Complementarity, Search, and Price Dispersion

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Abstract. We study an equilibrium sequential search model where buyers have potentially downward-sloping demand and with bilateral heterogeneity in buyers' search costs and firms' production costs. We show that downward-sloping demand and heterogeneity in production costs are necessary to ensure price dispersion when buyers have finite willingness to pay. We then show that firms' profits display increasing differences with respect to price and the distribution of prices when the distribution of search costs is "uniform-like". It follows that the set of equilibrium price distributions is nonempty and has a largest and smallest element, in the sense of first-order stochastic dominance. Since the high-price equilibrium is strongly focal, equilibrium is therefore effectively unique.

Keywords: complementarity, price dispersion, search, supermodularity.

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1. Introduction

Equilibrium search models have been used to study a wide variety of issues in industrial organization, labor economics, microeconomics, and macroeconomics, including:

- price dispersion – Reinganum (1979), Burdett and Judd (1983), Rob (1985), Baye and Morgan (2001, 2004), and Rauh (2001);
- the transaction costs of inflation – Bénabou (1988, 1992) and Bénabou and Gertner (1993);
- advertising – Robert and Stahl (1993);
- market microstructure and the role of middlemen – Spulber (1999);
- wage and price controls – Fershtman and Fishman (1994) and Rauh (2004);
- and internet pricing — Baye and Morgan (2001, 2004).

For a survey of this literature, see Baye, Morgan, and Scholten (2005). Despite the relative importance of this class of models, little is known about their general structure. In particular, the existence of search market equilibrium is problematic in models where buyers use sequential search strategies, because their reservation levels are based on the *distribution* of prices. As a result, the pricing game involves firms whose payoffs depend on the distribution of prices induced by the strategy profile, rather than direct dependence on the latter, so the classic Nash-Debreu existence theorem and related results are inapplicable. To complicate matters, the equilibrium set is contained in the infinite-dimensional space of cumulative distribution functions of prices, making further analysis difficult. In particular, one cannot rule out multiple equilibria in general.

In Reinganum (1979), these issues simply do not arise, since the equilibrium distribution of prices is induced in a straightforward manner by the distribution of production costs. Although Rob's (1985) existence proof is ingenious, it relies heavily on the assumption that firms are identical and does not seem to extend to models with heterogeneous production costs. Rauh (1997) proves existence of search market equilibrium assuming buyers' sequential search strategies depend on only finitely many moments of the distribution of prices (as opposed to all of them, generally required for rational expectations), and Rauh (2005) for a very general equilibrium search model with a hyperfinite Loeb space of firms (a particular type of nonstandard measure space).

In this paper, we consider an equilibrium search model where buyers search sequentially with potentially downward-sloping demand and bilateral heterogeneity in buyers' search costs and firms' production costs. The model is essentially the same as Bénabou (1993), which generalizes the Reinganum (1979) and Rob (1985) models. We show that when demand is perfectly inelastic, the degenerate price distribution where almost all firms charge buyers' maximum willingness to pay is an equilibrium, regardless of heterogeneity in search and production costs. In other words, mere heterogeneity is insufficient to overcome the Diamond (1971) Paradox. In contrast, Rob (1985) shows that dispersion can occur with perfectly inelastic demand, identical firms, and heterogeneous search costs, provided buyers are prepared to pay any price for the good, no matter how high. We then show that when firms are identical, the degenerate price distribution where almost all firms charge the common monopoly price is an equilibrium, even with downward-sloping demand and heterogeneous search costs. In this model, the only way to ensure price dispersion is to assume downward-sloping demand and heterogenous production costs. Dispersion is therefore of the Reinganum (1979) variety, although in our model buyers actively search in equilibrium.

We then turn to issues related to the existence and structure of equilibrium, using ideas from the literature on supermodularity and complementarity [e.g., see Topkis (1998) and Vives (1999)]. We show that firms' profits display increasing differences in price and the distribution of prices when the density of buyers' search costs is "uniform-like," in the sense that the density is (weakly) increasing on the relevant range and *either* (a) the ratio of the minimum of its height to the maximum of its slope is sufficiently large, *or* (b) the density is (weakly) convex and satisfies a hazard condition on the relevant range. The canonical example of such a density is the uniform distribution. It follows that the set of equilibrium price distributions is nonempty, with a largest and smallest element in the sense of first-order stochastic dominance. Since the high-price equilibrium is strongly focal, we may regard it as effectively unique, although there may in fact be multiple equilibria.

