Manipulation via Endowments in a Market with Profit Maximizing Agents

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Abstract

In this paper we show that it is possible to manipulate market equilibria in an economy with profit maximizing agents (or exchange economies with quasi-linear utilities and hence interest free money as a means of transfer among agents) by either destroying or withholding ones initial endowments. Example 1 of this paper is an instance of a very simple economy, where every competitive extended allocation is vulnerable to manipulation via both misrepresentation of productive capabilities as well as destruction or withholding of endowments.

1. Introduction: The most well known solution prescribed by modern economic theory for decentralized allocation of resources is Walrasian equilirium. Agents are initially endowed with non-negative quantities of resources and are assumed to engage in mutually compatible exchanges, after maximizing utility or preferences subject to a budget constraint. The budget constraint that an agent faces in such problems imposes a liquidity constraint on their transactions. The Pareto Optimality of such solutions make them very appealing both as prescriptive and descriptive models of market behavior. Campbell (1987) contains a contemporary discussion of issues related to the Walrasian mechanism.

The Walrasian mechanism of resource allocation suffers from severe incentive problems. The possibility of an agent being able to improve itself by misrepresenting its preferences, has often been cited as the main drawback of the Walrasian solution. However, if the Walrasian solution were to depend on an agent's ability to compute an optimal solution to a constrained optimization problem, then its applicability as a solution in resource allocation problems would be rather suspect. It is unreasonable to assume that market participants are or should be equipped with the necessary mathematical expertise to enter into mutually compatible trades, in much the same way as the knowledge of partial differential equations is superfluous for being able to ride a bicycle. Further, the issue of misrepresenting individual preferences, rests on (i) the realization of an economic environment allowing such a misrepresentation; (ii) the agent who stands from the deviation being aware of such a possibility as also of the entire profile of preferences and initial endowments. This latter information being prohibitive and dispersed among the agents, adds to the computational complexity of the exercise of preference misrepresentation. Thus, while the problem of preference misrepresentation is surely a theoretical possibility, the sheer computational complexity of the entire exercise makes the threat posed by it somewhat unlikely.

A more likely possibility is that an agent may tamper around with its available endowments, in order to secure an improvement in welfare for itself. Non-disclosure or false disclosure of taxable income is an extremely common phenomenon that governments often need to deal with. While money income is not the same as a bundle of goods initially endowed with an agent, they are related concepts. In any event, misrepresentation of ones physical assets for accounting or tax purposes, is not uncommon either. The question that naturally arises is the following: Is it possible to manipulate the Walrasian mechanism by either destroying or withholding a portion of ones initial endowment? The answer to this question is in the affirmative, as shown in Postlewaite (1979). In fact Postlewaite (1979) contains the more general result that any social choice mechanism in the pure exchange setting which satisfies Pareto Optimality and individual rationality is manipulable via endowments.

What concerns us here is the ability of a producer to influence prices by taking into consideration the effects of its decision on demand or supply. "The activities of the OPEC oil cartel have made a commonplace phenomenon, that of a group of producers withholding or destroying part of their output, even better known" (:Campbell (1987)). However, a producer in economic theory is assumed to maximize profits instead of utility subject to a budget constraint. A paradigm shift is thus required if agents are assumed to behave like profit maximizing producers and not like budget constrained consumers. The market clearing equilibrium price quantity configuration that now emerges is referred to here as a market equilibrium. Agents are equipped with concave production function, which converts bundles of inputs that the economy is initially endowed with, into a numeraire consumption good. The market prices of the inputs are measured in terms of this numeraire consumption good. Market equilibrium obtains when the total quantity of each input demanded is equal to its total availability within the economy. Sun and Yang (2004) show, that every such economy admits market equilibrium, when the inputs are divisible and under a max-convolution preservability property when the inputs are available in integer amounts only.

The assumption that technology available with a producer is usually common knowledge is certainly not one that can be made without reservations. However, the possibility of technology being more amenable to public scrutiny than individual preferences cannot be completely ruled out either. Hence misrepresenting the technology available with a producer, is less likely to succeed than preference misrepresentation, if at all. Thus, while the arguments concerning computational complexity may have less "bite" when the agents are profit maximizing producers, public knowledge about the technical capabilities of a producer, make misrepresentation of ones production function an unlikely possibility. On the other hand, the possibility of gaining by misrepresenting or manipulating ones initial endowments, remains and once again is not very uncommon in the real world. For manufacturing firms manipulation via endowments is especially problematic as firms can keep inputs idle or destroy them or not sell them on the market. We will show that a single productive agent can do this in a way that influences prices to its advantage. The difference between our analysis and that of Postlewaite (1979) is that we show that market equilibrium is manipulable via destroying as well as withholding initial endowments, whereas they exhibited a similar manipulability for solutions which are Pareto Optimal and individually rational. Individual rationality in the sense of Postlewaite (1979) would require each agent to produce at least as much as it could by using its own initial endowment. A market equilibrium allocation is easily verified to be Pareto Optimal. However, the assumption that agents are profit maximizers, fails to guarantee that a market equilibrium is individually rational. This is what prevents us from obtaining our result as a consequence of the results established by Postlewaite (1979). Our results are shown to be valid both when the inputs are perfectly divisible as well as when they are available in integer units only.

While the model that we consider here is a static one, it has some connection with the problem with hoarding. While hoarding is primarily a dynamic problem, manipulation via endowment is an approximation of such a phenomenon in the static context.

In a recent paper Atlamaz and Klaus (2005), consider exchange markets with heterogeneous indivisible goods and investigate the possibility of exchange rules that are efficient and immune to manipulation via endowments. They consider manipulability either with respect to withholding or destroying part of the initial endowment or transferring part of the endowment to another trader. They show that in general no exchange rule exists which is both efficient and immune to manipulation via endowments.

