On the existence of maximal elements: An impossibility theorem*

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Abstract

Most properties of binary relations considered in the decision literature can be expressed as the impossibility of certain "configurations." There exists no condition of this form which would hold for a binary relation on a subset of a finite-dimensional Euclidean space if and only if the relation admits a maximal element on every nonempty compact subset of its domain.

Keywords: Binary relation; Maximal element; Necessary and sufficient condition; Potential games

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1 Introduction

The choice of maximal elements of a binary relation is often used to model decision making in various contexts (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995). For a maximal element to be chosen, however, it must exist in the first place. Accordingly, there is a considerable literature studying what conditions should be imposed on a binary relation to ensure the existence of maximal elements on potential feasible sets (Gillies, 1959; Smith, 1974; Bergstrom, 1975; Mukherji, 1977; Walker, 1977; Kiruta et al., 1980; Danilov and Sotskov, 1983; Campbell and Walker, 1990). This paper demonstrates that there may be limits to such undertakings, independent of the ingenuity of particular authors.

As is well known, a binary relation admits a maximal element on every finite subset of its domain if and only if it is acyclic. Here we assume that every compact subset may emerge as the set of feasible alternatives. This, admittedly stylized, setting is a natural second step from a purely technical viewpoint, even though a narrower class of feasible sets may be justified in this or that particular context. Acyclicity obviously remains necessary, but not sufficient. Acyclicity plus open lower contours is sufficient, but too exacting as (weak) Pareto dominance, or lexicographic orders, show.

Smith (1974) found a condition necessary and sufficient for a *preference relation* (i.e., complete preorder) to attain a maximum on every compact subset of its domain. The restriction, however, need not be justified in every potential application, especially when the relation in question is supposed to reflect the preferences of several agents as well as their abilities to influence the outcome. For instance, a Nash equilibrium in a strategic game is naturally perceived as a maximal element of the individual improvement relation.

Here we put no *a priori* restriction on the relation. It turns out that no condition of a reasonable form can be equivalent to the existence of maximal elements on every nonempty compact subset of the universal domain.

The simplest admissible condition consists of, loosely speaking, one quantifier, either there exists or there does not exist, followed by a description of a "configuration," i.e., a list of points with the relation (or its absence) fixed between some pairs, and with the convergence (or the absence of it) fixed for some sequences. The exact definitions are semantic rather than syntactic. Then we allow conjunctions of such "elementary" conditions and, finally, disjunctive forms.

One justification for this particular notion of an admissible condition is its relative simplicity (necessary and sufficient conditions more complicated than the original property hardly make any sense). Another lies in the fact that all sufficient, as well as necessary, conditions found so far in our setting can be represented in the form. Still, our theorem does not cover more involved conditions (e.g., with several quantifiers), which are also used for various purposes in the literature.

The next section contains necessary formal definitions; Section 3, the main theorem and its proof. Implications for the theory of topological potential games are considered in Section 4. A discussion of related questions completes the paper.

2 Formulations

A *binary relation* on a set X is a Boolean function on $X \times X$; as usual, we write $y \rhd x$ whenever the relation $\rhd$ is true on a pair (y, x) and $y \not\rhd x$ whenever it is false. Let $Y \subseteq X$; $x \in Y$ is a *maximizer* for $\rhd$ on Y if $y \not\rhd x$ for every $y \in Y$. We denote $\mathbb{N} = \{0, 1, \dots\}$ the chain of natural numbers starting from zero.

An *abstract configuration* consists of $P_=, N_=, P_\rhd, N_\rhd \subseteq \mathbb{N} \times \mathbb{N}$ and $P_\to, N_\to \subseteq \mathbb{N}^\mathbb{N}$, where $\mathbb{N}^\mathbb{N}$ means the set of mappings $\mathbb{N} \to \mathbb{N}$, i.e., sequences in $\mathbb{N}$. Let $\rhd$ be a binary relation on a metric space X and C be an abstract configuration. A *realization* of C in X for $\rhd$ is a mapping $\mu : \mathbb{N} \to X$ such that: $\mu(k') = \mu(k)$ whenever $(k', k) \in P_=$; $\mu(k') \neq \mu(k)$ whenever $(k', k) \in N_=$; $\mu(k') \rhd \mu(k)$ whenever $(k', k) \in P_\rhd$; $\mu(k') \not\rhd \mu(k)$ whenever $(k', k) \in N_\rhd$; $\mu(\nu(k)) \to \mu(\nu(0))$ whenever $\nu \in P_\to$; $\mu(\nu(k)) \not\to \mu(\nu(0))$ whenever $\nu \in N_\to$.

Many properties of binary relations can be expressed as the impossibility to realize a certain configuration. For example, to define the reflexivity of $\rhd$, we can prohibit the realization of a configuration with $N_\rhd = \{(0, 0)\}$ and other sets empty; to define irreflexivity, with $P_\rhd = \{(0, 0)\}$; transitivity, with $P_\rhd = \{(1, 0), (2, 1)\}$ and $N_\rhd = \{(2, 0)\}$. Open lower contours (lower continuity) are described by the prohibition of a configuration with $P_\rhd = \{(0, 1)\}$, $N_\rhd = \{(0, k)\}_{k \geq 2}$, and $P_\to = \{\nu^+\}$, where $\nu^+(k) = k + 1$; weak lower continuity (Campbell and Walker, 1990), by $P_\rhd = \{(0, 1)\} \cup \{(k, 0)\}_{k \geq 2}$ and $P_\to = \{\nu^+\}$ with the same $\nu^+(k)$. To formalize acyclicity in this style, we prohibit the realization of each of a countable set of configurations parameterized with $m \in \mathbb{N}$: $P_\rhd^{(m)} = \{(1, 0), (2, 1), \dots, (m + 1, m)\}$ and $P_=^{(m)} = \{(0, m + 1)\}$. The list can easily be extended.

Remark. One can always dispense with $P_=$, but a symmetric definition seems preferable.

A *simplest configurational condition* is defined by a set of abstract configurations $\mathcal{N}$. We say that such a condition *holds* on a metric space X for a binary relation $\rhd$ if no configuration $C \in \mathcal{N}$ admits a realization in X for $\rhd$. The set of all simplest configurational conditions is denoted $\mathcal{S}_0$.

Every condition from the class $\mathcal{S}_0$ is "inherited" (Walker, 1977): if such a condition holds on X for $\rhd$, then it also holds on every $X' \subseteq X$ for the restriction of $\rhd$ to X'. It seems natural, therefore, to use such conditions when trying to characterize properties of binary relations which are inherited by their nature (like the existence of a maximizer on every compact subset). It is also worth noting that, e.g., all the properties of binary relations considered in Duggan (1999) belong to the class $\mathcal{S}_0$.

Our impossibility result can be proven almost as easily for a wider class of conditions, viz. for logical combinations of "negative" and "positive" conditions. We formalize such combinations as "(infinite) disjunctive forms" made of (infinite) conjunctions of positive and negative conditions.

A *general configurational condition* (*C-condition*) consists of a set A of indices, and two sets of abstract configurations, $\mathcal{P}(\alpha)$ and $\mathcal{N}(\alpha)$, for every $\alpha \in$ A. We say that such a condition $\mathcal{C}$ *holds* on X for $\rhd$ if there is $\alpha \in$ A such that every configuration $C \in \mathcal{P}(\alpha)$, and no configuration $C \in \mathcal{N}(\alpha)$, admits a realization in X for $\triangleright$. The set of all general configurational conditions is denoted $\mathcal{S}_1$; obviously, $\mathcal{S}_0 \subset \mathcal{S}_1$.

Remark. We put no restriction on the cardinality, or the complexity, of the sets involved in the definition of a C-condition: the more freedom is allowed, the more convincing our negative result is.

3 The Result

Theorem. *There exists no condition $\mathcal{C} \in \mathcal{S}_1$ such that $\mathcal{C}$ would hold on a subset X of a finite-dimensional Euclidean space for a binary relation $\triangleright$ on X if and only if $\triangleright$ admits a maximizer on every nonempty compact subset of X.*

Proof. Let $\mathcal{C}$ be such a condition. We consider $X = \{e^{\mathrm{i}t} | \ t \in \mathbb{R}\}$ (where $\mathrm{i} = \sqrt{-1}$); geometrically, X is a circle embedded into the plane of complex numbers. We define a binary relation $y \triangleright x \iff y = e^{\mathrm{i}} \cdot x$. Clearly, there is no maximizer for $\triangleright$ on X; since X is compact itself, $\mathcal{C}$ must not hold on X.

Picking $x^0 \in X$, we denote $X' = X \setminus \{x^0\}$. Let us show that every nonempty compact $Y \subseteq X'$ admits a maximizer for $\triangleright$. Supposing the contrary, we pick $y^0 \in Y$; since y^0 is not a maximizer, we can pick $y^1 \in Y$ such that $y^1 \triangleright y^0$; since y^1 is not a maximizer, we can pick $y^2 \in Y$ such that $y^2 \triangleright y^1$; etc. Since Y is compact, every limit point of $\{y^k\}_{k \in \mathbb{N}}$ must belong to Y. On the other hand, we have $y^{k+1} = e^{\mathrm{i}} \cdot y^k$; by the Jacobi theorem (see, e.g., Billingsley, 1965), $\{y^k\}_{k \in \mathbb{N}}$ is dense in X. Therefore, x^0 is a limit point, hence $x^0 \in Y \subset X \setminus \{x^0\}$. The contradiction proves our claim.

Therefore, $\mathcal{C}$ must hold on X', i.e., there is $\alpha \in \mathrm{A}$ such that every configuration $C \in \mathcal{P}(\alpha)$, and no configuration $C \in \mathcal{N}(\alpha)$, admits a realization in X' for $\triangleright$. Every realization $\mu : \mathbb{N} \to X' \subset X$ being simultaneously a realization in X, and $\mathcal{C}$ not holding on X, there must be $C \in \mathcal{N}(\alpha)$ admitting a realization μ in X. We pick $r \in X \setminus \{x^0/\mu(k)\}_{k \in \mathbb{N}}$ and define $\mu^* : \mathbb{N} \to X$ by $\mu^*(k) = r \cdot \mu(k)$. Clearly, $\mu^*(k) = \mu^*(h) \iff \mu(k) = \mu(h)$, $\mu^*(k) \triangleright \mu^*(h) \iff \mu(k) \triangleright \mu(h)$, and $\mu^*(\nu(k)) \to \mu^*(\nu(0)) \iff \mu(\nu(k)) \to \mu(\nu(0))$ for all $k, h \in \mathbb{N}$ and $\nu \in \mathbb{N}^{\mathbb{N}}$; besides, $x^0 \notin \mu^*(\mathbb{N})$ by the choice of r. Thus, μ^* is a realization of C in X', contradicting the choice of α. $\square$

4 Implications for the Theory of Potential Games

As usual, a *strategic game* Γ is defined by a finite set of players N, and strategy sets X_i and preference relations $\succeq_i$ on $X = \prod_{i \in N} X_i$ for all $i \in N$. With every strategic game, the *individual improvement relation* $\triangleright^{\mathrm{Ind}}$ on X is associated ($y, x \in X$, $i \in N$):

$$y \triangleright^{\mathrm{Ind}}{}_i \, x \iff [y_{-i} = x_{-i} \ \& \ y \succ_i x]; \tag{1a}$$

$$y \triangleright^{\mathrm{Ind}} x \iff \exists i \in N \, [y \triangleright^{\mathrm{Ind}}{}_i \, x]. \tag{1b}$$

By definition, a strategy profile $x \in X$ is a Nash equilibrium if and only if x is a maximizer for $\rhd^{\mathrm{Ind}}$.

Monderer and Shapley (1996) introduced several classes of *potential games*, the "most ordinal" being that of a "generalized potential game." For a finite game, the property amounts to the acyclicity of the individual improvement relation; every such game obviously possesses a Nash equilibrium. Moreover, the existence of an equilibrium is preserved if arbitrary restrictions are imposed on feasible choices of each player, although Takahashi and Yamamori (2002) showed that the existence of a Nash equilibrium under arbitrary restrictions on strategies does not imply the acyclicity of individual improvements.

An equivalence between the acyclicity of individual improvements and persistent existence of equilibria holds if a wider class of modifications of the original game is allowed (strictly speaking, a modification of the concept of Nash equilibrium is needed as well). We assume that any strategy may prove infeasible to the relevant player, and any strategy profile may prove unacceptable to all players (say, entail a global nuclear conflict).

A *finite restriction* Γ^0 of Γ is defined by the same set of players N, strategy sets $\emptyset \neq X_i^0 \subseteq X_i$, and a finite set of *acceptable* profiles $\emptyset \neq X^0 \subseteq \prod_{i \in N} X_i^0$; preferences in Γ^0 are somewhat modified: $y \succeq_i^0 x \iff y \succeq_i x$ whenever $x, y \in X^0$, while $y \succ_i^0 x$ whenever $x \notin X^0 \ni y$. An *acceptable Nash equilibrium* is $x^0 \in X^0$ such that $x^0 \succeq_i^0 (x_{-i}^0, x_i)$ for every $i \in N$ and $x_i \in X_i^0$. Acceptable Nash equilibria can be viewed as maximizers of the individual improvement relation in Γ^0 if we modify the definition (1), assuming $x \rhd^{\mathrm{Ind}^0} x$ whenever $x \notin X^0$.

Proposition. *Every finite restriction of Γ possesses an acceptable Nash equilibrium if and only if the individual improvement relation in Γ is acyclic (i.e., Γ is a generalized potential game).*

Proof. Let $\rhd^{\mathrm{Ind}}$ be acyclic and Γ^0 be a finite restriction of Γ. Then $\rhd^{\mathrm{Ind}}$ admits a maximizer on X^0, which is obviously an acceptable Nash equilibrium. Conversely, if there is a cycle $x^0, x^1, \ldots, x^m = x^0$ such that $x^{k+1} \rhd^{\mathrm{Ind}} x^k$, we define Γ^0 by $X_i^0 = \{x_i^0, x_i^1, \ldots, x_i^{m-1}\}$ for each $i \in N$ and $X^0 = \{x^0, x^1, \ldots, x^{m-1}\}$. Clearly, there is no acceptable Nash equilibrium in Γ^0. $\square$

The Proposition seems to justify the following way to extend the notion of a "generalized potential game" to infinite (topological) games: Replace "finite" with "compact" in the definition of a restriction of Γ, and consider strategic games every compact restriction of which possesses an acceptable Nash equilibrium. However, our theorem shows that this class of games admits no simple description (at least, it cannot be described by the absence of cycles in any sense of the individual improvement relation). On the one hand, the fact is disappointing; on the other hand, it may be an indirect argument for the notion of a "purely ordinal" potential game from Kukushkin (1999), where the existence of an acceptable Nash equilibrium in every compact restriction is ensured.

5 Concluding Remarks

5.1. In the proof, we did not have to assume that the hypothetical C-condition works for *any* subset of a Euclidean space, however complicated; it was only applied to a circle and to an open interval. Admittedly, our relation was exotic, but this property is quite usual for mathematical counterexamples. Actually, that relation can be interpreted as the best response improvement relation in a strategic game (Kukushkin, 1999, Example 2) although the game itself is certainly artificial. Unfortunately, the class of "natural" relations seems impossible to define.

5.2. Considering formal restrictions on the relation, there is a condition from $\mathcal{S}_0$ which is necessary and sufficient for an *interval order* to admit a maximizer on every compact subset of its domain (Kukushkin, 2005, Theorem 4). On the other hand, the restriction to transitive relations would leave our Theorem intact, only requiring a bit more complicated proof based on the transitive closure of the same relation $\rhd$.

5.3. There are quite natural conditions on binary relations admitting no (obvious) representation of the form allowed here. Consider, for instance, the existence of a maximizer on the whole X,

$$\exists x \ \nexists y \, [y \rhd x],$$

or (the key condition in the definition of) a semilattice order,

$$\forall a, b \, \exists c \left[c \geq a \ \& \ c \geq b \ \& \nexists d \, [d \geq a \ \& \ d \geq b \ \& \text{ not } d \geq c]\right].$$

Either formulation ends with a "normal" negative condition: something is impossible. However, that "something" is preceded with one or two extra quantifiers, and, in the latter case, also with positive requirements, which was not allowed by our definitions of Section 2. Thus, our Theorem does not preclude the possibility that the existence of a maximizer on every compact subset could be equivalent to a (combination of) condition(s) of such a form. Moreover, if there is no restriction on the use of quantifiers and logical operations, such a combination can be written down explicitly (Kukushkin, 2005, Section 5); however, it could not claim any usefulness in any applications.

5.4. Of greater interest to economists may be the existence of maximizers on convex, compact subsets. The current notion of a C-condition gives no means to express convexity, so it seems implausible that a condition of that form could be equivalent to the property. A simple modification of our notions changes the situation: let us add a ternary relation meaning "x is a convex combination of y and z." Apparently, every sufficient condition in the literature now belongs to $\mathcal{S}_0$. The question of whether a condition of the form can be *necessary* and sufficient remains open.

5.5. The impossibility statement ascribed by Walker (1977, last paragraph) to P. Fishburn can be viewed as a precursor of this result. There is a big difference, however: What was shown there was the impossibility of a condition sufficient for the existence of a maximizer on *every* compact subset and simultaneously necessary for the existence of a maximizer on a *single* compact set. Since much more was expected of the hypothetical condition, the impossibility result is much weaker. Actually, the requirements were so strong that there was no need to specify the class of admissible conditions.

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