The plan for the rest of the paper is as follows. In section 2, we briefly summarize the relevant definitions and results from the literature on supermodularity and complementarity. In section 3, we construct the equilibrium sequential search model. Section 4 presents results on price dispersion and section 5 studies the set of equilibrium price distributions. Section 6 concludes. All proofs not in the text are in the appendix.

2. Preliminaries

In this section, we collect some definitions and results related to supermodularity and complementarity. For more information, consult Topkis (1998) or Vives (1999). Let $\mathbf{R}$ and $\mathbf{R}_+$ denote the spaces of real and nonnegative real numbers, respectively.

Let X be a nonempty set and $\geq$ a binary relation on X. If $\geq$ is reflexive, transitive, and antisymmetric then $(X, \geq)$ is a *partially ordered set* (poset). We write $x > x'$ iff $x \geq x'$ and $x \neq x'$. A *lattice* is a poset which contains $\sup\{x, y\}$ and $\inf\{x, y\}$ for all $x, y \in X$. A lattice X is *complete* if every nonempty subset of X has a supremum and infimum in X. The *interval topology* on a poset X is generated by taking the sets $\{z \in X \,|\, z \leq x\}$ and $\{z \in X \,|\, z \geq x\}$ to be a subbasis for closed sets. In Euclidean spaces, the interval topology is the same as the usual one. A lattice is complete iff it is compact in the interval topology. A function $f : X \to \mathbf{R}$ on a lattice X is *supermodular* if

$$f(\inf\{x, y\}) + f(\sup\{x, y\}) \geq f(x) + f(y) \tag{1}$$

for all $x, y \in X$. Functions of one real variable are trivially supermodular. Let X and T be posets. A function $f : X \times T \to \mathbf{R}$ has *(strictly) increasing differences* on $X \times T$ if $f(x,t) - f(x,t')$ is (strictly) increasing in x for all $t \geq t'$ ($t > t'$). In Euclidean spaces, we can check this condition using derivatives (assuming differentiability).

Let $\mathcal{D}$ denote the set of cumulative distribution functions (cdfs) with support contained in $[a,b] \subseteq \mathbf{R}_+$, where $a < b$. We define a partial order on $\mathcal{D}$ as follows: $F \geq F'$ iff F first-order stochastically dominates F'.[1] Given $F, F' \in \mathcal{D}$, the function defined by $G_{F,F'}(x) = \min\{F(x), F'(x)\}$ satisfies $G_{F,F'} = \sup\{F, F'\}$ and $G_{F,F'} \in \mathcal{D}$. A similar statement holds for $\inf\{F, F'\}$, so $(\mathcal{D}, \geq)$ is a lattice. Echenique and Edlin (2004, p. 65) prove that the weak topology on $\mathcal{D}$ is finer than the interval topology. Since $\mathcal{D}$ is compact in the former, it is compact in the latter, so $(\mathcal{D}, \geq)$ is a complete lattice.

3. The Model

The model is similar to that in Bénabou (1993). Let $J = [0, L]$ with Lebesgue measure be the space of firms, where $L > 0$. Let $m : J \to C$ be the function which assigns a constant marginal cost of production to each firm, where $C = [\underline{c}, \overline{c}]$ and $0 \leq \underline{c} \leq \overline{c} < \infty$. We arrange firms' indices so that m is increasing.[2]

Let $[0, N]$ with Lebesgue measure be the space of buyers, where $N > 0$. Let $\theta = N/L$. All buyers have the same indirect utility function $U(p, I) = v(p) + I$, where p is price and I is income. We assume $v : P \to \mathbf{R}$ is twice differentiable on $P = [0, \overline{p}]$ (one-sided derivatives at the endpoints), where $\overline{p} > \overline{c}$. The demand curve is therefore $x(p) = -v'(p)$, which is differentiable on P (and therefore continuous).

Assumptions 1. *(i)* $x(p) > 0$ *for all* $p \in [0, \overline{p})$ *and* $x(p) = 0$ *for all* $p \in (\overline{p}, \infty)$. *(ii)* x *is decreasing on* P. *(iii) There exists* $\delta > 0$ *such that* v *can be smoothly extended*

[1] I.e., $F(x) \leq F'(x)$ for all $x \in [a, b]$.

