\tau(g)} \cdot \frac{\tau(f^n)}{\tau(f)} \cdot \eta(f^n)\right\}_{n\geq 1} \longrightarrow \frac{\tau(f)}{\tau(g)}\eta(f), \tag{31}$$

and from TC and (31) there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\frac{\tau(f)}{\tau(g)} \cdot \frac{\tau(f^n)}{\tau(f)} \cdot \eta(f^n) \succ \eta(g),$$

and therefore, for all $n \geq n_0$, $\tau(f^n)U(\eta(f^n)) > \tau(g)U(\eta(g))$, and therefore $f^n \hat{\succ} g$.

3. If $\tau(g) > \tau(f) = 0$, then $\frac{1}{\tau(g)} \cdot \eta(g)$. Similarly to (30), we have

$$\left\{\frac{1}{\tau(g)} \cdot \tau(f^n) \cdot \eta(f^n)\right\}_{n\geq 1} \longrightarrow \frac{1}{\tau(g)} \cdot \overline{y_0}, \tag{32}$$

and from TC and (32) there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $\frac{1}{\tau(g)} \cdot \tau(f^n) \cdot \eta(f^n) \succ \eta(g)$, and therefore for all such n, $\tau(f^n) \cdot \eta(f^n) \succ \tau(g) \cdot \eta(g)$, and $f^n \succ g$.

■(Lemma 15)

Proof of Theorem 3:

$\Longrightarrow$

All definitions and lemmas for the finite case (except the final one) hold for the infinite case also. Take $f \in F$. Assume without loss of generality that $f(t) \succ y_0 \;\; \forall t \in T$. Define $\{f^n\}_{n\geq 1}$ by

$$f^n(t) = \begin{cases} f(t) & \text{if } t \leq n \\ y_0 & \text{if } t > n \end{cases}$$

Thus $f^n \longrightarrow f$. From the definition of $\hat{Y}$, there exist $y_M, y_m \in \hat{Y}$ such that $\overline{y_M} \hat{\succ} f \hat{\succ} \overline{y_m}$. From CONT there exists $\alpha \in (0,1)$ such that $f \hat{\sim} \alpha \cdot \overline{y_M} + (1-\alpha) \cdot \overline{y_m}$. From Lemma 2, for all ϵ such that $\max\{\alpha, 1-\alpha\} > \epsilon > 0$, it is true that $\overline{(\alpha+\epsilon)y_M + (1-\alpha-\epsilon)y_m} \hat{\succ} f \hat{\succ} \overline{(\alpha-\epsilon)y_M + (1-\alpha+\epsilon)y_m}$. Thus, from TC,

$$\begin{aligned} &\exists n_0 \geq 1 \text{ such that } \forall n \geq n_0 \\ &\overline{(\alpha+\epsilon)y_M + (1-\alpha-\epsilon)y_m} \hat{\succ} f^n \hat{\succ} \overline{(\alpha-\epsilon)y_M + (1-\alpha+\epsilon)y_m} \end{aligned} \tag{33}$$

Thus, for all $n \geq n_0$, $(\alpha+\epsilon)u(y_M) + (1-\alpha-\epsilon)u(y_m) > U(f^n) > (\alpha-\epsilon)u(y_M) + (1-\alpha+\epsilon)u(y_m)$. From the proof of the main theorem in the finite case, we have that

$U(f^n) = u(f_1) + \sum_{t=2}^{n} \left[\delta_t^+(d_t(f))^+ + \delta_t^-(d_t(f))^-\right]$, with δ_t^+, δ_t^- defined as in the finite case. Thus, from (33), $\lim_{n\to\infty} U(f^n)$ converges and is finite, and therefore $U(f)$ as defined in the statement of the theorem also converges and is finite.

U represents $\succeq$ on F, since

$$\forall f, g \in F,\ \forall n \geq 1, \quad f^n \succ g^n \iff U(f^n) > U(g^n)$$

as in the finite case, (defining the sequence $\{g^n\}$ similarly to f^n above) and from TC if $f \succ g$ then $\exists n_0 \geq 1$ such that $\forall n \geq n_0,\ \ f^n \succ g^n$. From TC it is straightforward to show that the strict inequality also prevails in the limit.

Item (2) of the theorem follows directly from Lemma 13 and TC on $\{\overset{\nearrow}{x}_n\}_{n\geq 1}$ and $\{\overset{\searrow}{x}_n\}_{n\geq 1}$. Uniqueness of the deltas is straightforward, as in the finite case.

$\Longleftarrow$

From (2) of the theorem and Lemma 14, $U(f)$ converges and is finite $\forall f \in F$. Take any $f \in F$, any sequence $\{f^n\} \to f$ and any $g \in F$ such that $f \succ g$. Since U represents $\succeq$ on F, $U(f) > U(g) \Rightarrow \forall \epsilon > 0\ \ \exists n_0 \geq 1$ such that if $n \geq n_0$ then $U(f^n) > U(g)$, and thus $f^n \succ g$. This shows TC. That the other axioms hold is trivial. ■(Theorem 3)

6 Remarks

1. **Interpretation of the functional U:** The values of δ_t^+ and δ_t^- for $t > 1$ embody the attitude of the decision-maker to gains and losses, respectively. The functional of a "typical" decision-maker who likes increases will have positive values of δ_t^+ and δ_t^-, and if in addition she is loss-averse, her functional will satisfy $\delta_t^- \geq \delta_t^+$ for all $t > 1$. Higher gain-loss sensitivity is characterized by higher values of δ^+ and δ^-. In the result of Theorem 1 the values of the functions δ^+ and δ^- may be any real numbers. Constant-tail monotonicity is equivalent to requiring that $\delta_t^+, \delta_t^- > 0\ \ \forall t \in T \setminus \{1\}$. The axioms of CTM and Loss-aversion together are equivalent to the requirement that $\delta_t^- \geq \delta_t^+ > 0\ \ \forall t \in T \setminus \{1\}$, as shown in Theorem 2.

2. There are $(2 \cdot n) - 2$ parameters (which depend on $\succeq$) in the function U of Theorem 1, after u has been fixed. Algebraic manipulation gives us other representations, such as

$$U(f) = \sum_{t=1}^{n} w_t \cdot u(f(t)) + \sum_{t=2}^{n} \delta_t^+ \cdot (u(f(t)) - u(f(t-1)))^+$$

(which is very similar to the representation proved in Lemma 4 of Gilboa (1989), or

$$U(f) = \sum_{t=1}^{n} v_t \cdot u(f(t)) + \sum_{t=2}^{n} \delta_t^- \cdot (u(f(t)) - u(f(t-1)))^-$$

each with $(2 \cdot n) - 2$ degrees of freedom. These representations are equivalent and emphasize different aspects of the preferences represented by $\succeq$.

3. Apart from relaxing the monotonicity assumption of Gilboa (which prevented us from using Schmeidler's (1989) results in our proof), we formulate the utility function in terms of interperiod gains and losses. This enabled us to give a simple characterization of loss-aversion from the comparison of the weights given to gains ($\{\delta_t^+\}_{t \in T}$) relative to those given to losses ($\{\delta_t^-\}_{t \in T}$). Gilboa's functional formulation in terms of instantaneous payoffs and interperiod differences (the absolute value of changes in utility) enables characterization of variation aversion, which (given temporal monotonicity) is mathematically equivalent to loss-aversion in this context of multi-period decisions.

4. Note that if any one of the axioms of Theorem 1 (except CND) is not satisfied, the desired representation does not exist.

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