It is a well known fact that Walrasian equilibria in pure exchange contexts, exhibit what is known as the "transfer paradox", where an agent can benefit by transferring some of its initial endowment to another agent. A stronger result noted by Gale (1974) and subsequently established for strictly convex economies by Aumann and Peleg (1974), is that it is possible for a non-empty proper subset comprising at least two agents to reallocate their initial endowments within the subset and then trade to a new Walrasian equilibrium where each and every member of the redistributing coalition is better off than at the Walrasian equilibrium associated with the original initial endowments. The core compatibilty of Walrasian equilibrium would imply that at the new Walrasian equilibrium, the aggregate consumption of the redistributing coalition exceeds its aggregate initial endowment for at least one commodity. Goenka and Matta (2005), examine the underlying uncertainty inherent in the formation of redistributing coalitions and the consequent extrinsic uncertainty that it implies for the economy. They show that under certain conditions, self-improving redistributing coalitions can form if and only extrinsic uncertainty is significant for the economy, i.e. sunspots matter. Clearly such paradoxes are ruled out, once we consider market equilibria with profit maximizing agents as in our model.

It may be wondered whether the phenomena we are concerned with in this paper, is specific to the example that we have provided, or can be extended beyond its narrow confines. To answer this question it might be instructive to note that individual profit entails equating "marginal productivity" of the agents to the price of the input. The change in price that follows as a result of manipulation via endowments, is determined by the availability of the respective marginal productivities in the production function of the agents. This would allow us to accommodate within our framework of manipulation via endowments a considerably larger class of production functions, which share the same marginal productivities as the production function in our example, at the desired allocations. Even otherwise, constancy of marginal productivities within a certain interval, is not counter intuitive. Hence, the production function that we use can be replaced with more piece-wise linear segments than the two that we already have. However, all this involves avoidable complication, in so far as the main goal of this paper is concerned.

The manipulability results that we obtain here are not meant to be surprising at all. In fact the analysis reported here reflects a common threat to the smooth delivery of the market mechanism, as we have already indicated above. In a finite economy everyone has a little bit of monopoly power, which is often exercised by manipulating one's endowments. This gets translated in an enhancement of the scarcity value of the commodity in question, leading to a price rise. The interesting point that the existing literature highlights is that (and without speculating on what would be the consequences in a continuum artifact) in a finite competitive economy this effect is very small.

What perhaps is noteworthy about the phenomena we discuss here is the vulnerability of the competitive mechanism to manipulation via endowments, even when the agents are profit maximizers. This corresponds to the situation in a pure exchange economy, where agent's preferences are represented by quasi-linear preferences, and every agent can lend and borrow money at zero interest rate. In such economies, unlike Postlewaite (1979), it is not possible to show that every individually rational and efficient "extended" allocation is manipulable via endowments. In fact the dictatorial "extended" efficient allocation, where all but one agent consumes the output of its initial endowment, and this one agent i.e. the dictator, skims away the entire surplus from the other agents, is individually rational and efficient without being manipulable via destruction or withholding of endowments. Hence, the possibility of being able to manipulate market equilibrium via destruction of endowments in profit maximizing economies, cannot be considered to be a corollary of a much more general phenomenon. What is further noticeable about the present analysis, is that the possibility of being able to manipulate efficient and individually rational extended allocations via withholding of endowments and the possibility of a similar manipulation via unilateral misrepresentation of preference (see Moulin (1995), Lemma 3.8 for an exact statement of the result and its and proof), persists even if agents are profit maximizers and also if the non-monetary good that is being traded is indivisible, provided the extended allocation does not correspond to a dictatorial extended allocation such as the ones discussed above. The implication of such manipulations as we observe in this analysis, is the distinct possibility of the dictatorial extended allocation being realized. Hence, the vulnerability of the market mechanism to manipulation is definitely more real than merely academic in the present context. Thus the pedagogic contribution of this analysis, lies in being able to convey a considerable amount of economic theory concerned with the strategic behavior of individual agents but, using simple demand and supply curves, instead of an Edgeworth box diagram. Being able to improve oneself by destroying a portion of one's initial endowment is clearly a "moral hazard" problem. Being able to improve oneself by withholding a portion of one's initial endowment, is akin to what one observes in tax evasion. It appears as though, market and or competitive equilibria possess some of the drawbacks of insurance schemes and taxation. One may therefore be tempted to conclude that the institution of free market serves the dual purpose of insuring an agent against risk, as well as achieving a more equitable distribution of resources, making additional risk insurance superfluous and public policy interventions an undesirable add-on. While there may be striking similarities between the consequences of a market mechanism and public policy interventions, it may well turn out hasty to conclude on the basis of our results that the market mechanism is a credible substitute for risk insurance companies or public policy initiatives for that matter. On the contrary, our results point to the requirement of vigilant policing of the market mechanism by the government, the cost of which can only be recovered through adequate taxation. It also suggests that a purely trading mechanism needs to be supplemented by institutions that guarantee financial compensation against risk.

2. The Model: In this section we develop the general equilibrium model with inputs being perfectly divisible.
Consider an economy with H > 0 agents and L + 1 > 1 commodities. The first L commodities are used as inputs to produce the L+1th commodity, which is a numeraire consumption good. Each agent i = 1,…,H is equipped with a production function $f^i: \Re^L_+ \to \Re_+$ which is concave, non-decreasing and continuous. Further, $f^i(0) = 0$ for all i = 1,…,H. Each agent i is initially endowed with a vector $w^i \in \Re^L_+$ of inputs.
While this need not be the case for a more general statement of the model, ***<u>for the purposes of this paper we assume that</u>*** $w = \sum_{i=1}^{H} w^i \in \Re^L_{++}$.

An input consumption of agent i is denoted by an L-vector $X^i \geq 0$.
A price vector p is an element of $\Re^L_+ \backslash \{0\}$, where for j = 1,…,L, p_j denotes the price of input j. Clearly a price vector ***<u>does not allow</u>*** all inputs to be available for free.
At a price vector p, the objective of agent i is to maximize profits:
Maximize $[f^i(X^i) – p^T X^i]$

Note: If for some $i \in \{1,…,H\}$, f^i exhibits constant returns to scale, then a maximum profit if it exists for i, must be zero.

An allocation is an array X = <X^i/ i = 1,…,H> such that $X^i \in \Re^L_+$ for all i = 1,…,H.
An allocation X = <X^i/ i = 1,…,H> is said to be feasible if $\sum_{i=1}^{H} X^i = w$.
A market equilibrium is a pair <p*,X*> where p* is a price vector and for all i = 1,…,H, X^{*i} maximizes profits for agent i.

Observation due to Konstantin Makralov: Given a price-allocation pair <p*,X*>: [for all i = 1,…,H, X^{*i} maximizes profits for agent i] if and only if [X* maximizes aggregate profits at p* i.e. $\sum_{i=1}^{H} f^i(X^{*i}) - p^* \sum_{i=1}^{H} X^{*i} \geq \sum_{i=1}^{H} f^i(X^i) - p^* \sum_{i=1}^{H} X^i$, whenever <$X^i$/ i = 1,…,H> is any allocation].

Hence, a market equilibrium can alternatively be defined as a price-allocation pair <p*,X*> such that X* is a feasible allocation and maximizes aggregate profits at p*.

A feasible allocation X* $= \langle X^{*i}/i = 1,\ldots,H\rangle$ is said to be efficient if $\sum_{i=1}^{H} f^i(X^{*i}) \geq$ $\sum_{i=1}^{H} f^i(X^i)$, whenever $X = \langle X^i / i = 1,\ldots,H\rangle$ is any feasible allocation.

Proposition 1: Let <p*,X*> be a market equilibrium. Then X* is an efficient allocation.

Proof: Let $X = \langle X^i/i = 1,\ldots,H\rangle$ be any feasible allocation. Since <p*,X*> is a competitive equilibrium, for all i = 1,…,H: $f^i(X^{*i}) - p^{*T}X^{*i} \geq f^i(X^i) - p^{*T}X^i$, i.e. $f^i(X^{*i}) - f^i(X^i) \geq p^{*T}(X^{*i} - X^i)$.
Thus, $\sum_{i=1}^{H} f^i(X^{*i})$ - $\sum_{i=1}^{H} f^i(X^i) \geq p^{*T}(\sum_{i=1}^{H} X^{*i}$ - $\sum_{i=1}^{H} X^i) = 0$, since both X* and X are feasible allocations.
Hence, X* is efficient.
Q.E.D.

Proposition 2: Let <p*,X*> be a market equilibrium and let X be an efficient allocation f. Then, <p*,X> is also a market equilibrium.

Proof: $f^i(X^{*i}) - p^TX^{*i} \geq f^i(X^i) - p^TX^i$ for all $i \in I$, since <p*,X*> be a market equilibrium.
Thus, $\sum_{i\in I} f^i(X^{*i})$ - $p^T\sum_{i\in I} X^{*i} \geq \sum_{i\in I} f^i(X^i)$ - $p^T\sum_{i\in I} X^i$.
Since $\sum_{i\in I} X^{*i} = \sum_{i\in I} X^i = w$, we get $\sum_{i\in I} f^i(X^{*i}) \geq \sum_{i\in I} f^i(X^i)$.
Since X is efficient, it must be the case that $\sum_{i\in I} f^i(X^{*i}) = \sum_{i\in I} f^i(X^i)$.
Thus, $\sum_{i\in I} f^i(X^{*i})$ - $p^T\sum_{i\in I} X^{*i} = \sum_{i\in I} f^i(X^i)$ - p^T.
This combined with $f^i(X^{*i}) - p^TX^{*i} \geq f^i(X^i) - p^TX^i$ for all $i \in I$, yields $f^i(X^{*i}) - p^TX^{*i} = f^i(X^i) - p^TX^i$ for all $i \in I$.
Thus, <p*,X> is also a market equilibrium.
Q.E.D.

3. Competitive Equilibria and Market Equilibria: An extended allocation is a pair (X,Y), where X is an allocation and Y is an H-vector of real numbers. The i^{th} component of Y, denoted Y^i is the amount of the numeraire consumption good allocated to agent i.
An extended allocation (X,Y) is said to be feasible if X is a feasible allocation and $\sum_{i=1}^{H} Y^i = 0$.
A feasible extended allocation (X,Y) is said to be efficient if X is an efficient allocation.
A feasible extended allocation (X,Y) may be said to be ***individually rational*** if for all i = 1,…,H: $f^i(X^i) + Y^i \geq f^i(w^i)$.

Let (X^*, Y^*) be a efficient extended allocation such that for some $i \in \{1,\ldots,H\}$: $Y^{*i} = \sum_{k \neq i} [f^k(X^{*k}) - f^k(w^k)]$ whilst $Y^{*k} = f^k(w^k) - f^k(X^{*k})$ for $k \neq i$.

In this extended allocation, agent i, receives the entire output that is produced from X^* and pays the other agents the output that each could have produced from its initial endowment.

Clearly (X^*, Y^*) is individually rational as well. Further, agent i, can do no better than at (X^*, Y^*), and given the weak monotonicity assumption of the production functions, at (X^*, Y^*) agent i does not stand to gain by destroying any of its initial endowment either. Further, since any agent $k(\neq i)$ receives the output that it can produce by using its own initial endowments, and hence given the weak monotonicity of the production functions, cannot benefit by destroying its own initial endowment.

A feasible extended allocation (X^*, Y^*) is said to be competitive if there exists a price vector p^* such that for all $i = 1,\ldots,H$: (X^{*i}, Y^{*i}) solves
Maximize $f^i(X^i) + Y^i$
Subject to $p^{*T}X^i + Y^i \leq p^{*T}w^i$ (the budget constraint of agent i).
The pair $<p^*, (X^*, Y^*)>$ is then called a competitive equilibrium.

Note: If $<p^*, (X^*, Y^*)>$ is a competitive equilibrium, then for all $i = 1,\ldots,H$: $p^{*T}X^{*i} + Y^{*i} = p^{*T}w^i$ (i.e. the budget constraint of each agent is satisfied with equality). In the absence of equality, an agent could consume more of the consumption good, leading to an improvement for itself.

Theorem 1: $<p^*, (X^*, Y^*)>$ is a competitive equilibrium if and only if $<p^*, X^*>$ is a market equilibrium and $Y^{*i} = p^{*T}(w^i - X^{*i})$ for all $i = 1,\ldots,H$.

Proof: Suppose $<p^*, (X^*, Y^*)>$ is a competitive equilibrium. Since for all $i = 1,\ldots,H$: $p^{*T}X^{*i} + Y^{*i} = p^{*T}w^i$(i.e. $Y^{*i} = p^{*T}(w^i - X^{*i})$ for all $i = 1,\ldots,H$), it must be the case that for all $i = 1,\ldots,H$: (X^{*i}, Y^{*i}) solves
Maximize $f^i(X^i) + Y^i$
Subject to $p^{*T}X^i + Y^i = p^{*T}w^i$.
Hence, for all $i = 1,\ldots,H$: X^{*i} solves
Maximize $[f^i(X^i) - p^{*T}X^i - p^{*T}w^i]$.
Hence, for all $i = 1,\ldots,H$: X^{*i} solves
Maximize $[f^i(X^i) - p^{*T}X^i]$.
Thus, $<p^*, X^*>$ is a market equilibrium.
Now suppose that $<p^*, X^*>$ is a market equilibrium and Y^* is such that $Y^{*i} = p^{*T}(w^i - X^{*i})$ for all $i = 1,\ldots,H$. The feasibility of X^* guarantees that the extended allocation (X^*, Y^*) is feasible.
Towards a contradiction suppose that there exists $i \in \{1,\ldots,H\}$ and $(X^i, Y^i) \in \Re^L_+ \times \Re$ such that: $Y^i \leq p^{*T}(w^i - X^i)$ and $f^i(X^i) + Y^i > f^i(X^{*i}) + Y^{*i}$.
Thus, $f^i(X^i) + p^{*T}(w^i - X^i) \geq f^i(X^i) + Y^i > f^i(X^{*i}) + Y^{*i} = f^i(X^{*i}) + p^{*T}(w^i - X^{*i})$.
Thus, $f^i(X^i) - p^{*T}X^i > f^i(X^{*i}) - p^{*T}X^{*i}$, contradicting the assumption that $<p^*, X^*>$ is a market equilibrium.
Thus, $<p^*, (X^*, Y^*)>$ is a competitive equilibrium.

Q.E.D.

Note: If <p*, X*> is a market equilibrium and $Y^{*i} = p^{*T}(w^i - X^{*i})$ for all i = 1,…,H, then $f^i(X^{*i}) + Y^{*i}$ is the value that accrues to agent i, at <p*, X*>. Let S be a non-empty subset of {1,…,H} and for i∈S, let $X^i \in \Re^L_+$ be such that $\sum_{i\in S} X^i = \sum_{i\in S} w^i$. Since <p*,(X*,Y*)> is a competitive equilibrium, $f^i(X^i) - p^{*T}X^i \leq f^i(X^{*i}) - p^{*T}X^{*i}$ for all i∈S. Thus,
$\sum_{i\in S}[f^i(X^i) - p^* w^i] = \sum_{i\in S}[f^i(X^i) - p^* X^i] \leq \sum_{i\in S}[f^i(X^{*i}) - p^* X^{*i}]$.
Hence $\sum_{i\in S} f^i(X^i) \leq \sum_{i\in S}[f^i(X^{*i}) + Y^{*i}]$.
This shows that the total value accruing to agents in S at a market equilibrium is at least at much as they could generate for themselves by simply using their own resources. Thus, the pay-offs to the agents at a market equilibrium belong to the transferable utility core of the underlying market game as in Shapley and Shubik (1969, 1976), thereby establishing that the set of core allocations are non-empty.

4. Manipulation via endowments:
Example 1: Let f: $\Re_+ \to \Re$ be a function defined as follows for positive real numbers a, b and positive integer K where a > (K+1) b and K ≥ 2:
For all x = [0,K-1] let f(x) = ax; for x ≥ K - 1, let f(x) = (K-1)a + b(x-K + 1).
Clearly, f is concave.
Consider an economy with L = 1 and H = 2. Let $f^i = f$ for i = 1,2, $w^1 = 2K$ and $w^2 = 0$.
Thus, w = 2K.
Let p* = b.
Let X* = <X^{*i}/ i = 1,2> where K+1 ≥ X^{*i} ≥ K -1 for i = 1,2 and $X^{*1} + X^{*2} = 2K$.
Clearly <p*,X*> is a market equilibrium and <p*,X*> is a market equilibrium if and only if it satisfies the above properties.
The value accruing to agent 1 in units of the numeraire consumption good at <p*,X*> is
$f^1(X^{*1}) - bX^{*1} + 2Kb = (K-1)a + b(X^{*1} - K+1) - X^{*1}b + 2Kb = (a + b)K - (a - b)$.
The value accruing to agent 2 in units of the numeraire consumption good at <p*,X*> is
$f^2(X^{*2}) - bX^{*2} = (K-1)a + b(X^{*2} - K+1) - X^{*2}b = (a - b)(K-1)$.
Now suppose agent 1 destroys half its endowments so that its new initial endowment is $\overline{w}^1$ = K. Let agent 2's initial endowment remain unchanged.
Let $\overline{p}$ = a.
Let $\overline{X}$ = <$\overline{X}^i$ / i = 1,2> where K - 1 ≥ $\overline{X}^i$ ≥ 1 for i = 1,2 and $\overline{X}^1 + \overline{X}^2$ = K.
<$\overline{p}$, $\overline{X}$> is a market equilibrium for the economy with aggregate endowments being K.
In fact <$\overline{p}$, $\overline{X}$> is a market equilibrium for the economy with aggregate endowments being K, if and only if it satisfies these properties.
The value accruing to agent 1 in units of the numeraire consumption good at <$\overline{p}$, $\overline{X}$> is
$a\overline{X}^1 - a\overline{X}^1 + Ka = Ka > (a + b)K - (a - b)$, since a > b(K + 1).
The value accruing to agent 2 in units of the numeraire consumption good at <$\overline{p}$, $\overline{X}$> is
$a\overline{X}^2 - a\overline{X}^2 = 0$.

Thus, agent 1 is better off after having destroyed half its endowment.
Further, if agent 1 had withheld half its endowment instead of destroying it, then the value that accrues to agent 1 in units of the numeraire consumption good at $< \bar{p}, \bar{X} >$ is $a(K-1) + (K + \bar{X}^1 - K + 1)b - a\bar{X}^1 + aK = 2aK - (a - b)(\bar{X}^1 + 1) > 2aK > (a + b)K - (a - b)$, since in this situation agent 1 gets to use 3 units of the input for production.
This example clearly reveals that agent 1 benefits by destroying half its initial endowments, and considerably more by withholding it.
Discussion of Example 1: (a) In Example 1, the maximum in units of the numeraire consumption good that agent 1 can get corresponding to the original initial allocation profile is $2a(K-1) + 2b$ whereas, after destroying half its initial endowments, at any individually rational and efficient extended allocation it achieves Ka units of the numeraire consumption good. Agent 1 achieves this maximum by withholding half its initial endowment.
(b) In Example 1, if agent 1 instead of revealing the production function f, had misrepresented its productivity by revealing the production function $g: \Re_+ \rightarrow \Re$ where $g(x) = ax$; for $x \geq 0$, then there exists a competitive extended allocation after misrepresentation, at which it could assure for itself $2a(K-1) + 2b$ units of the numeraire consumption good.
Thus, Example 1 is an instance of a very simple economy, where every competitive extended allocation is vulnerable to manipulation via both misrepresentation of productive capabilities as well as destruction or withholding of endowments.
We now provide another example of manipulation via endowments using a two input production function exhibiting constant returns to scale. This function is available in Campbell (1987).
Example 2: As in Campbell (1987), let $f: \Re^2_+ \rightarrow \Re$ be a function defined as follows:
For all $x = (x_1,x_2) \in \Re^2_+$: (a) $x_2 \leq \frac{2}{3}x_1$ implies $f(x) = x_1 + 8x_2$; (b) $x_2 \geq \frac{2}{3}x_1$ implies $f(x) = \frac{19}{5}(x_1 + x_2)$.
Clearly, $f(0) = 0$.
Let $x = (x_1,x_2)$, $y = (y_1,y_2) \in \Re^2_+$ and $\alpha \in [0,1]$. Let $z = (z_1,z_2) = \alpha y + (1-\alpha)x$.
If $x_2 \leq \frac{2}{3}x_1$ and $y_2 \leq \frac{2}{3}y_1$ then $z_2 \leq \frac{2}{3}z_1$.
Thus, $f(z) = z_1 + 8z_2 = (\alpha y_1 + (1-\alpha)x_1) + 8(\alpha y_2 + (1-\alpha)x_2) = \alpha[y_1 + 8y_2] + (1-\alpha)[x_1 + 8x_2] = \alpha f(y) + (1-\alpha) f(x)$.
If $x_2 \geq \frac{2}{3}x_1$ and $y_2 \geq \frac{2}{3}y_1$ then $z_2 \geq \frac{2}{3}z_1$.
Thus, $f(z) = \frac{19}{5}(z_1 + z_2) = \frac{19}{5}(\alpha y_1 + (1-\alpha)x_1) + \frac{19}{5}(\alpha y_2 + (1-\alpha)x_2) = \alpha[\frac{19}{5}(y_1 + y_2)] + (1-\alpha)[\frac{19}{5}(x_1 + x_2)] = \alpha f(y) + (1-\alpha) f(x)$.
Hence suppose $x_2 \geq \frac{2}{3}x_1$ and $y_2 \leq \frac{2}{3}y_1$. Thus, $f(x) = \frac{19}{5}(x_1 + x_2)$ and $f(y) = (y_1 + 8y_2)$.

Case 1: $z_2 \leq \frac{2}{3} z_1$.

Thus, $\alpha y_2 + (1-\alpha)x_2 \leq \frac{2}{3}[\alpha y_1 + (1-\alpha)x_1]$ and $f(z) = z_1 + 8z_2 = [\alpha y_1 + (1-\alpha)x_1] + 8[\alpha y_2 + (1-\alpha)x_2]$.

Further, $f(z) – (1-\alpha)f(x) - \alpha f(y) = [\alpha y_1 + (1-\alpha)x_1] + 8[\alpha y_2 + (1-\alpha)x_2] – (1-\alpha)\frac{19}{5}(x_1 + x_2) - \alpha(y_1 + 8y_2) = (1-\alpha)[(x_1+8x_2) - \frac{19}{5}(x_1 + x_2)] = (1-\alpha)[\frac{21}{5}x_2 - \frac{14}{5}x_1] = \frac{7(1-\alpha)}{5}[3x_2 – 2x_1] \geq 0$, since $x_2 \geq \frac{2}{3}x_1$.

Case 2: $z_2 \geq \frac{2}{3} z_1$.

Thus, $\alpha y_2 + (1-\alpha)x_2 \geq \frac{2}{3}[\alpha y_1 + (1-\alpha)x_1]$ and $f(z) = \frac{19}{5}(z_1 + z_2) = \frac{19}{5}([\alpha y_1 + (1-\alpha)x_1] + [\alpha y_2 + (1-\alpha)x_2])$.

Further, $f(z) – (1-\alpha)f(x) - \alpha f(y) = \frac{19}{5}([\alpha y_1 + (1-\alpha)x_1] +[\alpha y_2 + (1-\alpha)x_2]) – (1-\alpha)\frac{19}{5}(x_1 + x_2) - \alpha(y_1 + 8y_2) = \alpha[\frac{19}{5}(y_1 + y_2) - (y_1+8y_2)] = \alpha[\frac{14}{5}y_1 - \frac{21}{5}y_2] = \frac{7\alpha}{5}[2x_1 – 3x_2] \geq 0$, since $x_2 \leq \frac{2}{3}x_1$.

Thus, f is concave. Also observe that f exhibits constant returns to scale.

Consider an economy with L = 2 and H = 2. Let $f^i = f$ for i = 1,2, $w^1 = (2,0)^T$ and $w^2 = (0,1)^T$. Thus, $w = (2,1)^T$.

Let $p^* = (1,8)^T$. If $X^i \in \Re^2_+$ is such that $X^i_2 > \frac{2}{3}X^i_1$, then $f^i(X^i) – p^{*T}X^i = \frac{19}{5}(X^i_1 + X^i_2) – (X^i_1 + 8X^i_2) = (\frac{14}{5}X^i_1 - \frac{21}{5}X^i_2) = \frac{7}{5}(2X^i_1 - 3X^i_2) < 0$.

However, if $X^i_2 \leq \frac{2}{3}X^i_1$, then $f^i(X^i) – p^{*T}X^i = (X^i_1 + 8X^i_2) – (X^i_1 + 8X^i_2) = 0$.

Let $X^* = \langle X^{*i} / i = 1,2\rangle$ where $X^{*1} = (\frac{2}{5}, \frac{1}{5})^T$ and $X^{*2} = (\frac{8}{5}, \frac{4}{5})^T$.

$\langle p^*, X^*\rangle$ is a market equilibrium.

The value accruing to agent 1 in units of the numeraire consumption good at $\langle p^*, X^*\rangle$ is $f^1(X^{*1}) – p^{*T}X^{*1} + p^{*T}w^1 = 2$.

The value accruing to agent 2 in units of the numeraire consumption good at $\langle p^*, X^*\rangle$ is $f^2(X^{*2}) – p^{*T}X^{*2} + p^{*T}w^2 = 8$.

Now suppose agent 1 destroys half its endowments so that its new initial endowment is $\overline{w}^1 = (1,0)^T$. Let agent 2's initial endowment remain unchanged.

Let $\overline{p} = (\frac{19}{5}, \frac{19}{5})^T$. If $X^i \in \Re^2_+$ is such that $X_2^i < \frac{2}{3} X_1^i$, then $f^i(X^i) - p^{*T}X^i = (X_1^i + 8X_2^i) - \frac{19}{5}(X_1^i + X_2^i) = (\frac{21}{5}X_2^i - \frac{14}{5}X_1^i) = \frac{7}{5}(3X_2^i - 2X_1^i) < 0$.

However, if $X_2^i \geq \frac{2}{3} X_1^i$, then $f^i(X^i) - p^{*T}X^i = \frac{19}{5}(X_1^i + X_2^i) - \frac{19}{5}(X_1^i + X_2^i) = 0$.

Let $\overline{X} = \langle \overline{X}^i / i = 1,2 \rangle$ where $\overline{X}^i = (\frac{1}{2}, \frac{1}{2})^T$ for i = 1,2.

$\langle \overline{p}, \overline{X} \rangle$ is a market equilibrium for the economy with aggregate endowments being $(1,1)^T$.

The value accruing to agent 1 in units of the numeraire consumption good at $\langle \overline{p}, \overline{X} \rangle$ is $f^1(\overline{X}^1) - \overline{p}^T\overline{X}^1 + \overline{p}^T\overline{w}^1 = \frac{19}{5} > 2$.

The value accruing to agent 2 in units of the numeraire consumption good at $\langle \overline{p}, \overline{X} \rangle$ is $f^2(\overline{X}^2) - \overline{p}^T\overline{X}^2 + \overline{p}^T w^2 = \frac{19}{5} < 8$.

Thus, agent 1 is better off after having destroyed half its endowment.

Further, if agent 1 had withheld half its endowment instead of destroying it, then the value that accrues to agent 1 in units of the numeraire consumption good at $\langle \overline{p}, \overline{X} \rangle$ is $\frac{11}{2} - \overline{p}^T\overline{X}^1 + \overline{p}^T\overline{w}^1 = \frac{11}{2} - \frac{19}{5} + \frac{19}{5} = \frac{11}{2} > \frac{19}{5} > 2$, since in this situation agent 1 gets to use $\frac{3}{2}$ units of the first and $\frac{1}{2}$ units of the second for production.

In the above, the equilibrium prices prescribed by Campbell (1987) and that are realized after agent 1 destroys or withholds half its endowments, had to be calibrated in order to yield profit maximizing solutions. With the price of both inputs being one as in Campbell (1987), a market equilibrium would clearly not exist.

This example again reveals that agent 1 benefits by destroying half its initial endowments, and considerably more by withholding it.

5. Discrete Concave Market Games: We now develop the general equilibrium model for the case where the inputs are available in integer amount only.

Let $N = \aleph \cup \{0\}$, where $\aleph$ denotes the set of natural numbers. As before let there be H > 0 agents and L+1 > 1 commodities. The first L commodities are used as inputs to produce the L+1th commodity, which is a numeraire consumption good. Each agent i is initially endowed with a commodity bundle $w(i) \in N^L$. Let 'w' denote the initial endowment function.

Suppose $\sum_{i=1}^{H} w(i) \in \Re^L_{++}$.

For j = 1,…,L, let $w_j = \sum_{i=1}^{H} w_j(i)$, i.e. the aggregate amount of commodity j that is available in the economy.

A function f: $N^L \rightarrow \Re$ is said to be monotonically non-decreasing if for all $x,y \in N^L$: $x \geq y$ implies $f(x) \geq f(y)$.
A function f: $N^L \rightarrow \Re$ is said to be discrete concave if there exists a continuous concave function $g: \Re_+^L \rightarrow \Re$ such that the restriction of g to N^L coincides with f.
Given functions f: $N^L \rightarrow \Re$ and $g: \Re_+^L \rightarrow \Re$, let graph(f) $\equiv \{(x, \alpha) \in N^L \times \Re / \alpha \leq f(x)\}$ and graph(g) $\equiv \{(x, \alpha) \in \Re_+^L \times \Re / \alpha \leq g(x)\}$.
Given a function f: $N^L \rightarrow \Re$ its canonical extension is the function $g^f: \Re_+^L \rightarrow \Re$ such that the graph(g^f) = convex hull of graph(f).
If f is discrete concave, then its canonical extension g^f is continuous and concave and the restriction of g^f to N^L coincides with f.
Each agent i has preferences defined over N^L which is representable by a monotonically non-decreasing discrete concave production function f^i.
The pair $<\{f^i / i = 1,\ldots,H\}, w>$ is called a discrete concave market game.
An input consumption vector of agent i is denoted by a vector $X^i \in N^L$.
A price vector p is an element of $\Re_+^L \backslash \{0\}$, where for j = 1,…,L, p_j denotes the price of input j.
At a price vector p, the objective of agent i is to maximize profits:
Maximize $[f^i(X^i) - p^T X^i]$
An allocation is an array $X = <X^i / i = 1,\ldots,H>$ such that $X^i \in N^L$ for all i = 1,…,H.
An allocation $X = <X^i / i = 1,\ldots,H>$ is said to be feasible if $\sum_{i=1}^{H} X^i = \sum_{i=1}^{H} w(i)$.
A market equilibrium is a pair $<p^*,X^*>$ where p* is a price vector and for all i = 1,…,H, X^{*i} maximizes profits for agent i.
Similar problems have been studied by Yang (2001), Sun and Yang (2004) and Inoue (2005). The case where $w_j = 1$ for j = 1,…,L has been investigated by Bhikchandani and Mamer (1997).
For j = 1,…,L, let e_j denote the j^{th} unit coordinate vector in $\Re^L$. Market equilibrium in the Shapley and Shubik (1972) assignment game is related to the case where:
(a) $w_j = 1$ for all j = 1,…,L; and
(b) for all i = 1,…,H and $x \in N^L$: $[\sum_{j=1}^{L} x_j > 1]$ implies $[f^i(x) = \max \{f^i(e_j) / x_j > 0\}]$.

A feasible allocation $X^* = <X^{*i} / i = 1,\ldots,H>$ is said to be efficient if $\sum_{i=1}^{H} f^i(X^{*i}) \geq \sum_{i=1}^{H} f^i(X^i)$, whenever $X = <X^i / i = 1,\ldots,H>$ is any feasible allocation.

A replication of the proof of Proposition 1 yields the following:

Proposition 3: Let $<p^*,X^*>$ be a market equilibrium. Then X* is an efficient allocation.

A replication of the proof of Proposition 2 yields the following:

Proposition 4: Let $<p^*,X^*>$ be a market equilibrium and let X be an efficient allocation f. Then, $<p^*,X>$ is also a market equilibrium.

An extended allocation is a pair (X,Y), where X is an allocation and Y is an H-vector of real numbers. The i^{th} component of Y, denoted Y^i is the amount of the numeraire consumption good allocated to agent i.
An extended allocation (X,Y) is said to be feasible if X is a feasible allocation and

$$\sum_{i=1}^{H} Y^i = 0.$$

A feasible extended allocation (X*,Y*) is said to be competitive if there exists a price vector p* such that for all i = 1,…,H: (X^{*i},Y^{*i}) solves
Maximize $f^i(X^i) + Y^i$
Subject to $p^{*T}X^i + Y^i \leq p^{*T}w^i$ (the budget constraint of agent i).
The pair $<p^*, (X^*,Y^*)>$ is then called a competitive equilibrium.

Note: If $<p^*,(X^*,Y^*)>$ is a competitive equilibrium, then for all i = 1,…,H: $p^{*T}X^{*i} + Y^{*i} = p^{*T}w^i$ (i.e. the budget constraint of each agent is satisfied with equality). In the absence of equality, an agent could consume more of the consumption good, leading to an improvement for itself.

The proof of the following theorem is identical to the proof of Theorem 1.

Theorem 2: $<p^*, (X^*,Y^*)>$ is a competitive equilibrium if and only if $<p^*, X^*>$ is a market equilibrium and $Y^{*i} = p^{*T}(w^i - X^{*i})$ for all i = 1,…,H.

6. Manipulation via endowments for discrete concave market games:
Example 3: Consider the discrete concave market game $<\{f^i/i= 1,2\}, w>$ with L = 1, $w^1 = 8$, $w^2 = 0$ and where each f^i is the restriction to N of the function f: $\Re \rightarrow \Re$ defined in Example 1 of section 4.
It is easily observed that $<p^*,X^*>$ is a market equilibrium, where p* = b, $5 \geq X^{*i} \geq 3$ for i = 1,2 and $X^{*1} + X^{*2} = 8$.
Now suppose agent 1 destroys half its endowments. Then its profits increase.
Thus, agent 1 is better off after having destroyed half its endowment.
Further, if agent 1 had withheld half its endowment instead of destroying it, then its profits increase even further.
Example 4: Consider the discrete concave market game $<\{f^i/i= 1,2\}, w>$ with L = 2, $w^1 = (20,0)^T$, $w^2 = (0,10)$ and where each f^i is the restriction to N^2 of the function f: $\Re^2_+ \rightarrow \Re$ defined in Example 2 of section 4.
It is easily observed that $<p^*,X^*>$ is a market equilibrium, where $p^* = (1,8)^T$, $X^{*1} = (4,2)^T$ and $X^{*2} = (16,8)^T$.
The value accruing to agent 1 in units of the numeraire consumption good at $<p^*,X^*>$ is $f^1(X^{*1}) - p^{*T}X^{*1} + p^{*T}w^1 = 20$.
The value accruing to agent 2 in units of the numeraire consumption good at $<p^*,X^*>$ is $f^2(X^{*2}) - p^{*T}X^{*2} + p^{*T}w^2 = 80$.

Now suppose agent 1 destroys half its endowments so that its new initial endowment is $\overline{w}^1 = (10,0)^T$. Let agent 2's initial endowment remain unchanged.
Let $\overline{p} = (\frac{19}{5}, \frac{19}{5})^T$ and $\overline{X} = <\overline{X}^i / i = 1,2>$ where $\overline{X}^i = (5,5)^T$ for i = 1,2.
$<\overline{p}, \overline{X}>$ is a market equilibrium for the economy with aggregate endowments being $(10,10)^T$.
The value accruing to agent 1 in units of the numeraire consumption good at $<\overline{p}, \overline{X}>$ is $f^1(\overline{X}^1) - \overline{p}^T\overline{X}^1 + \overline{p}^T\overline{w}^1 = 38 > 20$.
The value accruing to agent 2 in units of the numeraire consumption good at $<\overline{p}, \overline{X}>$ is $f^2(\overline{X}^2) - \overline{p}^T\overline{X}^2 + \overline{p}^T w^2 = 38 < 80$.
Thus, agent 1 is better off after having destroyed half its endowment.
Further, if agent 1 had withheld half its endowment instead of destroying it, then the value that accrues to agent 1 in units of the numeraire consumption good at $<\overline{p}, \overline{X}>$ is $55 - \overline{p}^T\overline{X}^1 + \overline{p}^T\overline{w}^1 = 55\text{-}38 + 38 = 55 > 38 > 20$, since in this situation agent 1 gets to use 15 units of the first and 5 units of the second for production.

7. Conclusion: The essence of Theorems 1 and 2 is that market equilibrium and competitive equilibrium are equivalent concepts. Thus in a market economy such as the one modeled above, there is no difference between assuming agents to be profit maximizers or as maximizing utility subject to a budget constraint. However, unlike a Walrasian equilibrium with quasi-linear utilities, we do not impose the requirement that consumption of the numeraire good has to be non-negative.
It is instructive to note that in our model, the set of market equilibria while being dependent on the aggregate initial endowment of the inputs w, is completely independent of the particular initial distribution of it among the agents, i.e. the array $<w^i/i=1,\ldots,H>$ itself. Thus two different initial distributions among the agents of the same aggregate initial endowment, leads to the same set of market equilibria. However, what changes along with a change in the initial endowment of an agent, is its final worth measured in terms of the numeraire consumption good.
If $<w^i/i=1,\ldots,H>$ is the array depicting the initial endowments of the agents, then corresponding to the market equilibria $<p^*,X^*>$, the final consumption of the numeraire good of agent $i \in \{1,\ldots,H\}$ is $f^i(X^{*i}) + p^{*T}(w^i - X^{*i})$. Thus if $t: \{1,\ldots H\} \rightarrow \Re^L$ with $\sum_{i=1}^{H} t(i) = 0$ is a function that, specifies a redistribution of the initial endowment among the agents, then the final allocation of value to agent $i \in \{1,\ldots,H\}$ is $f^i(X^{*i}) + p^{*T}(w^i + t(i) - X^{*i})$. The consequent change in welfare measured in units of the numeraire consumption good for agent i, is $p^{*T}t(i)$, for i = 1,..,H. Since $\sum_{i=1}^{H} t(i) = 0$, so is $p^{*T}\sum_{i=1}^{H} t(i)$. Thus, a redistribution of the initial endowment among the agents, leads to a redistribution of the aggregate final output of the consumption good among the agents. Hence, phenomena akin to the "transfer paradox" are ruled out in our setting.

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