[2] Throughout the paper, we use the terminology "increasing" and "strictly increasing," rather than "nondecreasing" and "increasing".

to $[-\delta, \overline{p}]$ *(the extension is* C^1 *on* $[-\delta, 0)$*). (iv) For all* $c \in C$*, the monopoly problem*

$$\max_{p \in [c, \overline{p}]} \pi^M(p, c) = (p - c)x(p) \tag{2}$$

has a unique solution $p_M(c) \in (c, \overline{p}]$ *and* $\pi^M_p \equiv \partial \pi^M / \partial p > 0$ *on* $(c, p_M(c))$*.*

A common assumption is that demand is perfectly inelastic at one unit up to some maximum willingness to pay $\overline{p}$, so $x = 1$ on P. In that case, $p_M(c) = \overline{p}$ for all $c \in C$ and $\pi^M_p = 1$ on P. The assumptions also hold when demand is downward-sloping, twice differentiable on P, and not too convex, so that π^M is strictly concave. Note that $p_M(c)$ is increasing on C by theorem 2.4 and its corollary in Vives (1999, p. 30). Since no firm will price below $\underline{c}$ (the lowest marginal cost) or above $\overline{p}$ (where demand is zero), we henceforth take $\mathcal{D}$ to be the set of all cdfs with support contained in $P' = [\underline{c}, \overline{p}]$.

We make the standard assumptions that buyers search sequentially with recall and the first search is free. Let $r \in P'$ be the lowest price observed so far. The marginal benefit of another search is

$$\int_0^r [v(p) - v(r)]\, dF(p) = \int_{-\delta}^r [v(p) - v(r)]\, dF(p), \tag{3}$$

where equality holds since $[-\delta, 0)$ is F-null for all $F \in \mathcal{D}$. Integrating the second integral in (3) by parts, we get

$$\int_{-\delta}^r [v(p) - v(r)]\, dF(p) = \int_{-\delta}^r x(p)F(p)\, dp = \int_0^r x(p)F(p)\, dp, \tag{4}$$

where the first equality uses the fact that $F(-\delta) = 0$ and the second that $F = 0$ on $[-\delta, 0)$. Let

$$\Gamma(r, F) = \int_0^r x(p)F(p)\, dp. \tag{5}$$

Note that the integral in (5) is well-defined since the integrand is a.e. continuous and bounded $0 \leq x(p)F(p) \leq x(0)$. Let

$$z(F) = \sup\{p \in \mathbf{R} \mid F(p) = 0\} \geq 0 \tag{6}$$

and $\Gamma : P \times \mathcal{D} \to \mathbf{R}_+$ be the function defined in (5). The following results will be repeatedly used throughout the paper, generally without mention.

Proposition 1. *For all $F \in \mathcal{D}$: (i) Γ is absolutely continuous (AC) and a.e. differentiable on P, with derivative $x(p)F(p)$ where it exists. (ii) $\Gamma = 0$ on $[0, z(F)]$, is positive on $(z(F), \overline{p}]$, and strictly increasing on $[z(F), \overline{p}]$. (iii) $0 \leq \Gamma(r, F) \leq V$ for all $r \geq \underline{c}$, where*

$$V \equiv \int_{\underline{c}}^{\overline{p}} x(r)\, dr. \tag{7}$$

Proof. (i) follows from theorem 5.5 and proposition 6.5 in Haaser and Sullivan (1991, p. 237, 240). (ii) is obvious. Since

$$0 \leq \Gamma(r, F) \leq \Gamma(\overline{p}, F) = \int_{\underline{c}}^{\overline{p}} x(p)F(p)\, dp \leq \int_{\underline{c}}^{\overline{p}} x(p)\, dp = V, \tag{8}$$

(iii) follows. ∎

We might call V the *value* since $V = v(\underline{c}) - v(\overline{p})$, where the former is the indirect utility corresponding to the Bertrand outcome (best case scenario for buyers) and the latter the monopolistic outcome under perfectly inelastic demand (worst case scenario).

A buyer with search cost s stops searching at all prices r such that the marginal benefit $\Gamma(r, F)$ of another search is less than or equal to the marginal cost s, so the reservation level is

$$r(s, F) = \sup\{r \in P \,|\, \Gamma(r, F) \leq s\}. \tag{9}$$

From Fig. 1 below, we see that $r(0, F) = z(F)$, $r(s_1, F) = r_1$, and $r(s, F) = \overline{p}$ for all buyers with search costs $s \geq \Gamma(\overline{p}, F)$.

