Bayes Contingent Plans

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Abstract

An intuitively natural consistency condition for contingent plans is necessary and sufficient for a contingent plan to be rationalized by maximization of conditional expected utility. One alternative theory of choice under uncertainty, the weighted-utility theory developed by Chew Soo Hong (1983) does not entail that contingent plans will generally satisfy this condition. Another alternative theory, the minimax theory as formulated by Savage (1972), does entail the consistency condition (at least for singleton-valued plans).

1. Introduction

Social scientists must undertake the formidable task of modelling agents who are trying simultaneously to learn about their environment and to change it. For those social scientists who impose on themselves the intellectual discipline of articulating their ideas as explicit formal models, L. J. Savage's ([1954] 1972) Bayesian decision-theoretic characterization of coherent decision making and learning under uncertainty has become the benchmark representation of this process of simultaneous learning and action. In the context of this symposium, it is fitting to mention the field of industrial organization as an example. The study of strategic interaction among producers in an industry has been conducted primarily in the context of Bayesian game-theoretic models during the past two decades. Cyert and DeGroot (1970) introduced the Bayesian model to this field.

In this paper, we consider the specification of Bayesian decision theory as a "behavioral" theory—a term strongly emphasized by Savage in statistics as well as by Cyert, Simon, and others in economics. A theory of decision is to be considered behavioralistic if it has to do with predicting how agents make decisions or with advising agents about how to make decisions, rather than with agents' introspection about their decisions. Savage emphasizes the sharp contrast in this respect between his approach and the "verbalistic" approach taken by most of his contemporaries in statistics.

Savage recognizes (especially in the concluding paragraphs of Chapter 2.5 and in Chapter 5.5 of [1954] 1972) that the practical employment of his theory by a researcher would require the researcher's own understanding of which consequences are the salient ones. This burden on the researcher is exacerbated by the status of acts in Savage's theory. Savage does not take acts to be logically primitive entities. Rather, he defines them in terms of two other types of entity: consequences and states of nature. He takes the position that the identification of salient consequences "is an operation in which we all have much experience, and one in which there is in practice considerable agreement," but he acknowledges that "what are often thought of as consequences ... are in reality highly uncertain ... [and] can perhaps never be well approximated." ([1954 1972, pp. 16–17, 84.)

In this paper, we address Savage's problem by formulating a new version of Bayesian subjectivist decision theory . In this theory, acts and observations of evidence are the logically primitive terms. States of nature are defined to be infinite sequences of observations. The utility of an act is defined formally to be a state-dependent function of the act itself, so that reference to consequences is avoided completely. Our main object of study is a *contingent plan* which completely specifies an agent's plan about which act to perform if the choice of act were to be based on various possible sequences of observations.

After having set forth this theory, we investigate its strength. First we provide a necessary and sufficient condition for a contingent plan to be rationalizable in terms of maximization of conditional expected utility (that is, in terms of the Bayesian decision-theoretic criterion). Then we show that this condition is also satisfied by single-valued plans (that is, plans that recommend a unique act for each observation sequence) that can be rationalized by either a minimax-loss or minimax-regret criterion. In view of the heavy emphasis that Savage placed on the distinction between Bayesian and minimax decision making, this result shows a clear sense in which our behavioral formulation of the Bayesian theory has only weak observational implications. However, we also show that the observational implications of this formulation can fail to be satisfied by a contingent plan determined according to Chew's (1983) weighted-utility criterion. Thus the formulation is sufficiently strong to serve as a basis for a study of whether an actual decision maker would conform more closely to the Bayesian theory or to a prominent alternative theory.

We wish to emphasize that the equivalence result of this paper is proved in a finite-state, finite-action setting. It is an open question whether or not this result extends to a wider class of models. Also, we wish to make it clear that we do not view behavioralistic theories as being necessarily the only theories of value for social science. (See Chomsky (1957), for instance, for an argument that agents' judgments and intuitions constitute important non-behavioral evidence to be predicted and explained.)

2. Bayesian concept of rationality in decisions

The Bayesian decision-theoretic concept of rationality incorporates two principles. First, in any situation an agent ought to maximize expected utility. Second, the probability measures with respect to which expected utility is computed in successive situations ought to be related by conditionalization on whatever evidence the agent may have observed in their interim. This paper provides a representation theory for such a two-part Bayesian concept of rationality. The theory contemplates an agent who has available a set of feasible actions. The agent makes a sequence of observations. Before he begins to make these observations, and subsequently after each observation that he makes, the agent reports which action he would be most inclined to take on the basis of what he has observed to date. (If the agent considers several actions to be tied for the best, he will report all of them.) A *contingent plan* is a record of such a sequence of reports for every possible sequence of observations. The main contribution of this paper is to show that a necessary and sufficient condition (to be called *consistency*) exists for a contingent plan to be rationalizable in terms of conditional expected utility.

A simple example will illustrate the sort of contingent plan that would violate this necessary and sufficient condition. Consider someone who can choose either to go to a concert or to go to the opera or to stay at home (the feasible actions). Suppose that this person knows what is the program of the concert, but is uncertain which of two operas is being performed. He is initially inclined to go to the opera. However, regardless of which opera is announced in the newspaper (the possible outcomes of making an observation), upon reflection, he is inclined to stay at home. The pattern of decisions that is envisioned in this example is very specific. Prior to making an observation, the agent would choose a particular action if he had to make an immediate choice. After making the observation, the agent would choose a different action which is always the same, regardless of exactly what he observes. Call a contingent plan *consistent* if it does not stipulate such a pattern of decisions at any point.

The example violates Bayes rationality because the prior expectation of conditional expected utility must be equal to prior expected utility. Regardless of what he reads in the newspaper, the agent's subsequent inclination would imply that the conditional expected utility of staying at home must be greater than the conditional expected utility of going to the opera. If that were always so, then the prior expected utility of staying at home should have been greater than the prior expected utility of going to the opera. Thus, contrary to the example, the agent should initially have been inclined to stay at home rather than to go to the opera.

The setting of these results is closely related to that of the characterization by Green and Osband (1991) of expected-utility maximization. The main difference is that, while Green and Osband assume directly that an agent's decision is a function of a probability measure, here it is only assumed that the agent's decision is a function of what the agent has observed. It may be helpful also to mention some ways in which the theorem to be proved here differs from representation theorems for subjective expected utility such as that of Savage ([1954] 1972). As mentioned in Section 1, actions are taken here to be primitive entities of the theory. Savage interprets them to be functions from a specified set of states of the world to a set of consequences. The contingent plans studied here specify only the agent's optimal actions. Savage specifies the agent's preferences among suboptimal actions as well. Conditionalization of probabilities is represented explicitly here. Savage represents conditionalization implicitly in the context of specific assumptions about features of preferences (including "state-independence") and about the structure of the set of actions. The probability measure and utility function shown here to rationalize a contingent plan that satisfies the sufficient condition for Bayes rationality are not necessarily unique. Savage's assumptions imply their uniqueness.

3. Formalization of the problem

Consider an agent who observes at each date $t \in \mathbb{N}_+$ a piece of evidence $x_t \in \mathbf{X}$, where $\mathbf{X} = \{1, \ldots, m\}$ is a finite set. At some date $t \in \mathbb{N}$, and after having observed the values $x_1, \ldots, x_t$, the agent must take an action a from a finite set A.

The Bayesian concept of rationality interprets the agent as having beliefs represented by a probability space $(\Omega, \mathcal{B}, \Pr)$ and a state-contingent utility function $u\colon A \times \Omega \to \mathbb{R}$, and interprets the observations x_t as the values of random variables $X_t\colon \Omega \to \mathbf{X}$. A Bayesian-rational agent always chooses an action that maximizes the posterior expected utility

$$U(a, \omega, t) = \frac{\int_{B(\omega,t)} u(a, \theta)\, d\Pr(\theta)}{\Pr(B(\omega, t))}, \tag{1}$$

where

$$B(\omega, t) = \{\theta \in \Omega | \forall s \leq t \;\; X_s(\theta) = X_s(\omega)\}. \tag{2}$$

(Note that $B(\omega, 0) = \Omega$.) That is, if $\alpha(X_1(\omega), \ldots, X_t(\omega)) \subseteq A$ is the set of actions that the agent might decide to take after having observed $X_1(\omega), \ldots, X_t(\omega)$, then

$$\begin{aligned} \forall \omega \in \Omega \;\; \forall t \in \mathbb{N} \;\; \forall a \in A \;\; \big[& a \in \alpha(X_1(\omega), \ldots, X_t(\omega)) \\ & \iff \forall a' \in A \quad U(a', \omega, t) \leq U(a, \omega, t) \big]. \end{aligned} \tag{3}$$

Note that this formulation does not represent the agent as using the information that t, rather than another date, is when the action is to be taken. Implicitly it is assumed that this date is determined independently of the random variables X_t and independently of the sections $u(a, \cdot)$ of the utility function.

Now define the set of *observation sequences*, $\mathbf{X}^* = \bigcup_{t \in \mathbb{N}} \mathbf{X}^t$, and define $\lambda: \mathbf{X}^* \to \mathbb{N}$ by $\lambda(\vec{x}) = t \iff \vec{x} \in \mathbf{X}^t$. Given any measurable space $(\Omega, \mathcal{B})$ and infinite sequence of $\mathcal{B}$-measurable functions $X_t: \Omega \to \mathbf{X}$, define

$$B^*(\vec{x}) = \{\omega | \forall t \leq \lambda(\vec{x}) \;\; X_t(\omega) = \vec{x}_t\}. \tag{4}$$

Note that a measurable space and sequence of random variables can always be constructed from $\mathbf{X}$ by taking $\Omega = \mathbf{X}^{\mathbb{N}}$ and $X_t(\omega) = \omega_t$, and by taking $\mathcal{B}$ to be the smallest σ-algebra with which all of the projection functions X_t are measurable. If $\forall t \leq \lambda(\vec{x}) \;\; X_t(\omega) = \vec{x}_t$, then $B^*(\vec{x}) = B(\omega, \lambda(\vec{x}))$. Define $\xi: \Omega \times \mathbb{N} \to \mathbf{X}^*$ by

$$\xi(\omega, t) = \big\langle X_1(\omega), \ldots, X_t(\omega) \big\rangle. \tag{5}$$

Throughout this paper, attention will be confined to stochastic processes for which every observation sequence occurs with positive probability. That is, it is assumed that

$$\forall \vec{x} \in \mathbf{X}^* \;\; \Pr(B^*(\vec{x})) > 0\,. \tag{6}$$

Condition (6) makes it possible to define posterior expected utility analogously to (1), but in terms of $\mathbf{X}^*$, by

$$U^*(a, \vec{x}) = \frac{\int_{B^*(\vec{x})} u(a, \theta) \, d\Pr(\theta)}{\Pr(B^*(\vec{x}))}. \tag{7}$$

A *contingent plan* is a correspondence $\alpha: \mathbf{X}^* \to\!\!\to A$. A *Bayes* contingent plan is one for which there exists a representation of form (1)–(6). Say that a contingent plan α is *Bayes for* $\langle \Omega, \mathcal{B}, \Pr, \{X_t\}_{t \in \mathbb{N}_+} \rangle$ (or simply *Bayes for* $\{X_t\}_{t \in \mathbb{N}_+}$) if α satisfies

$$\forall \vec{x} \in \mathbf{X}^* \;\; \forall a \in A \;\; \Big[a \in \alpha(\vec{x}) \iff U^*(a, \vec{x}) = \max_{a' \in A} U^*(a', \vec{x}) \Big] \tag{8}$$

with respect to some von Neumann-Morgenstern utility function and to that stochastic process.

4. A necessary condition

This section concerns the formulation of a condition on contingent plans, and the proof that this condition must necessarily be satisfied by a Bayes contingent plan. First some notation is introduced to formulate and analyze the condition.

Define the *immediate extensions* of $\vec{x} \in \mathbf{X}^*$ to be those observation sequences that consist of $\vec{x}$ followed by a single observation. Let $\vec{x} * y$ denote the immediate extension of $\vec{x}$ by y. The immediate extensions of all observation sequences can be represented by a correspondence $\epsilon\colon \mathbf{X}^* \to\to \mathbf{X}^*$ defined by $\vec{y} \in \epsilon(\vec{x}) \iff \exists y \in \mathbf{X}\ \ \vec{y} = \vec{x} * y$.

Now the necessary condition for a contingent plan to be Bayes can be stated in terms of three subsidiary conditions.

Contingent plan α satisfies *existence of an optimum at* $\vec{x}$ if

$$\alpha(\vec{x}) \neq \emptyset.$$

Contingent plan α satisfies *dominance-inclusiveness at* $\vec{x}$ if

$$\forall a \ \left\{ \left[\forall y \ \ a \in \alpha(\vec{x} * y) \right] \Longrightarrow a \in \alpha(\vec{x}) \right\}.$$

Contingent plan α satisfies *dominance-restrictiveness at* $\vec{x}$ if

$$\forall a \ \forall b \ \left\{ \left[\forall y \ \ a \in \alpha(\vec{x} * y) \quad \text{and} \quad \exists y \ \ b \notin \alpha(\vec{x} * y) \right] \Longrightarrow b \notin \alpha(\vec{x}) \right\}.$$

Now we come to the main definition. Define a contingent plan α to be *consistent at* $\vec{x}$ if it satisfies at $\vec{x}$ the three subsidiary conditions just presented: existence of optimum, dominance-inclusiveness, and dominance-restrictiveness. It is easily seen that the conjunction of these subsidiary conditions is equivalent to the set-theoretic condition that

$$\alpha(\vec{x}) \neq \emptyset \ \text{ and } \ \left[\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset \quad \text{or} \quad \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \alpha(\vec{x}) \right]. \tag{9}$$

Theorem 1. *A Bayes contingent plan must be consistent at every observation sequence.*

Proof: First, a Bayes contingent plan must be nonempty valued (that is, must satisfy $\alpha(\vec{x}) \neq \emptyset$ at every $\vec{x}$) because expected utility attains a maximum on a finite set of alternative actions. Suppose that α is Bayes for a stochastic process $\{X_t\}_{t \in \mathbb{N}_+}$ and that $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \neq \emptyset$. Specifically, suppose that $a \in \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. Consider any $a' \in A$. Since $a \in \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$, (3) implies that $\forall \vec{y} \in \epsilon(\vec{x})\ \ U^*(a', \vec{y}) \leq U^*(a, \vec{y})$.

By the law of iterated expectation,

$$U^*(a, \vec{x}) = \frac{\sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a, \vec{y})}{\Pr(B^*(\vec{x}))} \quad \text{and} \quad U^*(a', \vec{x}) = \frac{\sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a', \vec{y})}{\Pr(B^*(\vec{x}))}. \tag{10}$$

Therefore $U^*(a', \vec{x}) \leq U^*(a, \vec{x})$. Since this inequality holds for all $a' \in A$, (3) implies that $a \in \alpha(\vec{x})$. That is, either $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$ or else $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})$.

Now suppose instead that $a \in \alpha(\vec{x}) \setminus \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. By (3) and (10),

$$\forall a' \in A \sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a', \vec{y}) \leq \sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a, \vec{y}). \tag{11}$$

Since $a \notin \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$, suppose that $\vec{z} \in \epsilon(\vec{x})$ and $a \notin \alpha(\vec{z})$. Since $\alpha(\vec{z})$ is nonempty, suppose that $a' \in \alpha(\vec{z})$. Then (3) implies that $U^*(a, \vec{z}) < U^*(a', \vec{z})$. Therefore, in order for (11) to hold, there must be some other $\vec{w} \in \epsilon(\vec{x})$ such that $U^*(a', \vec{w}) < U^*(a, \vec{w})$. By (3), $a' \notin \alpha(\vec{w})$ and hence $a' \notin \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. Since a' is arbitrary except that $a' \neq a$, this argument shows that $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \{a\}$. However, $a \notin \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$ by assumption. Thus $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$.

What has just been established is that either $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$ or else $\alpha(\vec{x}) \subseteq \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. The set inclusion must in fact be an equality, since it has earlier been established that either $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$ or else $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})$. ∎

5. Consistent plans are Bayes

The converse of Theorem 1 is also true. Theorem 2 states that, if one requires only that a state-contingent expected-utility function should be bounded above, then each consistent contingent plan is Bayes for every stochastic process that satisfies (6). Theorem 3 states that, if one requires that a state-contingent expected-utility function should be bounded below as well as above, then a consistent plan is Bayes for some stochastic process (that may depend on the plan) that satisfies (6).

Theorem 2. *Suppose that α is consistent, and let $\langle \Omega, \mathcal{B}, \Pr, \{X_t\}_{t\in\mathbb{N}_+}\rangle$ be any stochastic process that satisfies (6). Then there is a state-contingent expected-utility function $u: A\times\Omega \to \mathbb{R}$ that makes α Bayes for $\{X_t\}_{t\in\mathbb{N}_+}$, and such that u is bounded above.*

Proof: For each $a \in A$, define a martingale $\{V_t^a: \Omega \to \mathbb{R}\}_{t\in\mathbb{N}_+}$ recursively as follows. Let $V_0^a(\omega) = 0$ if $a \in \alpha(\emptyset)$ and $V_0^a(\omega) = -1$ if $a \notin \alpha(\emptyset)$. Suppose that V_t^a has been defined. Then begin to define V_{t+1}^a by setting

$$V_{t+1}^a(\omega) = \begin{cases} \max\{V_t^{a'}(\omega) | a' \in A\}, & \text{if } a \in \alpha(\xi(\omega, t+1)) \text{ and } \bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y}) \neq \emptyset \quad (12) \\ \max\{V_t^{a'}(\omega) \\ \quad +2^{-t} | a' \in A\}, & \text{if } a \in \alpha(\xi(\omega, t+1)) \text{ and } \bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y}) = \emptyset \quad (13) \\ V_t^a(\omega), & \text{if } a \notin \bigcup_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y}) \,. \quad (14) \end{cases}$$

Condition (12) defines V_{t+1}^a except at those $\omega \in \Omega$ such that $\exists \vec{y} \in \epsilon(\xi(\omega, t))$ $[a \in \alpha(\vec{y})$ but $a \notin \alpha(\xi(\omega, t+1))]$. At these states, V_{t+1}^a will be defined by a martingale condition. To accomplish this, first define

$$G(a, \omega, t) = \{\theta | a \in \alpha(\xi(\theta, t+1))\} \cap B(\omega, t). \tag{15}$$

The condition that $\exists \vec{y} \in \epsilon(\xi(\omega, t))$ $[a \in \alpha(\vec{y})$ but $a \notin \alpha(\xi(\omega, t+1))]$ is equivalent to $\omega \notin G(a, \omega, t)$ and $G(a, \omega, t) \neq \emptyset$. In this case, define $V_{t+1}^a(\omega)$ by imposing the martingale condition

$$V_t^a(\omega) \Pr(B(\omega, t)) = \int_{G(a,\omega,t)} V_{t+1}^a(\theta)\, d\Pr(\theta) + V_{t+1}^a(\omega) \Pr(B(\omega, t) \backslash G(a, \omega, t)). \tag{16}$$

It is now proved by induction that each $\{V_t^a\}_{t\in\mathbb{N}}$ is a martingale and that for each date τ,

$$\forall a\ \forall \omega\ \left[a \in \alpha(\xi(\omega, \tau)) \iff V_\tau^a(\omega) = \max\{V_\tau^{a'}(\omega) | a' \in A\}\right]. \tag{17}$$

The martingale condition on $V_0^a, \ldots, V_t^a$ is vacuous for $t = 0$, and V_0^a has been defined explicitly to assure (17). For the induction step, suppose the hypothesis that $V_0^a, \ldots, V_t^a$

is a martingale for each a and that (17) holds. These conditions must be shown to hold also for the random variables $\{V^a_{t+1} | a \in A\}$.

Begin by verifying condition (17). Consider an action a and a state ω. First suppose that the condition $a \in \alpha(\xi(\omega, t+1))$ and $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y}) \neq \emptyset$ of (12) is satisfied. Since $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y}) \neq \emptyset$, condition (13) is not satisfied at ω for any action a^*. Therefore $V^{a^*}_{t+1}(\omega)$ is determined for every action a^* by either (12) or (14) or (16). In all of these three cases, $V^{a^*}_{t+1}(\omega) \leq V^a_{t+1}(\omega) = \max\{V^{a'}_t(\omega) | a' \in A\}$. Therefore the condition that $a \in \alpha(\xi(\omega, t+1))$ and $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y}) \neq \emptyset$ implies that $V^a_{t+1}(\omega) = \max\{V^{a'}_{t+1}(\omega) | a' \in A\}$.

Second suppose that the condition $a \in \alpha(\xi(\omega, t+1))$ and $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y}) = \emptyset$ of condition (13) is satisfied. Then $V^a_{t+1}(\omega)$ is the highest number that can be assigned to any $V^{a'}_{t+1}(\omega)$ by any of conditions (12)–(14), so again $V^a_{t+1}(\omega) = \max\{V^{a'}_{t+1}(\omega) | a' \in A\}$.

Third suppose that the condition $a \notin \bigcup_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y})$ of condition (14) is satisfied. Here there are two subcases. The first is that, if the condition $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y}) \neq \emptyset$ of (12) is satisfied, then this condition and the defining condition (9) of Bayes consistency imply that $a \notin \alpha(\xi(\omega, t))$ and that therefore $V^a_t(\omega) < \max\{V^{a'}_t(\omega) | a' \in A\}$. Some a^* is an element of $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y})$, and $V^a_{t+1}(\omega) < V^{a^*}_{t+1}(\omega) = \max\{V^{a'}_{t+1}(\omega) | a' \in A\}$. The second subcase is that the condition $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y}) = \emptyset$ of (13) is satisfied. Then (13) specifies a higher value of $V^{a^*}_{t+1}(\omega)$ for any $a^* \in \alpha(\xi(\omega, t+1))$ (of which (9) requires that there should be at least one) than the value of $V^a_{t+1}(\omega)$.

Finally consider the case that $\omega \notin G(a, \omega, t)$ and $G(a, \omega, t) \neq \emptyset$. There are two subcases, depending on whether or not $a \in \alpha(\xi(\omega, t))$. If $a \in \alpha(\xi(\omega, t))$ and $\omega \notin G(a, \omega, t)$, then $\bigcap_{\vec{y} \in \epsilon(\xi(\omega,t))} \alpha(\vec{y}) = \emptyset$ by the definition (9) of Bayes consistency and the definition (15) of $G(a, \omega, t)$. Therefore $V^a_{t+1}(\theta) = V^a_t(\theta) + 2^{-t}$ for $\theta \in G(a, \omega, t)$. The condition that $G(a, \omega, t) \neq \emptyset$ implies that $\Pr(G(a, \omega, t)) > 0$ by (6), so the martingale condition (15) entails that $V^a_{t+1}(\omega) < V^a_t(\omega)$. Also by (9), there is some action $a^* \in \alpha(\xi(\omega, t+1))$, and $V^a_t(\omega) < V^{a^*}_{t+1}(\omega)$ by (13). Together these inequalities establish that $V^a_{t+1}(\omega) < V^{a^*}_{t+1}(\omega)$. In the alternative subcase, that $a \notin \alpha(\xi(\omega, t))$, continue to let a^* denote an element of $\alpha(\xi(\omega, t+1))$, and also let a' denote an element of $\alpha(\xi(\omega, t))$ which is nonempty as well by the Bayes consistency condition (9). By the induction hypothesis, $V^a_t(\omega) < V^{a'}_t(\omega)$. The value of $V^{a^*}_{t+1}(\omega)$ must be determined by either (12) or (13), and in either case $V^{a'}_t(\omega) \leq$

$V^{a^*}_{t+1}(\omega)$. Also $V^a_{t+1}(\omega) \le V^a_t(\omega)$, either by (14) or else by an argument from (16) parallel to the one just given. Again in this subcase, then, $V^a_{t+1}(\omega) < V^{a^*}_{t+1}(\omega)$.

The foregoing argument shows that (17) holds at date $\tau = t+1$. Now consider the martingale condition that, for every $a \in A$ and every event B measurable with respect to $X_1, \ldots, X_t$,

$$\int_B V^a_t(\omega)\, d\Pr(\omega) = \int_B V^a_{t+1}(\omega)\, d\Pr(\omega). \tag{18}$$

It is sufficient to verify this condition on the sets $B(\xi(\omega,t))$. If $a \in \bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y})$, then (18) is guaranteed by (12). If $a \notin \bigcup_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y})$, then (18) is guaranteed by (14). Otherwise (18) is guaranteed by (16).

Thus it has been shown by induction that the processes $\{\{V^a_t\}_{t\in\mathbb{N}} | a \in A\}$ are martingales that satisfy (17). Each martingale $\{V^a_t\}_{t\in\mathbb{N}}$ is bounded above by 2, so the Martingale Convergence Theorem implies that there is an integrable random variable $u^a : \Omega \to \mathbb{R}$ bounded above by 2 and such that $u^a(\omega) = \lim_{t\to\infty} V^a_t(\omega)$ almost surely. Defining $u: A \times \Omega \to \mathbb{R}$ by $u(a,\omega) = u^a(\omega)$ completes the proof, since the Martingale Convergence Theorem implies that $V^a_t(\omega) = U(a,\omega,t)$ for all a, ω and t. ∎

Theorem 3. *If α is consistent, then there exists a stochastic process $\langle \Omega, \mathcal{B}, \Pr^*, \{X_t\}_{t\in\mathbb{N}_+}\rangle$ satisfying (6) and a bounded, state-contingent, expected-utility function $u^*: A \times \Omega \to \mathbb{R}$ that makes α Bayes for $\{X_t\}_{t\in\mathbb{N}_+}$.*

Proof: Begin with a stochastic process $\langle \Omega, \mathcal{B}, \Pr, \{\mathbf{X}_t\}_{t\in\mathbb{N}_+}\rangle$ satisfying (6) and with the utility function $u: A \times \Omega \to \mathbb{R}$ that was constructed in Theorem 2. Define a function $\delta: \Omega \to \mathbb{R}_+$ by

$$\delta(\omega) = -\min\{-1, \min\{u(a,\omega) | a \in A\}\}. \tag{19}$$

Now define

$$\Pr^*(B) = \frac{\int_B \delta(\omega)\, d\Pr(\omega)}{\int_\Omega \delta(\omega)\, d\Pr(\omega)} \quad \text{and} \quad u^*(a,\omega^*) = \frac{u(a,\omega^*)}{\delta(\omega^*)} \int_\Omega \delta(\omega)\, d\Pr(\omega)\,. \tag{20}$$

By (19) and (20), Pr is absolutely continuous with respect to $\Pr^*$ so $\Pr^*$ satisfies (6). By (20), $\int_B u^*(a,\omega)\, d\Pr^*(\omega) = \int_B u(a,\omega)\, d\Pr(\omega)$ for all $B \in \mathcal{B}$ and $a \in A$ so u^* makes α Bayes. By (19), u^* satisfies the bounds $\forall a\ \forall\omega\ \ -1 \le u^*(a,\omega) \le 2$. ∎

6. Comparing the Bayes and miminax criteria

A prominent alternative to maximization of expected utility in the history of decision theory has been minimization of maximum loss—that is, the minimax criterion. An important reason to have characterized Bayes contingent plans is to compare them with the contingent plans that alternative criteria such as minimax would recommend. In order to do so, a necessary condition to be a minimax contingent plan is now derived. This condition coincides almost exactly with the consistency condition (9). In particular, the condition is equivalent to consistency in the important case that a contingent plan is singleton valued.

The minimax criterion is formulated in terms of the loss $\ell(a, \omega)$ of an action in a state of nature. The loss may be conceived either as the negative of the state-contingent utility of an action,

$$\ell(a, \omega) = -u(a, \omega)\,, \tag{21}$$

or else as a state-contingent regret which is defined by

$$\ell(a, \omega) = \max_{a' \in A} u(a', \omega) - u(a, \omega)\,. \tag{22}$$

Action a is minimax, conditional on observation sequence $\vec{x}$, if

$$\sup_{\omega \in B^*(\vec{x})} \ell(a, \omega) = \min_{a' \in A} \sup_{\omega \in B^*(\vec{x})} \ell(a', \omega)\,. \tag{23}$$

A contingent plan α is *minimax* (for utility function $u: A \times \Omega \to \mathbb{R}$) if

$$\forall \vec{x} \in \mathbf{X}^* \;\; \alpha(\vec{x}) = \{a | \sup_{\omega \in B^*(\vec{x})} \ell(a, \omega) = \min_{a' \in A} \sup_{\omega \in B^*(\vec{x})} \ell(a', \omega)\}\,. \tag{24}$$

Theorem 4. *If a contingent plan $\alpha: \mathbf{X}^* \to A$ is minimax, then for every $\vec{x} \in \mathbf{X}^*$,*

$$\alpha(\vec{x}) \neq \emptyset \;\; \textit{and} \;\; \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})\,, \tag{25}$$

and

$$\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \neq \emptyset \implies \exists \vec{y} \in \epsilon(\vec{x}) \;\; \alpha(\vec{x}) \subseteq \alpha(\vec{y})\,. \tag{26}$$

A singleton-valued minimax contingent plan is consistent at every observation sequence $\vec{x}$ and hence is Bayes.

Proof: Suppose that α is minimax. Clearly α must be nonempty valued. Suppose that $a \notin \alpha(\vec{x})$. Then $\ell(a,\omega^*) > \sup_{\omega \in B^*(\vec{x})} \ell(a',\omega)$ for some $\omega^* \in B^*(\vec{x})$. Since $\omega^* \in B^*(\xi(\omega^*, \lambda(\vec{x})+1))$ and $\xi(\omega^*, \lambda(\vec{x})+1) \in \epsilon(\vec{x})$, $a \notin \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. This shows that $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})$, establishing (25).

Now suppose that $a \in \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$ and that $a' \in \alpha(\vec{x}) \setminus \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. Let $\gamma = \sup_{\omega \in B^*(\vec{x})} \ell(a',\omega)$. Since $a' \in \alpha(\vec{x})$, for every $n \in \mathbb{N}_+$ there exists a state of nature $\omega_n \in B^*(\vec{x})$ such that $\ell(a,\omega_n) < \gamma + 1/n$. Because $\epsilon(\vec{x})$ is finite, $\{n | \omega_n \in B^*(\vec{y}^*)\}$ is infinite for some $\vec{y}^* \in \epsilon(\vec{x})$. Thus $\sup_{\omega \in B^*(\vec{y}^*)} \ell(a',\omega) \leq \gamma \leq \sup_{\omega \in B^*(\vec{y}^*)} \ell(a,\omega)$. Since $a \in \alpha(\vec{y}^*)$, these weak inequalities must actually be equalities and therefore $a' \in \alpha(\vec{y}^*)$. Since the choice of $\vec{y}^*$ in this argument depends only on γ which is the same for all $a' \in \alpha(x)$, the argument actually establishes assertion (26).

If α is singleton-valued (that is, if the correspondence α is actually a single-valued function), then condition (25) is equivalent to consistency of α (that is, to condition (9)). Hence α is Bayes by theorems 2 and 3. ∎

7. Comparing the Bayes and weighted-utility criteria

Another illuminating comparison is between Bayes contingent plans and those that would satisfy the analogous optimality definition with respect to the "weighted" utility functions introduced by Chew (1983). Chew replaced the "Independence" axiom of the expected utility theory with weaker axioms ("Betweenness" axiom and "Substitution-Independence" axiom), and identified a class of preferences among uncertain prospects that are representable by a weighted mean of the utility values of sure prospects (weighted utility function). These utility functions are intended to generalize expected utility as narrowly as possible while accommodating Allais' paradox.

Suppose, according to an expected utility function or minimax criterion, an action is optimal for probability measures $\Pr^1$ and $\Pr^2$ on Ω. Then, the action remains to be optimal for any convex combination of $\Pr^1$ and $\Pr^2$. However, if the underlying preference is a weighted utility function, this is not generally true any more. (Park (1993) characterizes the mapping, from the probability simplex on a finite set of decision-relevant events to a finite set of actions, that may be rationalized by weighted utility. In particular, the boundary of the set of probability measures at which one action is preferred to another is the set of roots of a quadratic function of the probabilities.) As a consequence, weighted utility functions may produce contingent plans that violate consistency as is illustrated by an example below. This is not a degenerate example because any weighted utility function "sufficiently close" to the one in the example, produces the same phenomenon.

Chew's (1983) original definition of weighted utility is stated in von Neumann and Morgenstern's framework involving preferences among lotteries, but it can be paraphrased to define conditional weighted utility in a way that is analogous to the definition of conditional expected utility in (1). In addition to a state-contingent utility function u, a "weighing function" $\gamma\colon \mathbb{R} \to \mathbb{R}_{++}$ is used to characterize weighted utility. This function γ takes the state-contingent utility of an action as its argument. (In Chew's formulation, its argument is a lottery payoff which would correspond to a consequence in Savage's formulation of the expected-utility theory.) Expected utility is the special case of weighted utility in which the weighing function is constant. Here is the definition of conditional weighted utility provided that appropriate measurability conditions are satisfied.

$$\tilde{U}(a, \omega, t) = \frac{\int_{B(\omega,t)} \gamma(u(a,\theta)) u(a,\theta)\, d\Pr(\theta)}{\int_{B(\omega,t)} \gamma(u(a,\theta))\, d\Pr(\theta)} \tag{27}$$

Corresponding to the definition of a Bayes contingent plan, say that a contingent plan α is *weighted-rational for* $\langle \Omega, \mathcal{B}, \Pr, \{X_t\}_{t\in\mathbb{N}_+}\rangle$ (or simply *weighted-rational for* $\{X_t\}_{t\in\mathbb{N}_+}$) if α satisfies

$$\forall \vec{x} \in \mathbf{X}^* \;\; \forall a \in A \;\; \left[a \in \alpha(\vec{x}) \iff \tilde{U}^*(a, \vec{x}) = \max_{a' \in A} \tilde{U}^*(a', \vec{x}) \right] \tag{28}$$

with respect to some state-contingent weighted utility function and weighting function and to that stochastic process. ($\tilde{U}^*$ is defined analogously to U^*.)

Now we construct an example of a weighted-rational contingent plan that fails to be consistent. Let $\Omega = \{\omega_1, \omega_2\}$, with $\Pr(\omega_1) = \Pr(\omega_2) = 1/2$. Define $\mathbf{X} = \{1, 2\}$, $X_1(\omega_1) = 1$ and $X_1(\omega_2) = 2$.

Let $A = \{a, b\}$. Define $u(a, \omega_1) = 1$, $u(a, \omega_2) = 3$, $u(b, \omega_1) = 2$, and $u(b, \omega_2) = 4$. Select a continuous (or, smooth) weighting function γ such that $\gamma(1) = \gamma(4) = 1$ and $\gamma(2) = \gamma(3) = 4$. For the resulting weighted-utility preference, a simple calculation shows that $\tilde{U}(a, \omega, 0) = 13/5$ and $\tilde{U}(b, \omega, 0) = 12/5$ so that $\alpha(\emptyset) = \{a\}$.

Next, find the optimal action after the realization of X_1. For $X_1(w) = 1$, or equivalently, for $\omega = \omega_1$,

$$\tilde{U}(a, \omega, 1) = \frac{\gamma(u(a, \omega))u(a, \omega)}{\gamma(u(a, \omega))} = u(a, \omega_1) = 1,$$
$$\tilde{U}(b, \omega, 1) = \frac{\gamma(u(b, \omega))u(b, \omega)}{\gamma(u(b, \omega))} = u(b, \omega_1) = 2.$$

Similarly, for $X_1(w) = 2$, or equivalently, for $\omega = \omega_2$,

$$\tilde{U}(a, \omega, 1) = u(a, \omega_2) = 3,$$
$$\tilde{U}(b, \omega, 1) = u(b, \omega_2) = 4.$$

Hence, $\alpha(\langle 1 \rangle) = \alpha(\langle 2 \rangle) = \{b\}$. This is a violation of consistency at $\emptyset$, namely,

$$\bigcap_{\vec{y} \in \epsilon(\emptyset)} \alpha(\vec{y}) = \{b\} \notin \{\emptyset, \alpha(\emptyset)\}.$$

This example illustrates the violation of consistency described in section 2 when actions a and b are going to the opera and staying at home, respectively, and different values of X_1 indicate different operas being announced to be performed.

8. Conclusion

An intuitively natural consistency condition for contingent plans has been shown to be necessary and sufficient for a contingent plan to be rationalized by maximization of conditional expected utility. One alternative theory of choice under uncertainty, the weighted-utility theory developed by Chew Soo Hong (1983) does not entail that contingent plans will generally satisfy this condition. Another alternative theory, the minimax theory as formulated by Savage ([1954] 1972), does entail the consistency condition at least for single-valued plans.

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