Time, risk and utility: a role-play analysis *

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Abstract. This paper defines temporal risk aversion in the context of a simple choice framework: that of time varying utility of wealth. The attention is focused on a decision maker who acts as a buyer: temporal risk premium, instantaneous risk premium and time preference premium are defined.

KEYWORDS AND PHRASES: temporal utility; buyer-seller viewpoints; temporal risk premium; instantaneous risk premium; time preference premium.

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1 Introduction

During most of the post-war period, the Expected Utility (EU) model has been one of the main scientific results of modern economic analysis. Yet individual choices under uncertainty may be inconsistent with the hypothesis of EU model. Then, some attempts to develop alternative models have been proposed among which some highly promising generalizations of EU. Nevertheless, very often they don't keep the simple and tractable formulation of EU.

In the Arrow-Pratt framework the utility for wealth is assumed to be not changing with time, i.e. utility is timeless. Clearly preferences may change with time: Nachman ([4]) developed a theory for measuring aversion to risk that allows for time varying preferences. In his context, a decision maker at a fixed time must compare her present wealth position with some future wealth position: she has well-defined preferences over uncertain prospects where outcomes are date-wealth pairs.

Our work refers to this model: more precisely, we are interested in the study of the distinction between agents who assume risk and agents who dispose themselves of it, in the case of time varying preferences.

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Some Authors have studied this distinction, with reference to preferences not changing with time. In [3] the distinction between the two sides of a contract in which one gives a risk and a counterpart assumes it is well presented, with a lot of references to decision problems in Statistics. In ([6]) the certainty equivalent of a random gain from the viewpoint of the seller and of the buyer is introduced, both in case of random initial wealth. In ([1]) the analysis is carried out in the framework of the multidimensional version of EU model.

The paper is organized as follows. In Section 2 we first set the objects of preferences and the basic assumptions: definitions of temporal risk premium, instantaneous risk premium and time preference premium are proposed in the case of a buyer B. Section 3 is devoted to the analysis of temporal risk aversion: after the basic definitions of a buyer who exhibits temporal risk averse preferences, instantaneous risk averse preferences, and impatient preferences, some characterizations of these notions will be proposed. Finally in Section 4 some conclusive observations are presented.

2 Problem setting

A Decision Maker at a fixed time point must compare her present wealth position w with some future wealth position: for doing so, she must be assumed to have well-defined preferences over uncertain prospects, namely over prospects whose outcomes are date-wealth pairs (t, w).

Let $\mathbf{R}^+ = [0, +\infty)$ be the range of the variable t and let $\mathbf{R}$ be the range of the wealth w.

$F = \mathbf{R}^+ \times \mathbf{R}$ is the set of all the possible fortunes (t, w), which is endowed with the relative topology from $\mathbf{R}^2$ if it is necessary to consider F as a topological space, and with the σ-field of Borel sets generated by this topology in case F has to be considered a measurable space.

The Decision Maker is identified with a preference order on fortune valued random prospects, that is on probability measures on the Borel subsets of F. In the following, her preference ordering is assumed to admit an expected utility representation where the utility function $u : F \to \mathbf{R}$ satisfies the following set of conditions:

U1 (*continuity*) u is continuous on F;

U2 (*monotonicity*) u is strictly increasing as a function of w, namely

$$u(t, w_1) < u(t, w_2)$$

for each (t, w_1) and $(t, w_2) \in F$ where $w_1 < w_2$;

U3 (*comparability*) for each $t_1, t_2 \in \mathbf{R}^+$ where $t_1 < t_2$ and for each $w_2 \in \mathbf{R}$ there exists $w_1 \in \mathbf{R}$ such that

$$u(t_1, w_1) = u(t_2, w_2).$$

Note that the time section $u_t(w) = u(t, w)$ may be considered the instantaneous (at time t) utility function of the wealth w, in other words a utility function of the Arrow-Pratt framework. So, assumptions $U1$ and $U2$ ensure the existence and the uniqueness of the certainty equivalent in the static Arrow-Pratt framework, namely when u doesn't vary with time. By assumption $U3$ it follows the existence of a wealth level w_1 which is *equivalent* (in terms of utility) to a future value of wealth w_2: this is true for every $0 \leq t_1 < t_2$. It is possible to define the wealth section $u_w(t) = u(t, w)$ with the meaning of function representing preferences on the time in which to possess the wealth level w. As pointed out by Nachman ([4]), the preferences represented by the utility function u referee to two different preference relations: we consider preferences varying with time in the sense of time varying utility of wealth, that is with reference to the instantaneous utility u_t which may change with t.

Given the utility function u and two times s and t where $s < t$, we consider the random variable X and we interpret it as a random increment to present value w to be received immediately. The real-valued random variable X is defined on some probability space. Q is the set of nondegenerate random variables with finite support. It is possible to define the buyer's temporal risk premium such as follows.

Definition 1 *The buyer's temporal risk premium* $\tau^B_{st}(w, X)$ *is the solution of*

$$Eu(s, w + X - (E(X) - \tau^B_{st}(w, X))) = u(t, w). \tag{1}$$

The quantity $E(X) - \tau^B_{st}(w, X)$ is the buyer's temporal certainty equivalent: it is the real number that makes the Decision Maker indifferent between buying the random amount X at time s and possessing the amount w at time t. That is, $\tau^B_{st}(w, X)$ is the minimum price the Decision Maker will not pay at s in order to buy X.

As in the case of the seller's viewpoint, it is possible to determine one part of the temporal risk premium which is related to risk preferences and one part to time preferences.

Definition 2 *The buyer's instantaneous risk premium at time s, $\pi^B_s(w, X)$, is the solution of*

$$Eu(s, w + X - (E(X) - \pi^B_s(w, X))) = u(s, w). \tag{2}$$

Clearly, $\pi^B_s(w, X)$ is the usual buyer's risk premium for the instantaneous utility function u_s. The quantity $E(X) - \pi^B_s(w, X)$ is the buyer's instantaneous certainty equivalent of X at time s, given the wealth w.

If we consider the price the Decision Maker would pay at time s for a deterministic increment z to wealth w in order to receive z at the next time t, it is possible to refer to the definition of time preference premium.

Definition 3 *The time preference premium* $\nu_{st}^B(w,z)$ *is the solution of*

$$u(s, w + z - \nu_{st}^B(w,z)) = u(t, w + z). \tag{3}$$

Note that $w+z-\nu_{st}^B(w,z)$ is the present value of $w+z$, that is $z-\nu_{st}^B(w,z)$ is the present value at wealth w of the future increment z. By assumptions on the utility function u, existence and uniqueness of the time preference premium follow.

Finally, given the definition of buyer's temporal risk premium $\tau_{st}^B(w,X)$, buyer's instantaneous risk premium at time s, $\pi_s^B(w,X)$, and of time preference premium $\nu_{st}^B(w,z)$, it is possible to state the following decomposition of the temporal risk premium

Proposition 1

$$\tau_{st}^B(w,X) = \pi_s^B(w - \nu_{st}^B(w,0), X) - \nu_{st}^B(w,0). \tag{4}$$

Proof

Starting from the definition of time preference premium we have

$$u(t,w) = u(s, w - \nu_{st}^B(w,0))$$

and by the definition of $\pi_s^B(w-\nu_{st}^B(w,0),X)$ it follows

$$u(s, w - \nu_{st}^B(w,0)) = Eu(s, w - \nu_{st}^B(w,0) + X - (E(X) - \pi_s^B(w - \nu_{st}^B(w,0), X))).$$

By referring to the definition of temporal risk premium $\tau_{st}^B(w,X)$, it is possible to deduce the result.

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3 Temporal risk aversion

As the definition of risk aversion may be stated in terms of the sign of the risk premium, both in the static case both with respect to the seller viewpoint, now we set the definitions of a buyer who exhibits temporal risk averse preferences, instantaneous risk averse preferences, and impatient preferences. Moreover some characterizations of these notions will be proposed.

Definition 4 *A buyer B exhibits temporal risk aversion attitude if the temporal risk premium* $\tau_{st}^B(w,X)$ *is nonnegative*

$$\tau_{st}^B(w,X) \geq 0$$

for all $(t,w) \in F$, *for all* s *such that* $0 \leq s < t$ *and all* $X \in Q$.

As usual, the case of strict temporal risk aversion attitude refers to the positivity of the temporal risk premium $\tau_{st}^B(w, X)$

$$\tau_{st}^B(w, X) > 0$$

for all $(t, w) \in F$, for all s such that $0 \leq s < t$ and all $X \in Q$. The next definition consider also the strict case.

Definition 5 *A buyer B exhibits [strictly] instantaneous risk aversion attitude if the instantaneous risk premium $\pi_s^B(w, X)$ is nonnegative [positive]*

$$\pi_s^B(w, X) \geq [>] 0$$

for all $(s, w) \in F$ and all $X \in Q$.

Clearly the notion of instantaneous risk aversion is the same of risk aversion in the Arrow-Pratt framework, in which all the instantaneous utility functions u_t $(t \geq 0)$ of wealth are considered. The next result is an immediate consequence.

Lemma 1 *A buyer B exhibits instantaneous risk aversion if and only if u is concave in w for each fixed $t \in \mathbf{R}^*$.*

Proof The proof easily follows: in fact it is sufficient to apply to the instantaneous utility functions u_t $(t \geq 0)$ of wealth some of the results proposed in ([5]).

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Definition 6 *A buyer B exhibits [strictly] impatience attitude if the time preference premium $\nu_{st}^B(w, z)$ is nonnegative [positive]*

$$\nu_{st}^B(w, z) \geq [>] 0$$

for all $(t, w) \in F$, for all s such that $0 \leq s < t$ and all $z \in \mathbf{R}$.

Speaking of impatience we express the basic idea that an Euro today is worth more that an Euro tomorrow: this is true for every today and every tomorrow, so an Euro today is worth more that an Euro n days after today.

Lemma 2 *A buyer B exhibits impatience attitude if and only if u is decreasing in t for each fixed w.*

Proof The proof directly follows from the definition of time preference premium.

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The link (4) between temporal risk premium, instantaneous risk premium and time preference premium, explicitly suggests that the meaning of temporal risk aversion is related in part to the willingness of avoiding risk and in part to time preferences. The following results state the link between the different buyer's attitudes to temporal risk aversion, instantaneous risk aversion and impatience.

Proposition 2 *A temporal risk averse buyer B necessarily exhibits instantaneous risk aversion.*

Proof

Clearly

$$\lim_{t\to s} \nu_{st}^B(w,z) = 0.$$

The assumptions made on u ensure that $\nu_{st}^B(w,z)$ and $\pi_s^B(w,X)$ are continuous functions of s, t and w.

Then by (4) it is

$$\lim_{t\to s} \tau_{st}^B(w,X) = \lim_{t\to s} \pi_s^B(w-\nu_{st}^B(w,0),X) - \nu_{st}^B(w,0) = \pi_s^B(w,X)$$

thus the thesis follows immediately.

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This result is completely in line with the corresponding one in which a seller Decision Maker is analyzed (see [4]). The study of the link between temporal risk aversion and impatience in the case of the buyer viewpoint is almost different from the relative study in the case of a seller Decision Maker. In fact we can deduce that in case of temporal risk aversion a Decision Maker B surely will have a time preference premium which satisfies a particular inequality.

Proposition 3 *A temporal risk averse buyer B necessarily exhibits a time preference premium which satisfies the following inequality*

$$z - \nu_{st}^B(w,z) > -\nu_{st}^B(w,0) \geq 0, \qquad \forall z > 0.$$

Proof

Let $X_n \in Q$ be a sequence of random variables that converges a.s. to $z \in \mathbf{R}$.

Owing to the fact that the following limit

$$\lim_{n\to+\infty} Eu(s, w-\nu_{st}^B(w,0) + X_n - (E(X_n) - \pi_s^B(w-\nu_{st}^B(w,0),X_n)))-$$

$$-u(s,w-\nu_{st}^B(w,0)) =$$

$$= u(s, w-\nu_{st}^B(w,0) + \pi_s^B(w-\nu_{st}^B(w,0),z))) - u(s,w-\nu_{st}^B(w,0))$$

is equal to 0, then necessarily it is

$$\lim_{n\to+\infty} \pi_s^B(w,X_n) = 0.$$

So it is

$$\lim_{n\to+\infty} \tau_{st}^B(w,X_n) = -\nu_{st}^B(w,0).$$

Moreover, monotonicity assumption on u and definition of time preference premium imply that for every $z > 0$ it is

$$\nu_{st}^B(w,z) < \nu_{st}^B(w,0) + z.$$

Then the thesis follows.

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4 Concluding remarks

In this contribution we considered the problem of deriving a theory for measuring aversion to risk that allows for time varying preferences, in the particular case of a Decision Maker who acts as a buyer. First, we defined the notions of temporal risk premium, instantaneous risk premium and time preference premium; then we devoted the analysis to the study of temporal risk aversion: after the basic definitions of a buyer who exhibits temporal risk averse preferences, instantaneous risk averse preferences, and impatient preferences, some characterizations of these notions have been proposed. Differently from the case of a seller-Decision Maker, the temporal risk premium cannot be decomposed into the sum of two components which are related only to instantaneous risk aversion and to impatience: this is why the analysis of the links between the proposed attitudes is more complicated. In fact the temporal risk premium $\tau_{st}^B(w,X)$ is nonnegative if and only if the instantaneous risk premium $\pi_s^B(w - \nu_{st}^B(w,0), X)$ and the time preference premium $\nu_{st}^B(w,0)$ satisfy the following inequality

$$\pi_s^B(w - \nu_{st}^B(w,0), X) \geq \nu_{st}^B(w,0).$$

Some results may be obtained also in this case: this is the actual direction of study which stems from this approach.

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