Figure 1 Goes Here

Hence, the proportion of buyers with reservation level $\overline{p}$ is $1 - Q(\Gamma(\overline{p}, F))$, where Q is the cdf of buyers' search costs. Assume Q is AC with probability density function (pdf) q with support $S = [0, \overline{s}]$, where $0 < \overline{s} \leq \infty$. Note that we allow $\overline{s} = \infty$, in which case q has positive support on $\mathbf{R}_+$.

Assumptions 2. *(i) $q(s) > 0$ for all $s \in (0, \overline{s})$ and zero outside. (ii) q is bounded and C^1 on $(0, \overline{s})$. (iii) $|q'| \leq K$ on $(0, \overline{s})$.*

Allowing q to have potential discontinuities at 0 and $\overline{s}$ creates some irritating technical difficulties and necessitates fussy proofs, but we want to accommodate the uniform density.

The cdf G of buyers' reservation levels is given by

$$G(r|F) = \begin{cases} Q(\Gamma(r,F)) & 0 \leq r < \overline{p} \\ 1 & r \geq \overline{p}, \end{cases} \tag{10}$$

with potential atom at $\overline{p}$ of size $1 - Q(\Gamma(\overline{p}, F))$. Since Q is AC and $\Gamma(r, F)$ is AC and increasing, it is easy to show that G is AC on $[0, \overline{p})$ with pdf g defined a.e. by

$$g(r|F) = G'(r|F) = Q'(\Gamma(r,F))x(r)F(r) = q(\Gamma(r,F))x(r)F(r). \tag{11}$$

Since firms' demand curve is well-known [e.g., see Bénabou (1993, p. 143)], we present an informal derivation. Say some firm charges the price $p \in P$. The firm's potential customers are those for which $r(s, F) \geq p$. We first consider buyers with $r(s, F) = \overline{p}$. These buyers search exactly once, and are randomly and evenly distributed across all firms. The firm expects to sell

$$\frac{N[1 - Q(\Gamma(\overline{p}, F))]x(p)}{L} = \theta x(p)[1 - Q(\Gamma(\overline{p}, F))] \tag{12}$$

to them. We now consider buyers $r(s, F) < \overline{p}$. Let $Ng(r|F)dr$ be the mass of a buyer with $p \leq r < \overline{p}$. The firm competes for this buyer with all other firms whose prices are less than or equal to r. Hence, expected sales to this buyer are

$$x(p)\frac{Ng(r|F)dr}{LF(r)} = \theta x(p)\frac{g(r|F)dr}{F(r)}, \tag{13}$$

assuming $F(r) > 0$. Substituting from (11), we get

$$\theta x(p) q(\Gamma(r, F)) x(r) dr. \tag{14}$$

Summing over all buyers $p \leq r < \overline{p}$ and adding the potential atom at $\overline{p}$,

$$D(p, F) = \theta x(p) \left[\int_p^{\overline{p}} q(\Gamma(r, F)) x(r)\, dr + 1 - Q(\Gamma(\overline{p}, F)) \right], \tag{15}$$

which is essentially the same as equation (4) in Bénabou (1993). The parameter θ is of little importance, so we henceforth set $\theta = 1$ for simplicity. Note that the integral in (15) is well-defined, since the integrand is bounded and a.e. continuous.[3] Furthermore, demand is continuous and (weakly) decreasing in price for all $F \in \mathcal{D}$. Let $D : P \times \mathcal{D} \to \mathbf{R}_+$ be the function defined in (15).

Let $\pi : P \times C \times \mathcal{D} \to \mathbf{R}$ be defined by $\pi(p, c, F) = (p - c)D(p, F)$. A firm with cost c will not charge more than $p_M(c)$, since doing so reduces sales and is non-profit-maximizing for those customers that remain. The maximization problem of firm j is therefore

$$\max_{p \in [c, p_M(c)]} \pi(p, m(j), F). \tag{16}$$

4. Price Dispersion

We now identify conditions under which price dispersion obtains in equilibrium, a major impetus in the development of equilibrium search models. Note that the results in this section are valid whether or not the complementarity conditions in the next section hold. Let F_{p_0} denote the cdf of prices which is degenerate at some $p_0 \in P'$.

According to Proposition 2(i) below, $F_{\overline{p}}$ is an equilibrium cdf of prices when demand is perfectly inelastic, regardless of heterogeneity in search and production costs. In comparison, Rob (1985) showed that dispersion can occur with perfectly

[3] The only potential discontinuities are at $r = z(F)$ and if $\Gamma(r, F)$ takes the value $\overline{s}$.

inelastic demand, identical firms, and heterogeneity in search costs, provided that $\overline{p} = \infty$. In 2(ii), we show that $F_{p_M(c)}$ is an equilibrium cdf of prices when firms are identical, even with downward-sloping demand and heterogeneous search costs. Even if other, dispersed equilibria exist, there is little point in discussing them since the aforementioned ones are highly focal. In 2(iii), we prove the collateral result that $q(0) = 0$ is necessary for the expression in (15) to represent firms' demand function, as assumed in Assumptions 2(i). Although many papers in the equilibrium search literature study the existence and character of search market equilibrium as $q(0)$ varies, it must in fact be zero.

Proposition 2. *(i) If $x = 1$ on P then $F_{\overline{p}}$ is an equilibrium cdf of prices. (ii) If $m(j) = c \in C$ for all $j \in J$ (identical production costs), $F_{p_M(c)}$ is an equilibrium cdf of prices. (iii) (15) requires $q(0) = 0$.*

Proof. Let $p_0 \in P'$ and assume all firms are charging p_0. Since $z(F_{p_0}) = p_0$, all buyers have reservation levels greater than or equal to p_0 (see Fig. 1). Hence, all buyers search exactly once and are evenly distributed across firms. It follows that $D(p, F) = x(p)$ and $\pi = (p - c)x(p)$ for all $0 \le p \le p_0$. In case (i), $p_0 = \overline{p}$ and $x = 1$ on P. Hence, $\pi = p - c$ and $p = \overline{p}$ is profit-maximizing. Note that $\overline{p} > \overline{c}$, so this is profitable for all firms. In case (ii), $p_0 = p_M(c)$ so $\pi = \pi^M$ for all $p \in [c, p_M(c)]$. Since firms are identical, $p = p_M(c)$ is profit-maximizing for all firms. We now prove (iii). Since all buyers have reservation levels greater than or equal to p_0,

$$\int_p^{p_0 = z(F_{p_0})} q(\Gamma(r, F_{p_0}))x(r)\, dr = q(0) \int_p^{p_0} x(r) = 0, \tag{17}$$

for all $0 \le p < p_0$. Since the integral in (17) is positive, we must have $q(0) = 0$. ∎

According to Proposition 2, we need downward-sloping demand and heterogeneity in production costs to ensure price dispersion. The next proposition gives a simple sufficient condition for dispersion, similar to that in Reinganum (1979), although in our model buyers actively search in equilibrium.

Proposition 3. *If*

$$\{(j,j') \in J \times J \,|\, j > j' \text{ and } p_M(m(j')) < m(j)\} \tag{18}$$

has positive (Lebesgue) measure in $J \times J$*, then degenerate equilibria do not exist.*

Proof. Suppose F_{p_0} is an equilibrium cdf of prices for some $p_0 \in P'$. Hence, $p_0 \in [m(j), p_M(m(j))]$ for almost all $j \in J$. But this is impossible, since

$$0 \leq m(j') < p_M(m(j')) < m(j) < p_M(m(j)) \leq \overline{p} \tag{19}$$

on a subset of $J \times J$ with positive measure. ∎

5. Complementarity

We now turn to issues related to existence of search market equilibrium, and the structure of the equilibrium set E. From a game-theoretic perspective, a distinctive feature of equilibrium search models is that players' (firms') payoffs $\pi(p, m(j), F)$ depend on the cdf F of actions induced by the strategy profile, as opposed to direct dependence on the strategy profile itself. This is because buyers' decision rules $r(s, F)$ are based on F. Hence, the classic Nash-Debreu existence theorem and related results do not apply.

In Reinganum (1979) such difficulties do not arise, because buyers are identical with downward-sloping demand and firms have heterogeneous production costs. All firms price below buyers' common reservation level, with high-cost firms bunching at the latter and low-cost firms charging their individual monopoly prices. Hence, the equilibrium cdf of prices is induced in a straightforward manner by the cdf of production costs. In Rob (1985), demand is perfectly inelastic, firms are identical, and buyers have heterogeneous search costs. To prove existence, he first identifies a suitable Γ-type function by taking the limit of a sequence of approximate equilibria. Since Γ is AC, it has a representation of the form $\Gamma = \int_0^r F(p)\,dp$. He then shows that F can be made into an equilibrium cdf of prices. Although ingenious, the argument does not appear to extend to the case of heterogeneous production costs. Bénabou (1990, 1993) synthesizes the previous two models, allowing for downward-sloping demand and bilateral heterogeneity in search and production costs. He is, however, unable to prove existence for reasons discussed in Bénabou (1990).

The more recent literature on existence of search market equilibrium includes Rauh (1997, 2005). In Rauh (1997), it is assumed that buyers' search strategies depend on only finitely many moments of the cdf of prices. Existence then follows from the result in Rauh (2003) [also see the extension in Yu and Zhu (2005)], which states that pure-strategy Nash equilibria exist for games with an atomless probability space of players with actions in a nonempty, compact subset of Euclidean space (not necessarily convex), and whose payoffs are continuous in finitely many moments of the strategy profile (not necessarily quasi-concave). Rauh (2005) uses nonstandard analysis to prove existence in an equilibrium search model with heterogeneity in buyers' demand functions and search costs, as well as firms' cost functions, with very general demand and cost functions. In particular, existence follows from a result by Khan and Sun (1999), which implies that pure-strategy equilibria exist for games with a hyperfinite Loeb space of players with actions in a nonempty, compact subset of $\mathbf{R}$ and whose payoffs are continuous in the cdf of actions. Although this approach ensures existence of search market equilibrium under very general conditions, it sheds little light on the structure of E; in particular, one cannot rule out multiple equilibria.

In this section, we show that the objective in (16) has increasing differences when Q is "uniform-like" on the relevant range, and then exploit complementarity to show that E is nonempty, with a largest and smallest element. Although we cannot rule out multiple equilibria, the high-price equilibrium (the largest element of E) is highly focal, making it effectively unique. Let $\phi : J \times \mathcal{D} \to P$ be the best response correspondence defined by

$$\phi(j, F) = \arg \max_{p \in [c, p_M(c)]} \pi(p, m(j), F). \tag{20}$$

Proposition 4. *For all $F \in \mathcal{D}$: (i) π has increasing differences on $P \times C$. (ii) For each firm $j \in J$, $\phi(j, F) \subseteq P'$ is nonempty and compact, with largest $\overline{\phi}(j, F)$ and smallest $\underline{\phi}(j, F)$ elements, which are increasing in j.*

Proof. To prove (i), note that

$$\frac{\partial \pi(p, c, F)}{\partial c} = -D(p, F), \tag{21}$$

which is increasing. Since π is trivially supermodular on P and the cost map m is increasing, (ii) follows from Theorem 2.4 and its Corollary in Vives (1999, p. 30). ▮

We now address the more difficult question as to whether π has increasing differences in (p, F). This is far from obvious. E.g., say a firm is charging the price p_0 and reservation levels are sparse in $[p_0, p_0 + \epsilon)$ for some small $\epsilon > 0$. The marginal profitability of raising price is therefore relatively high, since the firm loses few customers in doing so. Now suppose all other firms raise price. From (5), it is clear that reservation levels rise. If there was a sizable mass of reservation levels just below p_0 before the price changes, a substantial portion of that mass could be pushed into $[p_0, p_0 + \epsilon)$, *lowering* the marginal profitability of increasing price. This example suggests a possible solution: we can minimize such bunching of reservation levels by taking Q to be roughly uniform over the relevant range.

Since $\mathcal{D}$ lacks a relative linear structure, we cannot differentiate with respect to F. Even if we considered $\mathcal{D}$ as a subset of some larger Banach space, say the space of bounded functions on P' in the supremum norm, there would be no relationship between the derivative and first-order stochastic dominance. The basic idea of the proof is to take $F > F'$ in $\mathcal{D}$ and define $F_t = tF + (1-t)F'$ for all $t \in [0, 1]$. We then prove increasing differences by differentiating marginal profitability with respect to t. To get the result, we need to assume Q is "uniform-like" on the relevant range, in the sense that q is (weakly) increasing and *either* (a) $q(s) \geq V \sup q'(s)$ *or* (b) q is also (weakly) convex and satisfies a hazard condition on the relevant range. In the case of the uniform distribution, $q' = 0$, so both (a) and (b) are satisfied.

Proposition 5. *Assume* $q'(s) \geq 0$ *on* $(0, \tilde{s})$, *where* $V < \tilde{s} \leq \overline{s}$. *If (a)* $q(s) \geq MV$ *for all* $s \in (0, V]$, *where*

$$M = \sup_{s \in (0,V]} q'(s), \tag{22}$$

or (b) $q''(s) \geq 0$ *on* $(0, \tilde{s})$ *and* $q'(s)V \leq q(s)$ *for all* $s \in (0, V]$, *then (i)* $\pi(p, c, F)$ *has increasing differences on* $[c, p_M(c)] \times \mathcal{D}$ *for all* $c \in C$ *and (ii) for each firm* $j \in J$, $\overline{\phi}(j, F)$ *and* $\underline{\phi}(j, F)$ *are increasing on* $\mathcal{D}$.

Proof. By theorem 2.4 and its corollary in Vives (1999, p. 30), (ii) follows from (i). We now prove (i). Fix $c \in C$. Choose any $F, F' \in \mathcal{D}$ such that $F > F'$ and let $F_t = tF + (1-t)F'$ for all $t \in [0, 1]$. Note that $F > F'$ implies $F' \neq F_{\overline{p}}$ and $F_t \neq F_{\overline{p}}$ for all $t \in (0, 1)$. It follows that

$$0 < \Gamma(\overline{p}, F_t) \leq V < \tilde{s}. \tag{23}$$

By Lemmas A.1 and A.2 in the appendix, we can differentiate $\pi(p, c, F_t)$ with respect to $t \in (0, 1)$ to get

$$\pi^M \left[\int_p^{\overline{p}} q'(r, t) h(r) x(r)\, dr - q(\overline{p}, t) h(\overline{p}) \right], \tag{24}$$

where

$$h(r) \equiv \frac{\partial \Gamma(r, F_t)}{\partial t} = \int_0^r [F(p) - F'(p)] x(p)\, dp \leq 0 \tag{25}$$

and $q'(r, t) \equiv q'(\Gamma(r, F_t))$. Moreover, we are permitted to define $q'(0) = 0$, albeit for integration purposes only. We must show that (24) is increasing in p on $[c, p_M(c)]$. Since (24) is continuous on $[c, p_M(c)]$, by theorem 5.11 in Rudin (1976, p. 108) it is enough to show that (24) is differentiable with nonnegative derivative in the interior. For now, assume $\Gamma(r, F_t) > 0$ on $(c, p_M(c))$ (we address the other possibilities in the appendix). Since $x(r)$ and $h(r)$ are continuous on P,

$$0 < \Gamma(r, F_t) \leq \Gamma(\overline{p}, F_t) \leq V < \tilde{s} \leq \overline{s}, \tag{26}$$

and q' is continuous on $(0, \overline{s})$, it follows that (24) is differentiable on $(c, p_M(c))$ with derivative

$$-\pi^M q'(p,t)h(p)x(p) + \pi_p^M \left[\int_p^{\overline{p}} q'(r,t)h(r)x(r)\,dr - q(\overline{p},t)h(\overline{p}) \right] \geq 0, \tag{27}$$

where the inequality remains to be shown. The first term in (27) is nonnegative because $\pi^M, x(p), q'(p,t) \geq 0$ and $h(p) \leq 0$. Since $\pi_p^M > 0$, it suffices to prove

$$\int_p^{\overline{p}} q'(r,t)h(r)x(r)\,dr \geq q(\overline{p},t)h(\overline{p}). \tag{28}$$

Since $h(r) \leq 0$ and decreasing, $h(\overline{p}) \leq h(r) \leq 0$. A sufficient condition is therefore

$$h(\overline{p}) \int_p^{\overline{p}} q'(r,t)x(r)\,dr \geq q(\overline{p},t)h(\overline{p}). \tag{29}$$

If $h(\overline{p}) = 0$, there is nothing more to prove. Otherwise,

$$\int_p^{\overline{p}} q'(r,t)x(r)\,dr \leq q(\overline{p},t). \tag{30}$$

is sufficient. We want (30) to hold for all $p \in (c, p_M(c))$ and all $c \in C$. The integrand in (30) is nonnegative, so

$$\int_p^{\overline{p}} q'(r,t)x(r)\,dr \leq \int_{\underline{c}}^{\overline{p}} q'(r,t)x(r)\,dr \leq q(\Gamma(\overline{p}, F_t)) \tag{31}$$

is sufficient. From (22),

$$\int_{\underline{c}}^{\overline{p}} q'(r,t)x(r)\,dr \leq MV, \tag{32}$$

and from (23) we get the condition in (a). If $q''(s) \geq 0$ on $(0, \tilde{s})$ then

$$\int_{\underline{c}}^{\overline{p}} q'(r,t)x(r)\,dr \leq q'(\overline{p},t) \int_{\underline{c}}^{\overline{p}} x(r)\,dr = q'(\overline{p},t)V, \tag{33}$$

and we get the hazard condition in (b). ▮

Theorem 1. *Under Assumptions 1 and 2 and the hypotheses of Proposition 5, the equilibrium set E is nonempty and has a largest $\overline{F}$ and smallest $\underline{F}$ element.*

Proof. By Proposition 4, $\overline{\phi}(j, F)$ is an increasing function from J to P' for all $F \in \mathcal{D}$. It is therefore measurable, and induces $\overline{\Phi}(F) \in \mathcal{D}$. We therefore have a map $\overline{\Phi} : \mathcal{D} \to \mathcal{D}$. From Proposition 5, $F \geq F'$ implies $\overline{\Phi}(F) \geq \overline{\Phi}(F')$, so $\overline{\Phi}$ is increasing. Since $\mathcal{D}$ is a complete lattice, we can apply the Tarski fixed point theorem [e.g., Vives (1999, p. 20)] to obtain $\overline{F}$. The argument for $\underline{F}$ is similar. ▮

6. Conclusion

In this paper, we studied an equilibrium sequential search model where buyers have potentially downward-sloping demand and with bilateral heterogeneity in buyers' search costs and firms' production costs. The model is essentially the same as that in Bénabou (1993), which generalizes Reinganum (1979) and Rob (1985). We showed that the degenerate price distribution where almost all firms charge buyers' maximum willingness to pay is an equilibrium when demand is perfectly inelastic, regardless of heterogeneity in search and production costs. Even when demand is downward-sloping, if firms are identical then the degenerate price distribution where almost all firms charge the common monopoly price is an equilibrium. Hence, the only way to rule out degenerate equilibria and ensure price dispersion is to assume downward-sloping demand and heterogeneous production costs. The model therefore generates Reinganum-type (1979) price dispersion, although in our model buyers actively search in equilibrium.

Since firms' payoffs depend on the distribution of prices, the classic Nash-Debreu existence theorem and related results are inapplicable. In this paper, we used ideas from the literature on supermodularity and complementarity to study the set of equilibrium price distributions. We showed that firms' profits have increasing differences with respect to price and the distribution of prices when the distribution of search costs is "uniform-like," in the sense that the density of search costs is (weakly) increasing on the relevant range and either (a) the ratio of the minimum height of the density to its maximum slope is sufficiently large or (b) the density is (weakly) convex and satisfies a hazard condition on the relevant range. Under these conditions, the set of equilibrium price distributions is nonempty and has a largest and smallest element, in the sense of first-order stochastic dominance. This result is significant because, even if there are multiple equilibria, the high-price equilibrium will be strongly focal.

Appendix

Lemma A.1. $\Gamma(r, F_t)$ *is differentiable with respect to* $t \in (0,1)$ *with derivative as in (25). Furthermore,* $h : P \to \mathbf{R}$ *as defined in (25) is decreasing, AC, and* $|h(r)| \leq V$ *for all* $p \in P$.

Proof. We have already noted that the integrand in (5) is indeed integrable on P for all $F \in \mathcal{D}$, including F_t. For all $p \in P$, $x(p)F_t(p) = x(p)[tF(p) + (1-t)F'(p)]$ is smooth in t with derivative $x(p)[F(p) - F'(p)]$. Furthermore, $|x(p)[F(p) - F'(p)]| \leq x(0)$, so (25) follows from theorem 20.4 in Aliprantis and Burkinshaw (1990, p. 151). