Convergence of the Aumann-Davis-Maschler and Geanakoplos Bargaining Sets

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Abstract

Geanakoplos [14] defined a notion of bargaining set, and proved that his bargaining set is approximately competitive in large finite transferable utility (TU) exchange economies with smooth preferences. Shapley and Shubik [20] showed that the Aumann-Davis-Maschler bargaining set is approximately competitive in replica sequences of TU exchange economies with smooth preferences. We extend Geanakoplos' result to nontransferable utility (NTU) exchange economies without smooth preferences, and we extend the Shapley and Shubik result to non-replica sequences of NTU exchange economies with smooth preferences.

1 Introduction

The bargaining set was originally defined by Aumann and Maschler [6]. Several different definitions have been subsequently proposed; the most frequently used definition was proposed by Davis and Maschler [8]. In the exchange economy context we consider here, the core consists of all allocations such that no coalition can propose an alternative set of trades which is feasible for the coalition on its own and which makes all of its members better off. All definitions of the bargaining set restrict the ability of coalitions to block ("object to") an allocation, by taking into account the possibility that a second coalition might propose yet another set of trades ("counterobject") and thereby cause some members to defect from the first coalition. In the Aumann-Maschler and Davis-Maschler definitions, the original objection is proposed by a "leader;" any counterobjecting coalition must exclude this leader.

Geanakoplos [14] considered sequences of TU exchange economies with smooth preferences. He modified the Davis-Maschler definition so that the "leader" was a group of agents containing a fixed (but small) fraction of the number of agents in the economy; thus, as the number of agents grew along the sequence of economies, the number of individuals in the "leader" grew proportionately. He showed that this Geanakoplos bargaining set becomes asymptotically competitive as the number of agents grows; the proof, which uses Nonstandard Analysis, is quite lengthy.

In Section 3, we present two theorems showing that the Geanakoplos bargaining set is approximately competitive in large finite *NTU* exchange economies. In addition to dropping the assumption of transferable utility, we weaken certain other assumptions (notably smoothness of preferences) required in Geanakoplos [14]. The first theorem (Theorem 3.4) requires assumptions similar to those needed for certain core convergence results (see Anderson [4]); no rate of convergence is established. The hypotheses of the second theorem (Theorem 3.6) are incomparable to those of the first. The main additional hypothesis in the second theorem is that a positive fraction of the agents have uniformly bounded marginal ero. The second theorem includes a rate of conve

arbitrarily close to the inverse of the number of agents. Both theorems are derived from Proposition 3.10.

Mas-Colell [19] considered exchange economies with a continuum of agents but without transferable utility or smooth preferences. He proposed a definition of the bargaining set which does not involve the concept of a leader. Under hypotheses similar to those of Aumann's core equivalence theorem, he showed that the Mas-Colell bargaining set coincides with the set of Walrasian allocations. Since models with a continuum of agents are thought of as ideali ations of large economies, it seemed reasonable to expect that Mas-Colell's bargaining set would become approximately competitive in sequ hou [5] show that Mas-Colell's ba

competitive in large finite exchange economies.

The proof of our main Proposition 3.10 is an adaptation of Mas-Colell's equivalence proof. The error terms that arise from substituting the Shapley-Folkman Theorem for Richter's Theorem can be absorbed in the Geanakoplos leaders' consumption at the objection, but they are sufficient to destroy the Mas-Colell objection.

Finally, we apply our convergence argument to sequences of economies with smooth preferences. Anderson [3], Kim [16] and Geller [15] have previously shown that economies with smooth preferences exhibit faster core convergence rates than non-smooth economies. In the bargaining set context, the rate of convergence is expressed in terms of the si e of the competitive gap and the si e of the leader set. The faster convergence afforded by smoothness allows us to restrict the leader set to be a single individual, and thereby obtain a convergence theorem for the Aumann-Davis-Maschler bargaining set. Shapley and Shubik [20] had previously shown that the Aumann-Davis-Maschler bargainin*not* in the Mas-Colell bargaining set tends

to ero. It is remarkable that the designation of a single leader should make such a profound difference in the resulting bargaining set.

Smoothness of preferences is a strong assumption because it requires that indifference surfaces not cut the boundary of the consumption set; in particular, it implies that at every Walrasian equilibrium, all agents with positive income consume positive amounts of all goods. It may be possible to weaken smoothness to a linkedness condition, as in Cheng [7], Propositions 7.4.12 and 7.4.16 of Mas-Colell [18], and Kim [16].

2 Preliminaries

We begin with some notation and definitions which will be used throughout. Suppose $x, y \in \mathbf{R}^k, B \subset \mathbf{R}^k$. x^i denotes the ith component of x; $x \geq y$ means $x^i \geq y^i$ for all i; $x > y$ means $x \geq y$ and $x \neq y$; $x \gg y$ means $x^i > y^i$ for all i; $\|x\|_1 = \sum_{i=1}^k |x^i|$; $\|x\|_\infty = \max\{|x^1|, \cdots, |x^k|\}$; $\mathbf{R}_+^k = \{x \in \mathbf{R}^k : x \geq 0\}$; $\mathbf{R}_{++}^k = \{x \in \mathbf{R}^k : x \gg 0\}$. If $t \in \mathbf{R}$, $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

A preference is a binary relation $\succ$ on $\mathbf{R}_+^k$ satisfying the following conditions:

1. continuity: $\{(x, y) \in \mathbf{R}_{++}^k : x \succ y\}$ is open;

2. transitivity: if $x \succ y$ and $y \succ z$, then $x \succ z$;

3. negative transitivity: if $x \not\succ y$ and $y \not\succ z$, then $x \not\succ z$; and

4. irreflexivity: $x \not\succ x$.

Let $\mathcal{P}$ denote the set of preferences. If $\succ \in \mathcal{P}$, *define* $x \sim y$ if $x \not\succ y$ and $y \not\succ x$, $x \succeq y$ if $x \succ y$ or $x \sim y$. Note that the indifference relation $x \sim y$ is defined from the underlying strict preference relation $x \succ y$, and is not one of the primitives of the specification of the economy. For some purposes, we will need additional assumptions on preferences:

5. (a) strong monotonicity: $x > y \Rightarrow x \succ y$; or
   (b) i. strong monotonicity on $\mathbf{R}_{++}^k$: $x > y \in \mathbf{R}_{++}^k \Rightarrow x \succ y$;
       ii. weak monotonicity on $\mathbf{R}_+^k$: $x \gg y \Rightarrow x \succ y$;
       iii. boundary condition: if $y \in \mathbf{R}_{++}^k$, $\{x \in \mathbf{R}_+^k : y \not\succ x\} \subset \mathbf{R}_{++}^k$; and

iv. semismoothness: if $x \in \mathbf{R}^k_{++}$, there is an open ball B with $B \succ x$ such that x lies on the boundary of B; let $\rho_{\succ x}$ denote the supremum of the radii of such balls.

6. strong convexity: if $x \neq y$, then either $\frac{x+y}{2} \succ x$ or $\frac{x+y}{2} \succ y$.

Let

$$\begin{aligned}\mathcal{P}_{mo} &= \{\succ \in \mathcal{P} : \succ \text{ satisfies } 5a\} \\ \mathcal{P}_{ss} &= \{\succ \in \mathcal{P} : \succ \text{ satisfies } 5b\} \\ \mathcal{P}_{sc} &= \{\succ \in \mathcal{P} : \succ \text{ satisfies } 6\}.\end{aligned} \tag{1}$$

A set $P \subset \mathcal{P}_{ss}$ is said to be *equisemismooth* if, for every compact $K \subset \mathbf{R}^k_{++}$, $\inf\{\rho_{\succ x} : x \in K, \succ \in P\} > 0$.

An *exchange economy* is a map $\chi : A \to \mathcal{P} \times \mathbf{R}^k_+$, where A is a finite set. For $a \in A$, let $\succ_a$ denote the preference of a (i.e. the projection of $\chi(a)$ onto $\mathcal{P}$) and $e(a)$ the initial endowment of a (i.e. the projection of $\chi(a)$ onto $\mathbf{R}^k_+$). An *allocation* is a map $f : A \to \mathbf{R}^k_+$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$; let $\mathcal{A}(\chi)$ denote the set of allocations of the economy χ. A *coalition* is a non-empty subset of A.

A price p is an element of $\mathbf{R}^k_+$ with $\|p\|_1 = 1$. Δ denotes the set of prices, $\Delta^0 = \{p \in \Delta : p \gg 0\}$. If $p \in \Delta$, let $D(p,a) = \{x \in \mathbf{R}^k_+ : p \cdot x \leq p \cdot e(a), y \succ_a x \Rightarrow p \cdot y > p \cdot e(a)\}$. Let $M^m_\chi = \max\{\|e(a_1) + \ldots + e(a_m)\|_\infty : a_1, \ldots, a_m$ are distinct elements of $A\}$.

Given $x \in \mathbf{R}^k_+$, $(\succ, e) \in \mathcal{P} \times \mathbf{R}^k_+$, and $p \in \Delta$, define $\phi(p, x, (\succ, e)) = |p \cdot (x - e)| + |\inf\{p \cdot (y - x) : y \succ x\}|$. Note that, if $\succ$ is continuous and $p \gg 0$, then $\phi(p, x, (\succ, e)) = 0$ implies that x is in the demand set at price p of an agent with characteristics $(\succ, e)$. ϕ gives a quantitative measurement of the extent to which x fails to satisfy the definition of demand. By a slight abuse of notation, we let $\phi(p, f, a) = \phi(p, f(a), (\succ_a, e(a)))$ if f is an allocation, and $\phi(p, x, a) = \phi(p, x, (\succ_a, e(a)))$ if $x \in \mathbf{R}^k_+$. Given $f \in \mathcal{A}(\chi)$, the *average competitive gap* of f is defined to be $\frac{1}{|A|} \sum_{a \in A} \phi(p, f, a)$.

3 Convergence of the Geanakoplos Bargaining Set

The Davis-Maschler definition (and implicitly, the Aumann-Maschler definition) of the bargaining set require that an objection be put forward by a *leader*, an individual who proposes the objection. A counterobjecting coalition is then required to exclude the leader. Geanakoplos modified this definition to require that an objection be proposed by a *group* of leaders, *none* of whom can be included in a counterobjecting coalition. If $\delta \in [0,1]$, a δ-objection is one in which the number of leaders is at most $\delta|A|$. Formally, we have the following definition:

Definition 3.1 (Geanakoplos) Let $\chi : A \to \mathcal{P} \times \mathbf{R}_+^k$ be an exchange economy, f an allocation. (S,U,g) is a δ-objection to f if S is a coalition, $U \subset S$, $\frac{|U|}{|S|} \le \delta$, $g : S \to \mathbf{R}_+^k$, $\sum_{a\in S} g(a) \le \sum_{a\in S} e(a)$, and

$$g(a) \succeq_a f(a) \text{ for all } a \in S \text{ with strict preference for at least one } a. \tag{2}$$

(T,h) is a counterobjection to (S,U,g) if T is a coalition, $T \cap U = \emptyset$, $h : T \to \mathbf{R}_+^k$, $\sum_{a\in T} h(a) \le \sum_{a\in T} e(a)$, and

$$h(a) \succeq_a \begin{cases} g(a) & \text{if } a \in T \cap S \\ f(a) & \text{if } a \in T \setminus S. \end{cases} \tag{3}$$

with strict preference for at least one agent a. A δ-objection is justified if there is no counterobjection. f is in the δ-bargaining set, denoted $\mathcal{B}_\delta(\chi)$, if every δ-objection to f has a counterobjection.[1]

We will present two main convergence theorems; the hypotheses of the theorems are incomparable, while the conclusions of the second theorem are stronger than the conclusions of the first. In the first (Theorem 3.4), preferences are assumed to be strongly monotone and tight, i.e. for every $\sigma > 0$,

[1] It may be natural to impose additional restrictions on counterobjections. Zhou [22] has proposed adding three restrictions: $T \cap S \neq \emptyset$; $T \not\subset S$; and $S \not\subset T$. The third of these conditions is automatically satisfied provided $U \neq \emptyset$. Note that imposing additional restrictions on counterobjections makes it easier to propose a justified objection, and thus makes the bargaining set smaller. Thus, the convergence results in this paper would still apply if Zhou's additional restrictions were imposed.

there is a compact set K of strongly monotone preferences such that the proportion of agents with preferences in K is at least $1-\sigma$. Note that a compact set of strongly monotone preferences is equimonotone (see Anderson [4]). This first theorem also requires uniformly integrable endowments, though it does not require that the social endowment of each good be bounded away from 0. It establishes no rate of convergence. The second theorem (Theorem 3.6) assumes that there is a set P of preferences exhibiting bounded marginal rates of substitution such that the proportion of agents with preferences in P is bounded away from 0. Instead of uniform integrability of endowments, we only require that the largest individual endowment be small compared to the number of agents; however, we do require that the *per capita* social endowment be bounded away from 0 in each commodity. In the event that endowments are uniformly bounded, the rate of convergence is arbitrarily close to the inverse of the number of agents. The essence of the proof of both theorems is contained in a main proposition (Proposition 3.10), which analyzes a fixed finite economy. The proof of this proposition is conceptually the same as Mas-Colell's proof in the continuum case, with the Shapley-Folkman Theorem substituting for Richter's Theorem; however, the derivation of the needed estimates is quite complex. The derivation of the second theorem from Proposition 3.10 is quite straightforward, while the derivation of the first theorem is somewhat indirect.

Definition 3.2 Given an allocation f and a price p, define

$$L(p,f) = \sum_{a\in A} \max\{0, p\cdot e(a) - \inf\{p\cdot x : x \succ_a f(a)\}\}. \tag{4}$$

Our results establish bounds on $L(p,f)$ when f is in the bargaining set. Their significance comes from the following proposition.

Proposition 3.3 *Given an allocation f,*

$$\begin{aligned} \textstyle\sum_{a\in A} |p\cdot(f(a)-e(a))| &\leq 2L(p,f) \\ \textstyle\sum_{a\in A} |\inf\{p\cdot(x-e(a)) : x\succ_a f(a)\}| &\leq 2L(p,f). \end{aligned} \tag{5}$$

Proof: This is contained in the proof of Theorem 1 of Anderson [1]. ∎

Theorem 3.4 *Suppose*

$$\chi_n : A_n \to \mathcal{P}_{mo} \times \mathbf{R}^k_+ \tag{6}$$

is a sequence of economies satisfying the following conditions:

1. $|A_n| \to \infty$;

2. e_n *is uniformly integrable, i.e.*

$$\frac{|S_n|}{|A_n|} \to 0 \Longrightarrow \frac{1}{|A_n|} \sum_{a \in S_n} e_n(a) \to 0; \tag{7}$$

   and

3. *for all* $\rho > 0$, *there is a compact*[2] *set* $K \subset \mathcal{P}_{mo}$ *such that*

$$\frac{|\{a \in A_n : \succ_a \in K\}|}{|A_n|} > 1 - \rho. \tag{8}$$

Then there exists $\delta_n \to 0$ *such that for all* $f_n \in \mathcal{B}_{\delta_n}(\chi_n)$ $\exists p_n \in \Delta^0$ *such that*

$$\frac{L(p_n, f_n)}{|A_n|} \to 0. \tag{9}$$

Suppose χ_n *satisfies Assumptions 1, 2 and 3, and in addition*

4. $$\liminf_{n \to \infty} \frac{1}{|A_n|} \sum_{a \in A_n} e_n(a) \gg 0. \tag{10}$$

Then there is a compact set $D \subset \Delta^0$ *such that* $p_n \in D$ *for all* n.

Suppose χ_n *satisfies Assumptions 1, 2 and 4, and in addition*

5. *Assumption 3 holds for some compact set* $K \subset \mathcal{P}_{mo} \cap \mathcal{P}_{sc}$.

[2] We give the set of preferences the topology of closed convergence (Hildenbrand [17], page 96.)

Then

$$\frac{1}{|A_n|}\left\|\sum_{a\in A_n} f_n(a) - D(p_n,a)\right\|_\infty \le \frac{1}{|A_n|}\sum_{a\in A_n}\|f_n(a) - D(p_n,a)\|_\infty \to 0. \quad (11)$$

Suppose χ_n satisfies

6. *endowments are uniformly bounded, i.e.*

$$\sup\{\|e_n(a)\|_\infty : \ a\in A_n, \ n\in \mathbf{N}\} < \infty;. \quad (12)$$

Then Equation (9) (under Assumptions 1, 2, 3 and 6) and Equation (11) (under Assumptions 1, 2, 4, 5 and 6) hold for all *sequences δ_n such that $\delta_n|A_n|\to\infty$.*

Definition 3.5 We say that a set $P\subset \mathcal{P}_{mo}$ exhibits bounded marginal rates of substitution if

$$\begin{aligned}\exists\alpha\ \forall \succ\in P\ \forall x,y,z\in \mathbf{R}^k_+\ \|z\|_1 < \alpha\|y\|_1, x+y-z\in \mathbf{R}^k_+ \\ \Longrightarrow x+y-z\succ x.\end{aligned} \quad (13)$$

Note that if P exhibits bounded marginal rates of substitution, there exists a compact set K such that $P\subset K\subset \mathcal{P}_{mo}$.

Theorem 3.6 *Suppose*

$$\chi_n : A_n \to \mathcal{P}_{mo}\times \mathbf{R}^k_+ \quad (14)$$

is a sequence of economies satisfying the following conditions:

1. $|A_n|\to\infty$;
2. $$\sup_n \frac{1}{|A_n|}\|\sum_{a\in A_n} e(a)\|_\infty < \infty; \quad (15)$$

   and

   $$\frac{\max\{\|e(a)\|_\infty : a\in A_n\}}{|A_n|}\to 0; \quad (16)$$

   and

3. *there exists* $\sigma > 0$ *and a set* $P \subset \mathcal{P}_{mo}$ *exhibiting bounded marginal rates of substitution such that for all* i

$$\frac{|\{a \in A_n : \succ_a \in P,\ e(a)^i > \sigma\}|}{|A_n|} > \sigma. \tag{17}$$

Then there exists $\delta_n \to 0$ *and a compact set* $D \subset \Delta^0$ *such that for all* $f_n \in \mathcal{B}_{\delta_n}(\chi_n)$ *there exists* $p_n \in D$ *such that*

$$\frac{L(p_n, f_n)}{|A_n|} \to 0. \tag{18}$$

Suppose χ_n *satisfies Assumptions 1, 2, and 3, and in addition*

4. *for all* $\rho > 0$*, there is a compact set* $K \subset \mathcal{P}_{mo} \cap \mathcal{P}_{sc}$ *such that*

$$\frac{|\{a \in A_n : \succ_a \in K\}|}{|A_n|} > 1 - \rho. \tag{19}$$

Then $f_n(a) - D(p_n, a)$ *converges to* 0 *in measure, i.e.*

$$\forall \epsilon > 0 \ \frac{|\{a \in A_n : \|f_n(a) - D(p_n, a)\| > \epsilon\}|}{|A_n|} \longrightarrow 0. \tag{20}$$

Suppose χ_n *satisfies Assumptions 1, 2, and 3, and in addition*

5. *endowments are uniformly bounded, i.e.*

$$\sup\{\|e_n(a)\|_\infty :\ a \in A_n,\ n \in \mathbf{N}\} < \infty;. \tag{21}$$

Then Equation (18) holds for all *sequences* δ_n *such that* $\delta_n |A_n| \to \infty$*. Moreover, given any sequence* $J_n \to \infty$*, there exists* $\delta_n \to 0$ *such that*

$$\forall f_n \in \mathcal{B}_{\delta_n}(\chi_n)\ \exists p_n \in D\ \frac{L(p_n, f_n)}{|A_n|} \le \frac{J_n}{|A_n|}. \tag{22}$$

We need first to prove two lemmas.

Lemma 3.7 *Suppose there is a compact set* $K \subset (\mathcal{P}_{mo} \cup \mathcal{P}_{ss}) \times \mathbf{R}^k_+$*, a sequence* $p_n \in \Delta^0$ *with* $p_n \to p \in \Delta \setminus \Delta^0$*, and a sequence of characteristics* $(\succ_n, e_n) \in K$ *such that* $\inf p_n \cdot e_n > 0$*. Then*

$$\inf\{\|x\|_\infty : x \in D(p_n, \succ_n, e_n)\} \to \infty \tag{23}$$

as $n \to \infty$*.*

Proof: If the conclusion is false, we can find a subsequence such that $\succ_n \to \succ \in (\mathcal{P}_{mo} \cup \mathcal{P}_{ss})$, $e_n \to e \in \mathbf{R}^k_+$, and there exists $x_n \in D(p_n, \succ_n, e_n)$ with $x_n \to x \in \mathbf{R}^k_+$.

$$p \cdot x = \lim_{n\to\infty} p_n \cdot x_n \leq \lim_{n\to\infty} p_n \cdot e_n = p \cdot e > 0. \tag{24}$$

If there exists y with $p \cdot y \leq p \cdot e$ and $y \succ x$, then there exists z with $p \cdot z < p \cdot e$ and $z \succ x$. For n sufficiently large, $p_n \cdot z < p_n \cdot e_n$ and $z \succ_n x_n$, which contradicts the assumption that $x_n \in D(p_n, \succ_n, e_n)$. Therefore, $x \in D(p, \succ, e)$. We consider three cases:

1. If $\succ \in \mathcal{P}_{mo}$, then $D(p, \succ, e) = \emptyset$, since $p \in \Delta \setminus \Delta^0$, contradiction.

2. If $\succ \in \mathcal{P}_{ss}$ and $x \in \mathbf{R}^k_{++}$, then strong monotonicity on $\mathbf{R}^k_{++}$ and $p \in \Delta \setminus \Delta^0$ implies $x \notin D(p, \succ, e)$, contradiction.

3. If $\succ \in \mathcal{P}_{ss}$ and $x \in \mathbf{R}^k_+ \setminus \mathbf{R}^k_{++}$, note there exists $y \in \mathbf{R}^k_{++}$ such that $p \cdot y = p \cdot e$. By the boundary condition, $y \succ x$, contradiction.

Since all three cases lead to a contradiction, the proof is complete. ■

Lemma 3.8 *Suppose* $w \in \mathbf{R}^k_+$*,* $B_n \subset \mathbf{R}^k_+ - w$ *and*

$$\inf\{\|x\|_\infty :\ x \in B_n\} \to \infty. \tag{25}$$

Then

$$\inf\{\|x\|_\infty :\ x \in \text{con } B_n\} \to \infty. \tag{26}$$

Proof: If not, then we may (by taking a subsequence) find $M \in \mathbf{R}$ and $x_n \in \text{con } B_n$ with $\|x_n\|_\infty \leq M$. By Caratheodory's Theorem, for each n, there exist convex coefficients $\lambda_{n0}, \ldots, \lambda_{nk}$ and $x_{ni} \in B_n (0 \leq i \leq k)$ such that $\sum_{i=0}^k \lambda_i x_{ni} = x_n$. Given any $y \in \mathbf{R}^k_+$ and $y_1, \ldots, y_m \in \mathbf{R}^k_+$ with

$y = y_1 + \cdots + y_m$, $\frac{1}{k}(\|y_1\|_\infty + \cdots + \|y_m\|_\infty) \leq \|y\|_\infty \leq \|y_1\|_\infty + \cdots + \|y_m\|_\infty$. Therefore,

$$\begin{aligned} \|x_n + w\|_\infty &\geq \tfrac{1}{k} \textstyle\sum_{i=0}^{k} \lambda_i \|x_{ni} + w\|_\infty \\ &\geq \tfrac{1}{k} \min\{\|x_{n0} + w\|_\infty, \ldots, \|x_{nk} + w\|_\infty\} \\ &\geq \tfrac{1}{k} \min\{\|x + w\|_\infty : x \in B_n\} \to \infty. \end{aligned} \tag{27}$$

■

Definition 3.9 Suppose $\chi : A \to \mathcal{P} \times \mathbf{R}_+^k$ and $f \in \mathcal{B}_\delta(\chi)$. Let $m = \lfloor |A|\delta \rfloor$. If $a \in A$ and $p \in \Delta$, define

$$\begin{aligned} I(p,a) &= \inf\{p \cdot x : x \succeq_a f(a)\} \\ L(p,f,a) = L(p,a) &= \max\{p \cdot e(a) - I(p,a), 0\} \\ L(p)(= L(p,f)) &= \textstyle\sum_{a \in A} L(p,a) \\ A(p) &= \{a \in A : |\{b \in A : L(p,b) \geq L(p,a)\}| \leq m\} \\ B(p) &= \{a \in A : |\{b \in A : L(p,b) > L(p,a)\}| < m\} \\ C(p,a) &= D(p, (\succ_a, (I(p,a), \ldots, I(p,a)))) \end{aligned} \tag{28}$$

where D is the demand as defined in section 2. Thus, $I(p,a)$ is the minimum income needed for a to achieve the utility level of $f(a)$. $L(p,a)$ is a's income loss at the price vector p, i.e. the amount of income a could forego while still achieving the utility level of $f(a)$; $L(p)$ is the aggregate income loss over all agents. $B(p)$ is the set of agents with the m biggest income losses (including agents tied for the mth biggest loss if there is a tie), while $A(p)$ is the set of agents with the m biggest losses, excluding agents tied for the dth biggest loss.[3] Furthermore, $A(p) \subset B(p)$ with equality unless the mth and $m+1$st

[3] More precisely, fix $t \in \mathbf{R}_+$ and let $S = \{a \in A : L(p,a) = t\}$, $T = \{a \in A : L(p,a) > t\}$. Then $S \subset B(p)$ iff $|T| < m$ and $S \cap B(p) = \emptyset$ iff $|T| \geq m$. $S \subset A(p)$ iff $|T \cup S| \leq m$ and $S \cap A(p) = \emptyset$ iff $|T \cup S| > m$. Notice that $S \cap A(p) \neq \emptyset \Leftrightarrow S \subset A(p)$ and $S \cap B(p) \neq \emptyset \Leftrightarrow S \subset B(p)$.

largest losses are equal. $C(p,a)$ is the demand set of an agent with preference $\succ_a$ and income $I(p,a)$. For each $a \in A$, define $\gamma_a : \Delta^0 \to \mathbf{R} \times \mathbf{R}^k$ by

$$\gamma_a(p) = \begin{cases} \{(0,0)\} & \text{if } f(a) \succ_a D(p,a) \\ & \text{and } a \notin B(p) \\ \{(0,0),(1,0)\} & \text{if } f(a) \succ_a D(p,a) \\ & \text{and } a \in B(p) \setminus A(p) \\ \{(1,0)\} & \text{if } f(a) \succ_a D(p,a) \\ & \text{and } a \in A(p) \\ \{0\} \times ((D(p,a) - e(a)) \cup \{0\}) & \text{if } f(a) \sim_a D(p,a) \\ & \text{and } a \notin B(p) \\ \{0,1\} \times ((D(p,a) - e(a)) \cup \{0\}) & \text{if } f(a) \sim_a D(p,a) \\ & \text{and } a \in B(p) \setminus A(p) \\ \{1\} \times ((D(p,a) - e(a)) \cup \{0\}) & \text{if } f(a) \sim_a D(p,a) \\ & \text{and } a \in A(p) \\ (\{0\} \times (D(p,a) - e(a))) & \text{if } D(p,a) \succ_a f(a) \\ & \text{and } a \notin B(p) \\ (\{0\} \times (D(p,a) - e(a))) & \text{if } D(p,a) \succ_a f(a) \\ \quad \cup (\{1\} \times (C(p,a) - e(a))) & \text{and } a \in B(p) \setminus A(p) \\ (\{1\} \times (C(p,a) - e(a))) & \text{if } D(p,a) \succ_a f(a) \\ & \text{and } a \in A(p) \end{cases} \tag{29}$$

and

$$\Gamma(p) = \sum_{a \in A} \gamma_a(p). \tag{30}$$

Let

$$\Gamma'(p) = \text{con}\ \{x \in \mathbf{R}^k : (m,x) \in \Gamma(p)\} \tag{31}$$

and

$$\Gamma''(p) = \{x - (p \cdot x, \cdots, p \cdot x) : x \in \Gamma'(p)\}. \tag{32}$$

Proposition 3.10 *Suppose* $\chi : A \to (\mathcal{P}_{mo} \cup \mathcal{P}_{ss}) \times \mathbf{R}^k_+$. *If* $f \in \mathcal{B}_\delta(\chi)$, *then either*

$$\exists p \in \Delta^0 \;\; L(p, f) \le M_\chi^{\lfloor |A|\delta \rfloor} + 1 \tag{33}$$

or

$$\exists p \in \Delta^0 \left[0 \in \Gamma''(p) \;\&\; L(p, f) \le \left(\frac{|A|}{\lfloor |A|\delta \rfloor} \right) \left(\frac{M_\chi^{k+1}}{C_p} \right) \right], \tag{34}$$

where $C_p = \min\{p^1, \ldots, p^k\}$.[4] *Moreover, if there is a set* $P \subset \mathcal{P}_{mo}$ *of preferences which exhibits bounded marginal rates of substitution such that*

$$\exists \tau > \delta \; \exists \sigma > 0 \; \forall i \; \frac{|\{a \in A : e(a)^i > \sigma, \succ_a \in P\}|}{|A|} > 3\tau, \tag{35}$$

then there is a positive constant C *depending only on* $\|\frac{1}{A} \sum_{a \in A} e(a)\|_\infty$, σ, τ, *and* P, *such that Equation (34) holds and* $C_p > C$.

Proof:

1. Recall that

$$\Gamma'(p) = \text{con}\; \{x \in \mathbf{R}^k : (m, x) \in \Gamma(p)\}. \tag{36}$$

   We will show that Equation (33) is satisfied or there exists $p \in \Delta^0$ and $s \ge 0$ such that $x = (-s, \cdots, -s) \in \Gamma'(p)$. The argument proceeds in several steps:

   (a) If $a \in B(p)$, then there exists x_a such that $(1, x_a) \in \gamma_a(p)$. Since $|B(p)| \ge m$ and $|A(p)| \le m$, $\Gamma'(p) \ne \emptyset$.

   (b) If $x \in \Gamma'(p)$, then $p \cdot x \le 0$.

   (c) $\Gamma'(p)$ is convex-valued for all $p \in \Delta^0$.

   (d) For all $p \in \Delta^0$, $\Gamma'(p) \ge -\sum_{a \in A} e(a)$.

[4] Recall $M_\chi^m = \max\{\|e(a_1) + \cdots + e(a_m)\|_\infty : a_1, \ldots a_m$ are distinct elements of A$\}$.

(e) We show that γ_a is upper hemicontinuous on Δ^0 for each $a \in A$. Suppose $p \in \Delta^0$. Then there exists a neighborhood V of p and $\alpha > 0$ such that $q \in V \Rightarrow \min\{p^1, \cdots p^k\} \geq \alpha$. If $(t, x) \in \gamma_a(q)$ for some $q \in V$, we have $t \in \{0, 1\}$, $q \cdot x \leq 0$ and $x \geq -\sum_{a \in A} e(a)$. Accordingly, there is a compact set K containing $\cup_{q \in V} \gamma_a(q)$. Therefore, it suffices to show that γ_a has closed graph at p. Accordingly, suppose that $p_n \to p$, $(t_n, x_n) \in \gamma_a(p_n)$, and $(t_n, x_n) \to x$. Note that if $a \notin B(p)$, then $a \notin B(p_n)$ for n sufficiently large; and if $a \in A(p)$, then $a \in A(p_n)$ for n sufficiently large.

i. If $f(a) \succ_a D(p, a)$, then $f(a) \succ_a D(p_n, a)$ for sufficiently large n.

A. If $a \notin B(p)$, then $a \notin B(p_n)$ for n sufficiently large, so $\gamma_a(p_n) = \{(0, 0)\} = \gamma_a(p)$.

B. If $a \in B(p) \backslash A(p)$, then $\gamma_a(p_n)$ is defined by either the first, second, or third line of Equation (29), so $\gamma_a(p_n) \subset \gamma_a(p)$ for all sufficiently large n.

C. If $a \in A(p)$, then $a \in A(p_n)$ for all sufficiently large n, so $\gamma_a(p_n) = \{(1, 0)\} = \gamma_a(p)$.

ii. If $f(a) \sim_a D(p, a)$, there are three subcases to consider:

A. If $a \notin B(p)$, then $a \notin B(p_n)$ for n sufficiently large. Accordingly, $\gamma_a(p_n)$ is defined by the first, fourth, or seventh line of Equation (29). Since demand is upper hemicontinuous, $\limsup \gamma_a(p_n) \subset \gamma_a(p)$ in all three cases, so $(t, x) \in \gamma_a(p)$. [5]

B. If $a \in B(p) \setminus A(p)$, $\gamma_a(p_n)$ could be defined by *any* of the nine cases in Equation (29). However, $\limsup D(p_n, a) \subset D(p, a)$. Moreover, $I(p_n, a) \to I(p, a) = p \cdot e(a)$ since $f(a) \sim_a D(p, a)$, so $\limsup C(p_n, a) \subset D(p, a)$. From this, it follows that $\limsup \gamma_a(p_n) \subset \gamma_a(p)$.

C. If $a \in A(p)$, then $a \in A(p_n)$ for n sufficiently large, so $\gamma_a(p_n)$ is defined by the third, sixth or ninth line of Equation (29). As in item 1(e)iiB, $\limsup C(p_n, a) \subset D(p, a)$, so $\limsup \gamma_a(p_n) \subset \gamma_a(p)$.

[5] Given a sequence of sets $C_n \subset \mathbf{R}^k$, $\limsup C_n = \{x \in \mathbf{R}^k$: there is a subsequence n_i and $c_i \in C_{n_i}$ such that $c_i \to x\}$.

iii. If $D(p,a) \succ_a f(a)$, then $D(p_n,a) \succ_a f(a)$ for all n sufficiently large.

A. If $a \notin B(p)$, then $a \notin B(p_n)$ for all sufficiently large n, so $\gamma_a(p_n)$ and $\gamma_a(p)$ are both defined by the seventh line of Equation (29). Since D is upper hemicontinuous, $\limsup \gamma_a(p_n) \subset \gamma_a(p)$.

B. If $a \in B(p) \setminus A(p)$, then $\gamma_a(p_n)$ is defined by the seventh, eighth, or ninth line of Equation (29). Since D and C are upper hemicontinuous, $\limsup \gamma_a(p_n) \subset \gamma_a(p)$.

C. If $a \in A(p)$, then $a \in A(p_n)$ for all sufficiently large n, so $\gamma_a(p_n)$ and $\gamma_a(p)$ are both defined by the ninth line of Equation (29). Since C is upper hemicontinuous, $\limsup \gamma_a(p_n) \subset \gamma_a(p)$.

(f) We will show that $\Gamma'(p)$ is upper hemicontinuous as a function of $p \in \Delta^0$. Suppose $p_n \to p \in \Delta^0$, and $x_n \in \Gamma'(p_n)$. Observe that $p_n \cdot x_n \leq 0$ and $x_n \geq -\sum_{a\in A} e(a)$, so there is a compact set containing all x_n for n sufficiently large. Thus, it is sufficient to show that

$$x_n \to x \Longrightarrow x \in \Gamma'(p). \tag{37}$$

By Caratheodory's Theorem, there exist convex coefficients $\lambda_{ni}(0 \leq i \leq k)$, and $x_{ni} \in \mathbf{R}^k$ with $(m, x_{ni}) \in \Gamma(p_n)$ such that $x_n = \sum_{i=0}^k \lambda_{ni} x_{ni}$. Consequently, there exist $(t_{nia}, x_{nia}) \in \gamma_a(p_n)$ such that

$$m = \sum_{a\in A} t_{nia} \text{ and } x_{ni} = \sum_{a\in A} x_{nia}. \tag{38}$$

By taking a subsequence, we may assume without loss of generality that $\lambda_{ni} \to \lambda_{0i}$, $t_{nia} \to t_{0ia}$, and $x_{nia} \to x_{0ia}$ as $n \to \infty$. Since $(t_{0ia}, x_{0ia}) \in \gamma_a(p)$ for each $i \in \{1, \ldots, k\}$ and each $a \in A$; moreover, $\sum_{a\in A} t_{0ia} = m$ for each i and a. Therefore, letting

$$x_0 = \sum_{i=0}^k \lambda_{0i} \sum_{a\in A} x_{0ia}, \tag{39}$$

we see that $x_0 \in \Gamma'(p)$ and $x_0 = \lim_{n\to\infty} x_n$. This shows that Γ' is upper hemicontinuous on Δ^0.

(g) Suppose $p_n \to p \in \Delta \setminus \Delta^0$. We will show that either Equation (33) is satisfied or

$$\min\{\|x\|_\infty : x \in \Gamma'(p_n)\} \to \infty. \tag{40}$$

Let $(m, x_n) \in \Gamma(p_n)$. Then $m = \sum_{a\in A} t_{na}$ and $x_n = \sum_{a\in A} x_{na}$ with $(t_{na}, x_{na}) \in \gamma_a(p_n)$.

i. Choose a sequence a_n with $t_{na_n} = 0$ and

$$L(p_n, a_n) \geq L(p_n, a) \text{ for all } a \text{ with } t_{na} = 0. \tag{41}$$

A. If there is a subsequence p_{n_i} such that $L(p_{n_i}, a_{n_i}) \to 0$, then

$$\begin{aligned} L(p_{n_i}) &= \textstyle\sum_{a\in A} L(p_{n_i}, a) \\ &\leq \textstyle\sum_{a\in A} t_{n_i a} L(p_{n_i}, a) + (|A| - m) L(p_{n_i}, a_{n_i}) \\ &\leq \textstyle\sum_{a\in A} t_{n_i a} p_{n_i} \cdot e(a) \; + \; |A| L(p_{n_i}, a_{n_i}) \\ &\leq M_\chi^m + 1 \end{aligned} \tag{42}$$

for n sufficiently large, which establishes Equation (33).

B. If there exists $\rho > 0$ such that $L(p_n, a_n) \geq \rho$ for all n, then $p_n \cdot e(a_n) \geq \rho$ for all n. Since $L(p_n, a_n) > 0$, $D(p_n, a_n) \succ_{a_n} f(a_n)$; since $t_{na_n} = 0$, $x_{na_n} \in D(p_n, a_n) - e(a_n)$. Since $\succ_{a_n}$ comes from the finite family $\{\succ_a : a \in A\} \subset (\mathcal{P}_{mo} \cup \mathcal{P}_{ss})$ and $p_n \cdot e(a_n) \geq \rho$, $\min\{\|x\|_\infty : x \in D(p_n, a_n)\} \to \infty$ by Lemma 3.7. Since in addition $x_{na} \geq -e(a)$ for all $a \in A$,

$$\min\{\|x\|_\infty : (m, x) \in \Gamma(p_n)\} \to \infty \tag{43}$$

Then

$$\min\{\|x\|_\infty : x \in \Gamma'(p_n)\} \to \infty \tag{44}$$

by Lemma 3.8.

ii. Suppose Equation (35) is satisfied. Find P which exhibits bounded marginal rates of substitution (with constant α) and $\tau > \delta$, $\sigma > 0$, such that

$$\forall i \; \frac{|\{a \in A : e(a)^i > \sigma, \succ_a \in P\}|}{|A|} > 3\tau. \tag{45}$$

Let

$$M = \frac{1}{\tau|A|}\|\sum_{a\in A} e(a)\|_1$$

$$\begin{array}{c} S_n = \{a \in A : p_n \cdot e(a) \geq \frac{\sigma}{k}, \\ \|f(a)\|_1 < M,\ t_{na} = 0,\ \succ_a \in P\}. \end{array} \tag{46}$$

A. We claim that $\frac{|S_n|}{|A|} > \tau$. To see this, note that there exists i_n such that $p_n^{i_n} \geq \frac{1}{k}$. Then

$$\begin{aligned} |S_n| &\geq |\{a \in A : e(a)^{i_n} > \sigma,\ \succ_a \in P\}| \\ &\quad -|\{a \in A : \|f(a)\|_1 \geq M\}| - |\{a \in A : t_{na} = 1\}| \\ &> 3\tau|A| - \frac{\|\sum_{a\in A} f(a)\|_1}{M} - m \\ &\geq 3\tau|A| - \frac{\tau|A|M}{M} - \tau|A| = \tau|A|. \end{aligned} \tag{47}$$

B. For n sufficiently large, there exists i_n such that

$$p_n^{i_n} < \frac{\sigma}{k\alpha M}. \tag{48}$$

Let

$$y_n = (0, \ldots, 0, \alpha M, 0, \ldots 0), \tag{49}$$

where the non-zero entry occurs in the $(i_n)^{th}$ component. Then

$$p_n \cdot y_n \leq \left(\frac{\sigma}{k\alpha M}\right)\alpha M = \frac{\sigma}{k}. \tag{50}$$

If $a \in S_n$, $y_n = f(a) + y_n - f(a) \succ_a f(a)$, since $\|y_n\|_1 = \alpha M > \alpha\|f(a)\|_1$. Therefore, if $a \in S_n$, $D(p_n, a) \succ_a f(a)$ and $t_{na} = 0$, so $x_{na} \in D(p_n, a) - e(a)$. Therefore,

$$\begin{aligned} x_n = \sum_{a\in A} x_{na} &\geq \sum_{a\in S_n} x_{na} - \sum_{a\in A\setminus S_n} e(a) \\ &= \sum_{a\in S_n}(x_{na} + e(a)) - \sum_{a\in A} e(a). \end{aligned} \tag{51}$$

Observe that

$$\begin{aligned} &\|\sum_{a\in S_n} x_{na} + e(a)\|_\infty \\ &\geq \frac{\tau|A|}{k}\inf\{\|x\|_\infty : x \in D(p_n, \succ, e),\ p_n \cdot e > \frac{\sigma}{k},\ \succ \in P\} \end{aligned} \tag{52}$$

so

$$\|\tfrac{1}{|A|}x_n\|_\infty$$

$$\geq \tfrac{\tau}{k}\inf\{\|x\|_\infty : x \in D(p_n, \succ, e),\ p_n \cdot e > \tfrac{\sigma}{k},\ \succ \in P\} \\ -\|\tfrac{1}{|A|}\textstyle\sum_{a\in A} e(a)\|_\infty \tag{53}$$

which tends to infinity by Lemma 3.7. Therefore,

$$\min\{\frac{\|x\|_\infty}{|A|} : (m,x) \in \Gamma(p_n)\} \to \infty, \tag{54}$$

so

$$\min\{\frac{\|x\|_\infty}{|A|} : (m,x) \in \Gamma'(p_n)\} \to \infty \tag{55}$$

by Lemma 3.8. Note that the rate depends only on σ, τ, P, and $\frac{1}{|A|}\|\sum_{a\in A} e(a)\|_\infty$.

(h) Recall that

$$\Gamma''(p) = \{x - (p \cdot x, \cdots, p \cdot x) : x \in \Gamma'(p)\}. \tag{56}$$

The following properties of Γ'' follow from the corresponding properties of Γ':

i. If $x \in \Gamma''(p)$, then $p \cdot x = 0$.

ii. $\Gamma''(p)$ is nonempty and convex-valued for all $p \in \Delta^0$.

iii. $\Gamma''(p)$ is upper hemicontinuous as a function of $p \in \Delta^0$.

iv. If $p_n \to p \in \Delta \setminus \Delta^0$, then $\min\{\|x\|_\infty : x \in \Gamma''(p_n)\} \to \infty$.

Therefore, by Theorem 8 on page 722 of Debreu [9], there exists $p \in \Delta^0$ such that $0 \in \Gamma''(p)$. But then there exists $x \in \Gamma'(p)$ such that $x = (-s, \ldots, -s)$ for some $s \geq 0$.

(i) Suppose Equation (35) holds. By item 1(g)ii, there is a positive constant C (depending only on σ, τ, P and $\frac{1}{|A|}\|\sum_{a\in A} e(a)\|_\infty$) such that if $C_p < C$, then $x \in \Gamma'(p)$ implies $\|x\|_\infty > \|\sum_{a\in A} e(a)\|_\infty$. But if p is chosen as in item 1h, there exists $x \in \Gamma'(p)$ with $x \leq 0$. Since $x \geq -\sum_{a\in A} e(a)$, $\|x\|_\infty \leq \|\sum_{a\in A} e(a)\|_\infty$, contradiction. Hence, $C_p \geq C$.

2. Let $\lambda = \sup\{s : (-s, \ldots, -s) \in \Gamma'(p)\}$, $z = (-\lambda, \ldots, -\lambda)$, $n = |A|$.[6] We claim that

$$L(p) \leq -\frac{n}{m} p \cdot z. \tag{57}$$

By the definition of z, there exists λ' such that $(\lambda', \ldots, \lambda') \in \Gamma'(p)$ and $p \cdot z \leq p \cdot (\lambda', \cdots, \lambda') = \lambda'$. By the definition of $\Gamma'(p)$ (Equation 31),

$$(\lambda', \ldots, \lambda') \in \text{con}\ \{x \in \mathbf{R}^k : (m, x) \in \Gamma(p)\}. \tag{58}$$

Therefore, there exists $(m, x) \in \Gamma(p)$ such that $p \cdot x \geq p \cdot (\lambda', \ldots, \lambda') = \lambda' \geq p \cdot z$. There exist $(t_a, x_a) \in \gamma_a(p)$ such that $(m, x) = \sum_{a \in A}(t_a, x_a)$. If $t_a = 0$, then $x_a \in (D(p, a) \cup \{0\})$, so $p \cdot x_a = 0$. We will show that, whenever $t_a = 1$, $p \cdot x_a = -L(p, a)$. Suppose $t_a = 1$.

   (a) If $D(p, a) \succ_a f(a)$, then $x_a \in C(p, a) - e(a)$ and thus $p \cdot x_a = I(p, a) - p \cdot e(a) = -L(p, a)$.

   (b) If $f(a) \sim_a D(p, a)$, then $x_a \in (D(p, a) - e(a)) \cup \{0\}$, so $p \cdot x_a = 0 = -L(p, a)$.

   (c) If $f(a) \succ_a D(p, a)$, then $x_a = 0$, so $p \cdot x_a = 0 = -L(p, a)$.

Number the agents in A so that

$$L(p, a_1) \geq L(p, a_2) \geq \cdots \geq L(p, a_n). \tag{59}$$

If $B(p) \setminus A(p) \neq \emptyset$, let L be the common value of $L(p, a)$ for $a \in B(p) \setminus A(p)$.

$$\begin{aligned} p \cdot z \leq p \cdot x = -\textstyle\sum_{a \in A} t_a L(p, a) &= -\left[\textstyle\sum_{a \in A(p)} L(p, a) + (m - |A(p)|)L\right] \\ &= -\textstyle\sum_{i=1}^{m} L(p, a_i) \leq -\frac{m}{n} L(p), \end{aligned} \tag{60}$$

so $L(p) \leq -\frac{n}{m} p \cdot z$.

3. We claim there exists $(t, x) \in \Gamma(p)$ with $0 \leq t \leq m$ such that

$$x \leq z + \left(\frac{M_\chi^{k+1}}{C_p}, \ldots, \frac{M_\chi^{k+1}}{C_p}\right). \tag{61}$$

[6] One can show that $z \in \Gamma'(p)$, but we will only use the fact that $z \in \overline{\Gamma'(p)}$.

(a) Fix $\epsilon > 0$, and choose $y \in \Gamma'(p)$ with $y \leq z+(\epsilon, \ldots, \epsilon)$. Thus, there exist convex coefficients $\lambda_0, \ldots, \lambda_k$ and $(m, y_0), \ldots, (m, y_k) \in \Gamma(p)$ such that

$$y = \sum_{i=0}^{k} \lambda_i y_i, \tag{62}$$

so

$$(m, y) = \sum_{i=0}^{k} \lambda_i (m, y_i) \in \text{con} \sum_{a \in A} \gamma_a(p). \tag{63}$$

(b) By the Shapley-Folkman Theorem, we can find $\{a_0, \ldots, a_{k'}\} \subset A$ with $k' \leq k$ and

$$(S_a, Y_a) \in \begin{cases} \gamma_a(p) & \text{if } a \notin \{a_0, \ldots, a_{k'}\} \\ \text{con } \gamma_a(p) & \text{if } a \in \{a_0, \ldots, a_{k'}\} \end{cases} \tag{64}$$

such that

$$(m, y) = \sum_{a \in A} (S_a, Y_a). \tag{65}$$

Choose

$$(t_{a_i}, x_{a_i}) \in \gamma_{a_i}(p) \text{ with } t_{a_i} \leq S_{a_i} \text{ if } i \in \{0, \ldots, k'\} \tag{66}$$

and let

$$(t_a, x_a) = (S_a, Y_a) \text{ if } a \notin \{a_0, \ldots a_{k'}\}. \tag{67}$$

Let $t = \sum_{a \in A} t_a$ and $x = \sum_{a \in A} x_a$. Then $0 \leq t = \sum_{a \in A} t_a \leq \sum_{a \in A} S_a = m$ and $(t_a, x_a) \in \gamma_a(p)$ for all $a \in A$. Therefore, $(t, x) \in \Gamma(p)$.

(c)

$$\begin{aligned} x = y + \textstyle\sum_{i=0}^{k'} (x_{a_i} - Y_{a_i}) &= y + \textstyle\sum_{i=0}^{k'} ((x_{a_i} + e(a_i)) - (Y_{a_i} + e(a_i))) \\ &\leq y + \textstyle\sum_{i=0}^{k'} (x_{a_i} + e(a_i)) \leq y + \textstyle\sum_{i=0}^{k'} \left(\frac{p \cdot e(a_i)}{C_p}, \ldots, \frac{p \cdot e(a_i)}{C_p} \right) \\ &\leq z + (\epsilon, \ldots, \epsilon) + \left(\frac{M_\chi^{k+1}}{C_p}, \ldots, \frac{M_\chi^{k+1}}{C_p} \right). \end{aligned} \tag{68}$$

Since ϵ is arbitrary,

$$x \leq z + \left(\frac{M_\chi^{k+1}}{C_p}, \ldots, \frac{M_\chi^{k+1}}{C_p} \right). \tag{69}$$

4. Suppose there exists (t, x) with $0 \leq t \leq m$ and $x \ll 0$ with $(t, x) \in \Gamma(p)$. We will show that $f \notin \mathcal{B}_\delta(\chi)$, a contradiction.

   (a) There exists (t_a, x_a) in $\gamma_a(p)$ $(a \in A)$ such that $\sum_{a \in A}(t_a, x_a) = (t, x)$. Let

   $$S = \{a \in A : x_a + e(a) \in D(p, a) \cup C(p, a)\}, \qquad U = \{a \in S : t_a = 1\}. \tag{70}$$

   Since $x \neq 0$, $S \neq \emptyset$. Fix $a_0 \in S$ and define

   $$g(a) = \begin{cases} x_a + e(a) & \text{if } a \in S \setminus \{a_0\} \\ x_{a_0} + e(a_0) - x & \text{if } a = a_0 \end{cases} \tag{71}$$

   We will show that (S, U, g) is a justified δ-objection to f, and hence $f \notin \mathcal{B}_\delta(\chi)$, a contradiction.

      i. Observe that

      $$\frac{|U|}{|A|} \leq \frac{t}{|A|} \leq \frac{m}{|A|} = \frac{\lfloor \delta |A| \rfloor}{|A|} \leq \delta. \tag{72}$$

      Moreover,

      $$\sum_{a \in S} g(a) = \sum_{a \in S} (x_a + e(a)) - x = x + \sum_{a \in S} e(a) - x = \sum_{a \in S} e(a). \tag{73}$$

      We need to show that $g(a) \succeq_a f(a)$ for all $a \in S$, with strict preference for some $a \in S$. We need to consider two cases:

      A. *Case I:* $a \in S \setminus \{a_0\}$. Either

      $$g(a) = D(p, a) \succeq_a f(a) \text{ or } g(a) = C(p, a) \sim_a f(a), \tag{74}$$

      so $g(a) \succeq_a f(a)$.

      B. *Case II:* $a = a_0$. Either

      $$g(a_0) = D(p, a_0) - x \gg D(p, a_0) \succeq_{a_0} f(a_0) \text{ or} \tag{75}$$

      or

      $$g(a_0) = C(p, a_0) - x \gg C(p, a_0) \sim_{a_0} f(a_0). \tag{76}$$

Since $\succ_{a_0} \in \mathcal{P}_{mo} \cup \mathcal{P}_{ss}$, $\succ_{a_0}$ is weakly monotonic on $\mathbf{R}^k_+$; by Lemma 2.1 of Anderson, Trockel and Zhou [5], $g(a_0) \succ_{a_0} f(a_0)$.

Therefore (S, U, g) is a δ-objection to f.

ii. Suppose (T, h) is a counterobjection to (S, U, g), so $T \cap U = \emptyset$. Then

$$\begin{gathered} h(a) \succeq_a g(a) \in D(p,a) \\ \Rightarrow p \cdot h(a) \geq p \cdot e(a) \text{ for } a \in T \cap (S \setminus \{a_0\}) \\ \\ h(a_0) \succeq_{a_0} g(a_0) \in D(p,a_0) - x \\ \Rightarrow p \cdot h(a_0) > p \cdot e(a_0) \text{ if } a_0 \in T \\ \\ h(a) \succeq_a f(a) \succeq_a D(p,a) \\ \Rightarrow p \cdot h(a) \geq p \cdot e(a) \text{ for } a \in T \setminus S, \end{gathered} \tag{77}$$

with strict inequality for at least one $a \in T$. Accordingly,

$$p \cdot \sum_{a \in T} h(a) = \sum_{a \in T} p \cdot h(a) > \sum_{a \in T} p \cdot e(a) = p \cdot \sum_{a \in T} e(a). \tag{78}$$

Since $p \in \Delta$, this contradicts $\sum_{a \in T} h(a) \leq \sum_{a \in T} e(a)$.

Thus, (S, U, g) is a δ-objection to f with no counterobjection, and $f \notin \mathcal{B}_\delta(\chi)$, a contradiction.

5. Combining steps 3 and 4, we conclude that there exists i such that

$$z^i \geq -\frac{M_\chi^{k+1}}{C_p}. \tag{79}$$

But $z = (-\lambda, \ldots, -\lambda)$, so

$$z \geq \left(-\frac{M_\chi^{k+1}}{C_p}, \ldots, -\frac{M_\chi^{k+1}}{C_p}\right). \tag{80}$$

Therefore,

$$p \cdot z \geq -\frac{M_\chi^{k+1}}{C_p}, \tag{81}$$

so

$$L(p) \leq \frac{nM_\chi^{k+1}}{mC_p}, \tag{82}$$

by Equation (57), establishing Equation (34) and completing the proof.

■

Proof of Theorem 3.4: Suppose that Assumptions 1, 2 and 3 hold. Since $|A_n| \to \infty$, $\frac{k+1}{|A_n|} \to 0$. Since the endowment map is uniformly integrable,

$$\frac{M_{\chi_n}^{k+1}}{|A_n|} \to 0. \tag{83}$$

Therefore, we can choose $\delta_n \to 0$ sufficiently slowly that

$$\delta_n |A_n| \to \infty \text{ and } \frac{M_{\chi_n}^{k+1}}{\lfloor \delta_n |A_n| \rfloor} \to 0. \tag{84}$$

Notice further that if Assumption 6 is satisfied, then Equation (84) is satisfied for *any* sequence δ_n such that $\delta_n |A_n| \to \infty$.

Suppose we are given δ_n such that Equation (84) is satisfied and $f_n \in \mathcal{B}_{\delta_n}(\chi_n)$.

1. If $\delta_n \not\to 0$, we can choose $\delta'_n \leq \delta_n$ such that $\delta'_n \to 0$ and Equation (84) is still satisfied. Since $\delta'_n \leq \delta_n$, $\mathcal{B}_{\delta_n}(\chi_n) \subset \mathcal{B}_{\delta'_n}(\chi_n)$, so $f_n \in \mathcal{B}_{\delta'_n}$. Therefore, we can assume without loss of generality that $\delta_n \to 0$. Since the endowments are uniformly integrable,

$$\frac{M_{\chi_n}^{\lfloor \delta_n |A_n| \rfloor}}{|A_n|} \to 0. \tag{85}$$

2. Let p_n be chosen with respect to f_n as in Proposition 3.10. We say $L(p_n, f_n, \cdot)$ converges to 0 in measure if

$$\forall \rho > 0 \ \frac{|\{a \in A_n : \ L(p_n, f_n, a) > \rho\}|}{|A_n|} \to 0. \tag{86}$$

   There are two cases to consider:

(a) If $L(p_n, f_n, \cdot)$ does converge to 0 in measure, then since $L(p_n, f_n, a) \leq p_n \cdot e_n(a) \leq \|e_n(a)\|_\infty$ is uniformly integrable, $L(p_n, f_n, \cdot)$ converges to 0 in mean, i.e.

$$\frac{\sum_{a \in A_n} L(p_n, f_n, a)}{|A_n|} \to 0, \tag{87}$$

which establishes Equation (9).

(b) If $L(p_n, f_n, \cdot)$ does *not* converge to 0 in measure, we will use Proposition 3.10 to derive a contradiction. There exists $\rho > 0$ such that

$$\frac{|\{a \in A_n : L(p_n, f_n, a) > \rho\}|}{|A_n|} > 4\rho \tag{88}$$

for infinitely many n; by taking a subsequence, we may assume without loss of generality that Equation (88) holds for all n. From Proposition 3.10, we are in one of two cases:

i.
$$\frac{L(p_n, f_n)}{|A_n|} \leq \left(\frac{M_{\chi_n}^{\lfloor \delta_n |A_n| \rfloor} + 1}{|A_n|} \right) \to 0 \tag{89}$$

by Equation (85); but this implies Equation (86), contradiction.

ii.
$$\frac{L(p_n, f_n)}{|A_n|} \leq \left(\frac{1}{\lfloor |A_n| \delta_n \rfloor} \right) \left(\frac{M_{\chi_n}^{k+1}}{C_{p_n}} \right) \tag{90}$$

where

$$0 \in \Gamma''(p_n). \tag{91}$$

If we knew that C_{p_n} were bounded away from 0, Equations (84) and (90) would imply Equation (87), which in turn implies Equation (86), a contradiction. Thus, we shall show that C_{p_n} is bounded away from 0. If not, we may (by taking a further subsequence) assume without loss of generality that $p_n \to p \in \Delta \setminus \Delta^0$. If $(\lfloor \delta_n |A_n| \rfloor, x_n) \in \Gamma(p_n)$, there exist t_{na} and x_{na} such that

$$x_n = \sum_{a \in A_n} x_{na}, \tag{92}$$

where $(t_{na}, x_{na}) \in \gamma_a(p_n)$ and $\sum_{a \in A_n} t_{na} \le \delta_n |A_n|$. Since the endowments are uniformly integrable, there exists M such that for all n,

$$\frac{1}{|A_n|} \| \sum_{a \in A_n} e_n(a) \|_\infty \le M, \tag{93}$$

so that

$$\frac{|\{a \in A_n : \|e_n(a)\|_\infty \le \frac{kM}{\rho}\}|}{|A_n|} > 1 - \rho. \tag{94}$$

There is a compact set $K \subset \mathcal{P}_{mo}$ such that the following inequalities hold for sufficiently large n:

$$\begin{array}{c} \frac{|\{a \in A_n : L(p_n, f_n, a) > \rho\}|}{|A_n|} > 4\rho \\ \\ \frac{|\{a \in A_n : \succ_a \in K\}|}{|A_n|} > 1 - \rho \\ \\ \frac{|\{a \in A_n : \|e_n(a)\|_\infty \le \frac{kM}{\rho}\}|}{|A_n|} > 1 - \rho \\ \\ \frac{|\{a \in A_n : t_{na} = 0\}|}{|A_n|} > 1 - \rho. \end{array} \tag{95}$$

Therefore, if

$$\begin{array}{c} S_n = \{a \in A_n : L(p_n, f_n, a) > \rho, \succ_a \in K, \\ \|e_n(a)\|_\infty \le \frac{kM}{\rho}, t_{na} = 0\}, \end{array} \tag{96}$$

then

$$\frac{|S_n|}{|A_n|} > \rho. \tag{97}$$

For all $a \in S_n$, $x_{na} \in D(p_n, a) - e_n(a)$ and $p_n \cdot e_n(a) \ge L(p_n, f_n, a) > \rho$, so $\min\{\|x_{na}\|_\infty : a \in S_n\} \to \infty$ as $n \to \infty$ by Lemma 3.7. Since

$$\frac{1}{|A_n|} \sum_{a \in A_n \setminus S_n} x_{na} \ge -\frac{1}{|A_n|} \sum_{a \in A_n} e_n(a) \tag{98}$$

which is bounded, $\|x_n\|_\infty \to \infty$. Thus,

$$\inf\{\|x\|_\infty : (\lfloor \delta_n |A_n| \rfloor, x) \in \Gamma(p_n))\} \to \infty. \tag{99}$$

By Lemma 3.8, $\inf\{\|x\|_\infty \alpha x \in \Gamma''(p_n)\} \to \infty$, which shows that $0 \notin \Gamma''(p_n)$, a contradiction.

3. Step 2 established Equation (9). Now suppose Assumptions 1, 2, 3 and 4 hold. The proof[7] of Lemma 4 of Anderson [2] shows that there is a compact set $D \subset \Delta^0$ such that, for any sequence $(f_n, p_n) \in \mathcal{A}(\chi_n) \times \Delta^0$ satisfying Equation (9), $\{p_n : n \in \mathbf{N}\} \subset D$.

4. Now suppose that Assumptions 1, 2, 4 and 5 hold. The proof of Theorem 3 of Anderson [2] shows that for any sequence $(f_n, p_n) \in \mathcal{A}(\chi_n) \times D$ satisfying Equation (9), (f_n, p_n) satisfies Equation (11).

■

Proof of Theorem 3.6:

1. Suppose that assumptions 1-3 hold. Since

$$\frac{\max\{\|e(a)\|_\infty : a \in A_n\}}{|A_n|} \to 0, \tag{100}$$

$$\frac{M_{\chi_n}^{k+1}}{|A_n|} \to 0. \tag{101}$$

Therefore, we can choose $\delta_n \to 0$ sufficiently slowly that

$$\frac{M_{\chi_n}^{k+1}}{\lfloor \delta_n |A_n| \rfloor} \to 0. \tag{102}$$

Notice further that if Assumption 5 is satisfied, then Equation (102) holds for *any* sequence δ_n such that $\delta_n |A_n| \to \infty$.

2. Suppose we are given δ_n such that Equation (102) is satisfied and $f_n \in \mathcal{B}_{\delta_n}(\chi_n)$. As in the Proof of Theorem 3.4, we can assume without loss of generality that $\delta_n \to 0$. Therefore, for n sufficiently large, we have $\delta_n < \frac{\sigma}{3}$. Let $\tau = \frac{\sigma}{3}$.

[7] The statement of Lemma 4 assumes that preferences come from a equiconvex family. In the proof, equiconvexity is only applied to pairs of vectors of the form x and $x+(1,0,\ldots,0)$ to show there is a ball of radius bounded away from 0 around $x+(\frac{1}{2},0,\ldots,0)$ which is preferred to x; a compact family of monotone preferences directly implies the existence of such a ball. Lemma 4 also requires that there exist $\gamma > 0$ such that $\frac{|\{a \in A_n : e(a)^i > \gamma\}|}{|A_n|} > \gamma$, but this is implied by Assumptions 2 and 4.

3. By Proposition 3.10, because $\sigma = 3\tau$, we may choose p_n such that $0 \in \Gamma''(p_n)$ and

$$\frac{L(p_n, f_n)}{|A_n|} \leq \left(\frac{1}{\lfloor |A_n|\delta_n \rfloor}\right)\left(\frac{M_{\chi_n}^{k+1}}{C_{p_n}}\right) \tag{103}$$

where C_{p_n} is bounded below by a positive constant depending only on σ, P, and $\frac{1}{|A_n|}\|\sum_{a\in A_n} e(a)\|_\infty$. This shows that $\frac{L(p_n,f_n)}{|A_n|} \to 0$.

4. Suppose Assumptions 1, 2, 3, and 4 hold. The proof of Theorem 3 of Anderson [2] shows that for any sequence $(f_n, p_n) \in \mathcal{A}(\chi_n) \times D$ satisfying Equation (18), (f_n, p_n) satisfies Equation (20).

5. Suppose Assumptions 1, 2, 3 and 5 hold and $J_n \to \infty$. We can assume without loss of generality that $\frac{J_n}{|A_n|} \to 0$. Let

$$\delta_n = \frac{M_{\chi_n}^{k+1}}{J_n C_{p_n}}. \tag{104}$$

Since $M_{\chi_n}^{k+1}$ is bounded, C_{p_n} is bounded away from 0, and $J_n \to \infty$, it follows that $\delta_n \to 0$, and $\delta_n|A_n| \to \infty$. It follows from Equation (103) that

$$\frac{L(p_n, f_n)}{|A_n|} \leq \frac{J_n}{|A_n|}. \tag{105}$$

■

4 Convergence of the Aumann-Davis-Maschler Bargaining Set

Definition 4.1 Suppose $\chi : A \to \mathcal{P} \times \mathbf{R}_+^k$ is an exchange economy. The *Aumann-Davis-Maschler bargaining set*, denoted $\mathcal{B}_{ADM}(\chi)$, is the Geanakoplos bargaining set in which the set of leaders is required to consist of a single individual,

$$\mathcal{B}_{ADM}(\chi) = \mathcal{B}_{\frac{1}{|A|}}(\chi). \tag{106}$$

Remark 4.2 Aumann, Davis and Maschler actually require that an objection be proposed *against* a single person, who is then required to belong to the counterobjecting coalition. Since Definition 4.1 permits more potential counterobjections, it makes it harder to mount a justified objection, and thus results in a larger bargaining set than the original Aumann-Davis-Maschler definition. Thus, our convergence result inplies convergence of the Aumann-Davis-Maschler bargaining set, as originally defined.

Remark 4.3 Some comment on the relationship between the Aumann-Davis-Maschler and Geanakoplos bargaining sets is in order.

1. In a game with an atomless measure space of agents, an individual player makes no difference to the game. Hence, if one interprets the leader as a single individual, there is no difference between the Mas-Colell and Aumann-Davis-Maschler bargaining sets in atomless games. This motivated Geanakoplos to consider leader sets of small positive measure as the appropriate extension of the Aumann-Davis-Maschler bargaining set to continuum games. However, in the nonstandard hyperfinite games considered by Geanakoplos (as well, of course, in large finite games), an individual leader makes *sense*. What is surprising is that, given smooth preferences, an individual leader makes a *difference*. In the light of the nonconvergence example for the Mas-Colell bargaining set in Anderson, Trockel and Zhou [5], which satisfies all the assumptions of Theorem 4.4, we see that allowing a single leader can shrink the bargaining set from essentially every individually rational Pareto optimal allocation to a set of allocations which are approximately competitive. There appears to be no way to capture the importance of a single leader in a continuum model.

2. The leaders in an Aumann-Davis-Maschler or Geanakoplos objection play a coordinating role. An allocation not in the core may yet emerge from a bargaining process because a coalition which *can* object to it does not form, either because the members of the coaltion are unaware that it is feasible for them to object, or because the members of the coalition cannot agree on how to divide the surplus available to them from objecting. A leader's lot is not a happy one. In the proof of convergence of the Aumann-Davis-Maschler and Geanakoplos bargaining sets, the leaders achieve the same utility at the objection g as at

the prevailing allocation f. By sacrificing their interests in dividing the surplus to be gained by objecting, the leaders enhance the welfare of the remaining members of the objecting coalition, thus immunizing these non-leaders from the blandishments of a counterobjection. Moreover, the leaders precommit not to join a counterobjection, even if it would be in their interests to do so; in effect, like the signers of the American Declaration of Independence, the leaders pledge their "lives, fortunes and sacred honors" to the principle that the prevailing allocation is unfair. This creates a serious free rider problem; each player in the potential objecting coalition would prefer to have other people assume the leadership role. Given the dubious benefits of becoming a leader, a justified objection with a few leaders (or better yet, just one leader) seems inherently more likely to form than one requiring many leaders, even if the many leaders represent a small fraction of the population. Any unmodeled benefit of leadership (such as popular acclaim) will likely be greater to each leader if the num or these reasons, the Geanakoplos bargaini

too small as a positive solution concept for the bargaining problem, and establishing convergence of the Aumann-Davis-Maschler bargaining set is desirable.

Theorem 4.4 *Suppose*

$$\chi_n : A_n \to \mathcal{P}_{ss} \times \mathbf{R}^k_+ \tag{107}$$

is a sequence of economies satisfying the following conditions:

1. $|A_n| \to \infty$;
2. *endowments are uniformly bounded, i.e.*

$$\sup_{n \in \mathbf{N}} \max_{a \in A_n} \|e(a)\|_\infty < \infty; \tag{108}$$

and

3. there is a compact equisemismooth set $P \subset \mathcal{P}_{ss}$ and a compact set $K \subset \mathbf{R}^k_{++}$ such that

$$\frac{|\{a \in A_n : \succ_a \in P, e_n(a) \in K\}|}{|A_n|} \to 1. \tag{109}$$

Then there is a compact set $D \subset \Delta^0$ such that for all $f_n \in \mathcal{B}_{ADM}(\chi_n)$ $\exists p_n \in D$ such that

$$\frac{L(p_n, f_n)}{|A_n|} \to 0. \tag{110}$$

Suppose χ_n satisfies Assumptions 1 and 2, and in addition

4. Assumption 3 holds for some compact set $K \subset \mathcal{P}_{ss} \cap \mathcal{P}_{sc}$.

Then

$$\frac{1}{|A_n|} \left\| \sum_{a \in A_n} f_n(a) - D(p_n, a) \right\|_\infty \le \frac{1}{|A_n|} \sum_{a \in A_n} \|f_n(a) - D(p_n, a)\|_\infty \to 0. \tag{111}$$

Proof:

1. irst, we show that given $f_n \in \mathcal{B}_{ADM}(\chi_n)$, we can find $p_n \in \Delta^0$ such that Equation (110) holds.

   (a) By Proposition 3.10, there exists $p_n \in \Delta^0$ such that either

   i.
   $$\frac{L(p_n, f_n)}{|A_n|} \le \frac{M^1_{\chi_n} + 1}{|A_n|} \to 0 \tag{112}$$
   by Assumptions 1 and 2; or

   ii. $0 \in \Gamma''(p_n)$.

   Thus, for the purpose of establishing Equation (110), we can assume without loss of generality that $0 \in \Gamma''(p_n)$ for all n.

   (b) Let $\delta_n = \frac{1}{|A_n|}$. ollow item 2 of the proof of Theorem 3.4, substituting $\mathcal{P}_{ss}$ for $\mathcal{P}_{mo}$; since $\frac{M^{k+1}_{\chi_n}}{\lfloor |A_n|\delta_n \rfloor}$ need not tend to zero, this does not produce a contradiction, but it does show that there is a compact set $D \subset \Delta^0$ such that $p_n \in D$.

(c) Since $\frac{|\{a\in A_n:(\succ_a,e_n(a))\notin P\times K\}|}{|A_n|}\to 0$, we may find r_n such that

$$\begin{aligned} \tfrac{r_n}{|A_n|} &\to 0 \\ r_n - |\{a \in A_n : (\succ_a, e_n(a)) \notin P \times K\}| &\to \infty. \end{aligned} \tag{113}$$

By Assumption 2, $\frac{M_{\chi_n}^{r_n}}{|A_n|}\to 0$. If there exists $q_n \in \Delta^0$ such that $|\{a \in A_n : D(q_n, a) \succ_a f_n(a)\}| \le r_n$, then

$$\frac{L(q_n, f_n)}{|A_n|} \le \frac{M_{\chi_n}^{r_n}}{|A_n|} \to 0. \tag{114}$$

Hence, we can assume without loss of generality that

$$\forall n \in \mathbf{N} \;\; \forall q \in \Delta^0 \;\; |\{a \in A_n : D(p_n, a) \succ_a f_n(a)\}| > r_n. \tag{115}$$

(d) By item 3 of the proof of Proposition 3.10, there exists $(t_n, x_n) \in \Gamma(p_n)$ with $t_n \in \{0, 1\}$ such that

$$x_n \le z_n + \left(\frac{M_{\chi_n}^{k+1}}{C_{p_n}}, \ldots, \frac{M_{\chi_n}^{k+1}}{C_{p_n}}\right), \tag{116}$$

where z_n lies on the negative diagonal and $\frac{L(p_n,f_n)}{|A_n|} \le -p_n \cdot z_n$. Note that $p_n \cdot z_n \ge -M_{\chi_n}^1$, so $0 \ge z_n \ge -(M_{\chi_n}^1, \ldots, M_{\chi_n}^1)$. A slight rearrangement of the argument (in particular, in item 3c) shows that

$$x_n \ge z_n - \left(\frac{M_{\chi_n}^{k+1}}{C_{p_n}}, \ldots, \frac{M_{\chi_n}^{k+1}}{C_{p_n}}\right), \tag{117}$$

so

$$\|x_n\|_\infty \le \|z_n\|_\infty + \frac{M_{\chi_n}^{k+1}}{C_{p_n}}. \tag{118}$$

Therefore,

$$\sup_{n\in\mathbf{N}} \|x_n\|_\infty \le \sup_{n\in\mathbf{N}} M_{\chi_n}^1 + \frac{M_{\chi_n}^{k+1}}{C_{p_n}} < \infty. \tag{119}$$

(e) Since $(t_n, x_n) \in \Gamma(p_n)$, there exists $g_n : A_n \to \mathbf{R}^k_+$ and $t_n : A \to \{0,1\}$ such that $\sum_{a\in A_n} g_n(a) - e_n(a) = x_n$ and $\sum_{a\in A_n} t_n(a) = 1$. Let $S_n = \{a \in A_n : g_n(a) \neq e(a)\}$, and $\hat{\gamma}_n(a) = \{y \in \mathbf{R}^k_+ : y \succeq_a g_n(a)\}$ for $a \in S_n$, and $\hat{\Gamma}_n = \sum_{a\in S_n} \hat{\gamma}_n(a)$. Item 4 of the proof of Proposition 3.10 shows that there is no y in $\hat{\Gamma}_n$ with $y \ll 0$.

(f) $|S_n| > r_n$. Let

$$\hat{S}_n = \{a \in S_n : (\succ_a, e_n(a)) \in P \times K,\, t_n(a) = 0\}. \tag{120}$$

Then

$$|\hat{S}_n| > r_n - |\{a \in A_n : (\succ_a, e_n(a)) \notin P \times K\}| - 1 \to \infty. \tag{121}$$

Lemma 3.6 of Anderson [3] shows that

$$p_n \cdot z_n \geq -\frac{|x_n|^2}{2\sum_{a\in S_n} \rho_{\succ_a g_n(a)}} \geq -\frac{|x_n|^2}{2\sum_{a\in \hat{S}_n} \rho_{\succ_a g_n(a)}}. \tag{122}$$

If $a \in \hat{S}_n$, $g_n(a) = D(p_n, a) \succeq_a e_n(a)$, and $p_n \cdot g_n(a) \leq p_n \cdot e_n(a)$, so $g_n(a) \in$

$$\begin{array}{c} \{y \in \mathbf{R}^k_+ : \exists \succ \in P\ \exists e \in K\ y \succeq e, \\ \|y\|_\infty \leq \sup_n C_{p_n} \sup\{\|x\|_\infty : x \in K\}\} \end{array} \tag{123}$$

which is a compact subset of $\mathbf{R}^k_{++}$, since P is a compact subset of $\mathcal{P}_{ss}$ and K is a compact subset of $\mathbf{R}^k_{++}$. Since P is equisemismooth, there exists $\alpha > 0$ such that $\rho_{\succ_a g_n(a)} \geq \alpha$ for all $a \in \hat{S}_n$, so $p_n \cdot z_n \to 0$, so $\frac{L(p_n, f_n)}{|A_n|} \to 0$, establishing Equation (110).

2. Assumption (3) implies that $\liminf_{n\to\infty} \frac{1}{|A_N|} \sum_{a\in A_n} e_n(a) \gg 0$. The proof[8] of Lemma 4 of Anderson [2] shows that there is a compact set $D \subset \Delta^0$ such that, for any sequence $(f_n, p_n) \in \mathcal{A}(\chi_n) \times \Delta^0$ satisfying Equation (9), $\{p_n : n \in \mathbf{N}\} \subset D$.

3. The proof of Theorem 3 of Anderson [2] shows that for any sequence $(f_n, p_n) \in \mathcal{A}(\chi_n) \times D$ satisfying Equation (110), (f_n, p_n) satisfies Equation (111).

■

[8]See footnote 7 for further details.

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