Preliminary

Consistency and the Competitive Outcome Function

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Abstract

In this paper we are interested in the social choice theory of allocating resources, which are available and can be consumed in integer units only. Since goods are available in integer units only, the social choice theory for such problems cannot exploit any smoothness property, which may otherwise have been embedded in the preferences of the agents. This makes the outcome function approach for the study of such problems quite compelling. Our purpose here is to study outcome functions, which are efficient and consistent. We provide an example to show that the competitive social choice function may not be converse consistent. The competitive outcome function is easily observed to be efficient, consistent and converse consistent. What we are able to show here is that any efficient and consistent outcome function which is "reasonably well-behaved" for two-agent problems, must be a sub-correspondence of the competitive outcome function. Our proof of this result requires the converse consistency of the competitive outcome function.

1. Introduction: The axiomatic theory of resource allocation among a finite number of agents is concerned with the analysis of rules, which map preference profiles and initial endowments to desirable feasible allocations. While much of received theory has favored the rule, which selects competitive allocations, the associated competitive price vector has by and large not appeared as an explicit outcome of social choice, though it invariably plays a significant role in the relevant proofs. An exhaustive survey of the vast literature on this topic is the one by Moulin and Thomson (1997).

The reason why prices do not appear as explicit outcomes either in the statements of the theorems or axioms in received theory, is because the conventional model deals with perfectly divisible goods and convex preferences, which may also be smooth. Smooth convex preferences and the Second Fundamental Theorem of Welfare Economics are often invoked in the proofs to determine a set of unique price vectors. However the price vector being an intermediate device in the argument can usually be suppressed in the statement of the results.

An alternative approach for the study of resource allocation mechanisms, dating back to the seminal work of Hurwicz (1973), does require price to be an explicit outcome of interactive communication. A comprehensive account of developments in this direction is available in Campbell (1987). The idea there is agents announce messages, which an "outcome function" translates into allocations. Recent literature on mechanism design, often refer to the outcome function as a "game form". While mechanism design does not insist that messages should be preferences and/or endowments, it allows for mechanisms where such information may be required from the agents. Mechanisms where agents communicate preferences are known as revelation mechanisms. In this paper we use the term "outcome function" to refer to the rule which associates with each preference profile and initial endowment a set of price-allocation pairs.

We are here interested in the social choice theory of allocating resources, which are available and can be consumed in integer units only. Bundle auctions as discussed in Bhickchandani and Mamer (1997) is an example of such resource allocation, where exactly one unit of every resource is available. If in addition no bundle is preferred to its best item, i.e. the worth of a bundle is equal to the worth of its best item, then the analysis relates very naturally to the assignment game of Shapley and Shubik (1972).

Existence of competitive equilibrium in such models is especially problematic. Even in the special case investigated by Bhickchandani and Mamer (1997), competitive equilibrium need not exist. However, in the assignment game set up of Shapley and Shubik (1972) a competitive equilibrium always exists. The importance of competitive equilibrium in resource allocation problems, lies in that competitive outcomes belong to the core of the corresponding market game as defined and discussed in Shapley and Shubik (1969, 1976). Necessary and sufficient conditions for the existence of competitive equilibria in integer allocation problems, referred to here as discrete concave market games, are investigated in Yang (2001) and Sun and Yang (2004).

Since goods are available in integer units only, the social choice theory for such problems cannot exploit any smoothness property, which may otherwise have been embedded in the preferences of the agents. This makes the outcome function approach for the study of such problems quite compelling. Our purpose here is to study outcome functions, which are efficient and consistent.

That chosen alternatives include a statement about prices, does not automatically restrict the outcome function to agree with the one that selects competitive outcomes only. There are a large number of extremely reasonable and well-known non-competitive price mechanisms in the literature on rationing, which can be accommodated within this framework. In fact any feasible allocation along with any price vector is a possible outcome.

Efficiency requires a feasible allocation to yield the maximum possible output. Consistency of a chosen allocation requires that it continues to be chosen in any reduced or sub-problem. Fortunately, the competitive social choice function is both efficient as well as consistent. A third property known as converse consistency of a social choice function says that if a feasible allocation for a given problem is a chosen allocation for every two-agent sub-problem, then it is also a chosen allocation for the original problem. In the divisible goods case the Second Fundamental Theorem of Welfare Economics along with the smoothness of preferences of preferences is used to prove that the competitive social choice function is converse consistent. When goods are available in integer units only, such a method cannot be applied. We provide an example to show that the competitive social choice function may not be converse consistent. On the other hand, if we approach the problem using outcome functions, then we are able to retrieve a considerable amount of lost ground.

An outcome is efficient if its allocation is. A chosen outcome is consistent for a problem if it continues to be chosen for all sub-problems. A feasible outcome for a given problem is converse consistent if whenever it is a chosen outcome for all two-agent sub-problems, it becomes a chosen outcome for the original problem as well.

The competitive outcome function is easily observed to be efficient, consistent and converse consistent. What we are able to show here is that any efficient and consistent outcome function which is "reasonably well-behaved" for two-agent problems, must be a sub-correspondence of the competitive outcome function. Our proof of this result requires the converse consistency of the competitive outcome function. Hence, by invoking efficiency and consistency, we can do no better than the competitive outcome function, provided that the outcome function is "reasonably well-behaved", for two-agent problems.

There is a related literature on allocating goods available in integer amounts along with money, where competitive equilibrium correspond exactly with the set of envy free allocations. If the entire initial endowment is socially owned then at competitive prices each agent is required to pay the worth of its consumption of inputs and receives the average worth of the initial endowment. It is easily observed from a discussion available in Lahiri (2005) that the resulting pay-offs to the agents are envy-free: no agent could do better by seeking the input consumption bundle of another agent. In this framework, Bevia (1996) studies consistency in the multi-commodity framework, where one unit of each commodity is available for re-distribution. In the many input situation, Tadenuma and Thomson (1991) show that if a sub-solution of the envy-free solution satisfies neutrality (i.e. names of inputs don't matter) and consistency, then it is in fact the envy free solution. Further, Tadenuma and Thomson (1993) show that if there is one input available in integer units, then the winner's curse solution is the finest sub-solution of the no envy solution. A comprehensive survey of the use of consistency in axiomatic resource allocation is available in Thomson (2004), wherein a significant step in the proof that is used to establish our main result is referred to as "The Elevator Lemma".

2. The Model: This section builds on the framework developed in Lahiri (2005).

Let $N = \aleph \cup \{0\}$, where $\aleph$ denotes the set of natural numbers. A non-empty set P of $\aleph$ with cardinality greater than or equal to two denotes the set of potential agents. Suppose there are $L+1 > 1$ commodities. The first L commodities are used as inputs to produce the $L+1^{th}$ commodity, which is a numeraire consumption good.

A function f: $N^L \rightarrow \Re$ is said to be monotonically non-decreasing if for all $x,y \in N^L$: $x \geq y$ implies $f(x) \geq f(y)$.

A function f: $N^L \rightarrow \Re$ is said to be discrete concave if there exists a continuous concave function $g: \Re^L_+ \rightarrow \Re$ such that the restriction of g to N^L coincides with f.

A discrete concave market game is a pair $<\{f^i / i \in I\}, w>$ such that:

(i) I is a non-empty subset of P with cardinality greater than or equal to two, denoting the set of agents in G;

(ii) $w \in N^L$ is the aggregate initial endowment vector of inputs available to the agents;

(iii) for all $i \in I$: f^i is a monotonically non-decreasing discrete concave production function representing the preferences of agent $i \in I$.

The analysis in Lahiri (2005) was carried out in a fixed population framework and under the additional assumption that w $\in N^L \cap \Re^L_{++}$. Since the current investigation concerns a variable population framework, the axioms invoked will require that the initial endowment of one or more input may be zero.

Let Ω denote the set of all discrete market games, a generic element of which is denoted G.

Given a discrete concave market game G = <{f^i/ i∈I }, w>, for j = 1,…,L, let w_j be the aggregate amount of commodity j that is available in the economy.

Given a discrete concave market game G = <{f^i/ i∈I }, w>:

(a) An input consumption vector of agent i is denoted by a vector $X^i \in N^L$.

(b) An allocation is an array X = <X^i/ i∈I > such that $X^i \in N^L$ for all i∈I.

(c) An allocation X = <X^i/ i∈I > is said to be feasible if $\sum_{i\in I} X^i$ = w.

Given G = <{f^i/ i∈I}, w>∈Ω, an allocation X feasible for G and a non-empty subset J of I, the reduced game of G with respect to J at X, denoted G_J^X is defined to be the discrete concave market game <{f^i/ i∈J}, $\sum_{i\in J} X^i$ >.

Any non-empty subset Ω^0 of Ω is called an admissible domain if it satisfies the following property: [G = <{f^i/ i∈I }, w>∈Ω^0 , X feasible for G and (φ≠) J ⊂ I with |J| = 2] implies [$G_J^X \in \Omega^0$].

<u>Notation</u>: Given G∈Ω, we denote the set {G_J^X / (φ≠) J ⊂ I, X feasible for G} by Ω(G) and the set {G_J^X / (φ≠) J ⊂ I, X feasible for G} by Ω^2(G).

Clearly both Ω(G) and Ω^2(G) are admissible domains. Let us verify that Ω(G) is an admissible domain.

Let $G_J^X \in \Omega$(G) and let (φ≠) J' ⊂ J. Let X' be feasible for G_J^X . Thus, $\sum_{i\in J} X^i = \sum_{i\in J} X'^i$. Let X* be the allocation such that $X^{*i} = X'^i$ for all i∈J, $X^{*i} = X^i$ for all i∈I \ J. Thus, X* is feasible for G. Futher, $(G_J^X)_{J'}^{X'} = G_{J'}^{X^*} \in \Omega$(G).

Thus, Ω(G) is an admissible domain.

That Ω^2(G) is an admissible domain, follows from its definition.

A social choice function on an admissible domain Ω^0 is a correspondence S: $\Omega^0 \to \bigcup_{(\phi\neq)I\subset P} (N^L)^I$, such that for all G∈Ω^0 and X∈ S(G): X is a feasible allocation for G.

The domain of S which is Ω^0 in this case, is denoted dom(S).

We do not require a social choice function to be non-empty valued.

A price vector p is an element of $\Re^L_+ \backslash \{0\}$, where for j = 1,…,L, p_j denotes the price of input j. Clearly a price vector ***<u>does not allow</u>*** all inputs to be available for free.

An outcome for a discrete concave market game G is an ordered pair (X,p) where X is a feasible allocation for G and p is a price vector.

An outcome function on an admissible domain Ω^0 is a correspondence F: $\Omega^0 \to \bigcup_{(\phi\neq)I\subset P}(N^L)^I \times \Re_+^L\backslash\{0\}$, such that for all $G\in\Omega^0$ and $(X,p)\in F(G)$: (X,p) is an outcome for G.

The domain of F which is Ω^0 in this case, is denoted dom(F).

We do not require an outcome function to be non-empty valued.

Given an outcome function F, let S_F denote its projection on $\bigcup_{(\phi\neq)I\subset P}(N^L)^I$, i.e. for all $G\in$dom(F): $S_F(G)$ = {X/ there exists p such that $(X,p)\in F(G)$}.

Clearly, S_F is a social choice function.

An outcome (X,p) for $G = <\{f^i/ i\in I\}, w>\in\Omega$ is said to be a competitive outcome if for all $i\in I$: $f^i(X^i) - p^TX^i \geq f^i(x) - p^Tx$ for all $x\in N^L$.

If (X,p) is a competitive outcome for G, then X is said to be a competitive allocation for G and p is said to be a competitive price vector.

An outcome function F^* such that for all $G = <\{f^i/ i\in I\}, w> \in$dom(F*): F*(G) = {(X,p)/ (X,p) is a competitive outcome for G}, is said to be a competitive outcome function.

S_{F^*} is said to be a competitive social choice function.

A feasible allocation $X^* = <X^{*i}/ i\in I>$ is said to be efficient for $G = <\{f^i/ i\in I\}, w>$, if $\sum_{i\in I} f^i(X^{*i}) \geq \sum_{i\in I} f^i(X^i)$, whenever $X = <X^i/ i\in I>$ is any feasible allocation for G.

A social choice function S is said to be efficient if for all $G\in$dom(S): $X\in S(G)$ implies X is efficient for G.

An outcome function F is said to be efficient if the corresponding social choice function S_F is efficient.

If I and J are non-empty subsets of P with $I \subset J$ and $X \in(N^L)^I$, then X_J denotes the array $<X^i/i\in J>$.

3. Consistency and Converse Consistency of Social Choice and Outcome Functions: A social choice function S is said to be <u>Consistent</u> (or satisfy Con) if for all $G = <\{f^i/ i\in I\}, w>\in$dom(S), $(\phi\neq)J \subset I$: [$X\in S(G)$, $G_J^X \in$ dom(S)] implies [$X_J\in S(G_J^X)$].

A social choice function S is said to be <u>Converse Consistent</u> (or satisfy CCon) if for all $G = <\{f^i/ i\in I\}, w>\in$dom(S), and X feasible for G: [$X_J\in S(G_J^X)$, for all $(\phi\neq)J \subset I$, with |J| = 2] implies [$X\in S(G)$].

Proposition 1: S_{F^*} is efficient and satisfies Con. However, it does not satisfy CCon.

Proof: That S_{F^*} is efficient is easily established (Lahiri [2005]).

Let $G = <\{f^i/ i\in I\}, w>\in$dom($S_{F^*}$), $(\phi\neq)J \subset I$, $G_J^X \in$dom(S) and $X\in S_{F^*}(G)$. Thus, there exists a price vector p such that $(X,p)\in$ F*(G).

Hence, for all $i\in I$: $f^i(X^i) - p^TX^i \geq f^i(x) - p^Tx$ for all $x\in N^L$.

Thus, for all $i\in J$: $f^i(X^i) - p^TX^i \geq f^i(x) - p^Tx$ for all $x\in N^L$.

Since, $<X^i/i\in J>$ is feasible for G_J^X, $(X_J,p) \in F^*(G_J^X)$, i.e. $X_J \in S_{F^*}(G_J^X)$.

Thus, S_{F*} is consistent.

To show that that S_{F*} is not Converse Consistent we construct the following example, which has a similar motivation as the one used in the proof of Proposition 2 of Ergin (2000), in that both use a pattern similar to the one that is used to exhibit the Condorcet Paradox.

Let L = 3, I = {1,2.3} and w = $e_1 + e_2 + e_3$. Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Let $f^1(e_1) = f^2(e_3) = f^3(e_2) = 1.25$, $f^1(e_2) = f^2(e_1) = f^3(e_3) = 1$, $f^1(e_3) = f^2(e_2) = f^3(e_1) = 0$; $f^i(x) = 1.25$ for all $x \in N^3$ with $\sum_{j=1}^{L} x_j > 1$ and $i \in I$.

Let G = <$\{f^i/ i\in I\}$, w> and dom(S_{F*}) = $\Omega(G) \equiv \Omega^2(G)$.

Let X be the feasible allocation where $X^1 = e_2$, $X^2 = e_1$, $X^3 = e_3$.

Let J = {1,2} and $p^J = e_1 + 0.75e_2$. Then, $(X^J, p^J) \in F^*(G_J^X)$, i.e. $X_J \in S_{F*}(G_J^X)$.

Let J = {1,3} and $p^J = e_2 + 0.75e_3$. Then, $(X^J, p^J) \in F^*(G_J^X)$, i.e. $X_J \in S_{F*}(G_J^X)$.

Let J = {2,3} and $p^J = 0.75e_1 + e_3$. Then, $(X^J, p^J) \in F^*(G_J^X)$, i.e. $X_J \in S_{F*}(G_J^X)$.

However, X is not efficient for G, since $f^1(e_1) + f^2(e_3) + f^3(e_2) > f^1(X^1) + f^2(X^2) + f^3(X^3)$ and <e_i/ i = 1,2,3> is feasible for G. Thus, $X \notin S_{F*}(G)$. Q.E.D.

An outcome function F is said to be Consistent (or satisfy Con) if for all G = <$\{f^i/ i\in I\}$, w>$\in$dom(F), $(\phi\neq)J \subset I$: [$(X,p)\in F(G)$ and $G_J^X \in$ dom(F)] implies [$(X_J, p)\in F(G_J^X)$].

An outcome function F is said to be Converse Consistent (or satisfy CCon) if for all G = <$\{f^i/ i\in I\}$, w>$\in$dom(F), and outcome (X,p) for G: [$(X_J,p)\in F(G_J^X)$, for all $(\phi\neq)J \subset I$, with $|J| = 2$] implies [$(X,p)\in F(G)$].

Proposition 2: F* is efficient and satisfies Con and CCon.

Proof: That F* satisfies efficiency and Con are easily verified, by methods similar to the ones adopted in the proof of Proposition 1.

Let G = <$\{f^i/ i\in I\}$, w>$\in$dom(F*), and let (X,p) be an outcome for G. Suppose $(X_J,p)\in F^*(G_J^X)$, for all $(\phi\neq)J \subset I$, with $|J| = 2$. Thus, for all $i\in I$: $f^i(X^i) - p^TX^i \geq f^i(x) - p^Tx$ for all $x\in N^L$. Hence, $(X,p)\in F^*(G)$.

Thus, F* satisfies CCon. Q.E.D.

4. Efficiency and Competitive Outcomes: The proof of the following proposition is available in Lahiri (2005).

Proposition 3: Let G = <$\{f^i/ i\in I\}$, w>$\in$dom(F*), $(X^*,p)\in F^*(G)$ and let X be an efficient allocation for G. Then, $(X,p)\in F^*(G)$.

Proposition 3 implies the following:

Proposition 4: Let G = <$\{f^i/ i\in I\}$, w>$\in$dom(F*). Then [$F^*(G) \neq \phi$] implies F*(G) = {X/ X is efficient for G}$\times${p/ p is a competitive price vector for G}.

As a consequence of Proposition 3, we obtain the following important result:

Theorem 1: Let S be an efficient social choice function with dom(S) = dom(S_{F^*}). Then for all G∈dom(S) with F*(G) ≠ φ(or equivalently S_{F^*}(G) ≠ φ): S(G) ⊂ S_{F^*} (G).

A discrete concave market game need not have a competitive outcome.

Example 1: Let G = <{f^i/ i∈I}, w> with |I| = 2, w = 3, L = 1. For i∈I, let $f^i(x) = x$ for x ∈{0,1}; $f^i(x) = 1$ for x ≥ 1. If p > 0, then $f^i(x) – px = 0$ if x = 0,

= 1 – px for x ≥ 1.

Thus, for p∈(0,1), the profit of each agent is maximized at x = 1. For p > 1, profit is maximized at x = 0. Hence the total requirement of the commodity never exceeds 2. Clearly, G does not have a market equilibrium.

Yang (2001) and Sun and Zang (2004) provide necessary and sufficient conditions under which a discrete concave market game possesses a competitive outcome.

An admissible domain Ω^0 is said to be <u>Agreeable for Two-Agents</u> if for all G = <{f^i/ i∈I}, w>∈Γ^2 with |I| = 2, G has a competitive outcome.

F is said to be <u>Non-Empty valued for two-agents</u> if for all G = <{f^i/ i∈I}, w>∈ dom(F) with |I| = 2, F(G) ≠ φ.

F is said to satisfy <u>Non-discrimination for two-agents</u> if for all G = <{f^i/ i∈I}, w>∈ dom(F) with |I| = 2: [X∈S_F(G), X' feasible for G and $\sum_{i\in I} f^i(X^i) = \sum_{i\in I} f^i(X'^i)$] implies [X'∈S_F(G)].

Neither of the two properties mentioned above implies the other. The outcome function which is identically the empty set, satisfies Non-discrimination for two-agents, but violates Non-empty valued for two-agents.

Example 2: Let p* = $\sum_{j=1}^{L} e_j$. Let P = ℵ. Let Ω^0 be the set {G = <{f^i/ i∈I}, w>/ I is a finite subset of P with |I| ≥ 2, w∈N^L and for all i∈I and x∈N^L: $f^i(x) = \sum_{j=1}^{L} x_j$ }. Clearly Ω^0 is agreeable for two-agents.

For G = <{f^i/ i∈I}, w>∈ Ω^0 with |I| ≠ 2, let F(G) = F*(G). For G = <{f^i/ i∈I}, w>∈ Ω^0 with |I| = 2, if I = {i,k} with i < k, then let F(G) = (X,p*), where X^i = w and X^k = 0.

Let w = $10\sum_{j=1}^{L} e_j$. Then, X' = <X''^i,X'^k>, where $X''^i = X'^k = 5\sum_{j=1}^{L} e_j$ satisfies $\sum_{i\in I} f^i(X^i) = \sum_{i\in I} f^i(X'^i)$. However, X'∉S_F(G). Thus, F does not satisfy Non-Discrimination, although F is non-empty valued for all two-agent problems.

Observation: Let F be an outcome function with dom(F) = dom(F*) = Γ^2.

Suppose F is Efficient, Non-empty valued for two agents and satisfies Non-discrimination for two agents. Further suppose dom(F) = dom(F*) is agreeable for two agents.

Let $G = <\{f^i / i \in I\}, w> \in$ dom (F) with $|I| = 2$.
Suppose $X \in S_{F^*}(G)$. Since F is non-empty valued for two-agents $S_F(G)$ is non-empty. Since F satisfies efficiency, $X' \in S_F(G)$ implies X' is efficient for G. Since F* is efficient, X is efficient for G. Since F satisfies Non-discrimination for two-agents, $X \in S_F(G)$.
Hence, $S_{F^*}(G) \subset S_F(G)$.
Since dom(F) = dom(F*) is agreeable for two-agents, by Theorem 1, $S_F(G) \subset S_{F^*}(G)$.
Thus, $S_{F^*}(G) = S_F(G)$.

5. The Main Result: In this section we show that the competitive outcome function on domains which are agreeable for two-agents is the coarsest outcome function which satisfies efficiency, consistency and the following property.

F is said to satisfy the Two-Agent Price Property if for all $G = <\{f^i / i \in I\}, w> \in$ dom(F) with $|I| = 2$ and all $(X,p) \in F(G)$: [X is a competitive allocation] implies [(X,p) is a competitive outcome].

Theorem 2: Let F be an outcome function with dom(F) = dom(F*) being Agreeable for Two-Agents. If F is efficient and satisfies Con and Two-Agent Price Property then for all $G \in$ dom(F): $F(G) \subset F^*(G)$.

Proof: Suppose F is efficient and satisfies Con and Two-Agent Price Property. Further suppose dom(F) = dom(F*) being Agreeable for Two-Agents.
Let $G = <\{f^i / i \in I\}, w> \in$ dom (F).
Suppose $(X,p) \in F(G)$. By Con of F, $(X_J,p) \in F(G_J^X)$, for all $J \subset I$, with $|J| = 2$. Thus, $X_J \in S_F(G_J^X)$, for all $J \subset I$, with $|J| = 2$. Since dom(F) = dom(F*) is Agreeable for Two-Agents, by Theorem 1, $S_F(G_J^X) \subset S_{F^*}(G_J^X)$ for all $J \subset I$, with $|J| = 2$. Thus, $X_J \in S_{F^*}(G_J^X)$ for all $J \subset I$, with $|J| = 2$, i.e. X_J is a competitive allocation for all $J \subset I$, with $|J| = 2$. By Two-Agent Price property, $(X_J,p) \in F^*(G_J^X)$, for all $J \subset I$, with $|J| = 2$.
Since by Propostion 2, F* satisfies CCon, it must be the case that $(X,p) \in F^*(G)$.
Thus, $F(G) \subset F^*(G)$. Q.E.D.

Note: That the common domain of F and F* is agreeable for two-agents is ***not*** a superfluous assumption.
For instance if $\Omega^0 = \{G\}$ where $G = <\{f^i / i \in I\}, w>$ with $|I| = 2$, $w = 3$, $L = 1$, then Ω^0 is admissible though not agreeable, since as we saw in Example 1, there is no competitive outcome for G. Let $p^* = \sum_{j=1}^{L} e_j$. Then, F such that F(G) = {X/ X is an efficient allocation for G}$\times$\{p*\}, is efficient and satisfies Con and Two-Agent Price Property, since F*(G) = ϕ. However, it does not satisfy the conclusion of Theorem 1.

Proposition 5: There exists an outcome function F, with dom(F) = dom(F*) being Agreeable for Two-Agents, such that F is efficient and satisfies Con, CCon and Two-

Agent Price Property and yet F(G) is a non-empty proper subset of F*(G) for all G = <{f^i/ i∈I}, w> ∈dom(F), with w∈ $\Re^L_{++}$ and |I| ≥ 2.

Proof: Let p* = $\sum_{j=1}^{L} e_j$. Let Ω^0 be the set {G = <{f^i/ i∈I}, w>/ I is a finite subset of P with |I| ≥ 2, w∈N^L and for all i∈I and x∈N^L: f^i(x) = $\sum_{j=1}^{L} x_j$ }. Clearly Ω^0 is agreeable for two-agents.

For G = <{f^i/ i∈I}, w>∈ Ω^0, let F(G) = {(X,p*)}, where X^i = w if i = min{k/k∈I}, X^i = 0, otherwise.

Clearly, F(G) ⊂ F*(G) for all G∈Ω^0 and F satisfies all the desired properties including CCon.

For G = <{f^i/ i∈I}, w>∈ Ω^0, let X' be the feasible allocation such that X'^i = w if i = max{k/k∈I}, X'^i = 0, otherwise.

(X',p*)∈F*(G) for all G∈Ω^0.

However, for all G = <{f^i/ i∈I}, w> ∈dom(F), with w∈ $\Re^L_{++}$ and |I| ≥ 2, (X',p*) ≠ (X,p*).

Hence, F(G) is a non-empty proper subset of F*(G) for all G = <{f^i/ i∈I}, w> ∈dom(F), with w∈ $\Re^L_{++}$ and |I| ≥ 2. Q.E.D.

Observation: Suppose that F is an outcome function which is efficient, non-empty valued for two-agents and satisfies Non-discrimination for Two-Agents and Two-Agent Price Property. Suppose S_F satisfies Con and CCon.

Let G∈dom(F) and X∈S_{F*}(G).

By Con of S_{F*} (established in Proposition 1), X_J∈S_{F*}(G^X_J), for all J ⊂ I, with |J| = 2.

Thus, for all J ⊂ I, with |J| = 2: X_J is an efficient allocation for G^X_J . Since F and hence S_F is efficient and non-empty valued for two-agents for all J ⊂ I, with |J| = 2: S_F(G^X_J) is non-empty and every allocation in S_F(G^X_J) is efficient for G^X_J . By Non-discrimination of F, for all J ⊂ I, with |J| = 2: X_J∈ S_F(G^X_J). By CCon of S_F: X∈S_F(G).

Thus, S_{F*} ⊂ S_F.

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References

1. Bevia, C. (1996): "Identical preferences lower bound solution and consistency in economics with indivisible goods", Social Choice and Welfare 13: 113 – 126.
2. Campbell, D.E. (1987): "Resource Allocation Mechanisms", Cambridge University Press.
3. Ergin, H. (2000): "Consistency in House Allocation Problems", Journal of Mathematical Economics 34: 77-97.
4. Hurwicz, L. (1973): "The Design of Mechanism for Resource Allocation", American Economic Review, Vol. 63, No:2: 1 – 30.

5. Lahiri, S. (2005): “Manipulation via Endowments in a market with profit maximizing agents”, (mimeo).
6. Moulin, H. and W. Thomson (1997): “Axiomatic Analysis of resource allocation”, in *Social Choice Re-examined*, (K. Arrow, A. Sen, K. Suzumura, eds.), London: Mc Millan: 101-120.
7. Shapley, L. and M. Shubik (1969): “On Market Games”, Journal of Economic Theory, 1: 9-25.
8. Shapley, L. and M. Shubik (1972): “The assignment game 1: the core”, International Journal of Game Theory, 1: 111-130.
9. Shapley, L. and M. Shubik (1976): “Competitive Outcomes in the Cores of Market Games”, International Journal of Game Theory, 4: 229-237.
10. Sun, N. and Z. Yang (2004): “The Max-Convolution Approach to Equilibrium Models with Indivisibilities”, (mimeo).
11. Tadenuma, K. and W. Thomson (1991): “No-envy and consistency in economies with indivisible goods”, Econometrica 59: 1755-1767.
12. Tadenuma, K. and W. Thomson (1993): “The fair allocation of an indivisible good when monetary compensations are possible”, Mathematical Social Sciences 25: 117-132.

    Thomson, W. (2004): “Consistency and its Converse: an Introduction”, paper prepared for a NATO Advanced Research Institute entitled “Game Theory and Resource Allocation: the Axiomatic Approach”.
14. Yang, Z. (2001): “A Practical Competitive Market Model for Indivisible Commodities”, Cowles Foundation Discussion Paper No. 1317, Cowles Foundation For Research in Economics, Yale University, New Haven, Connecticut.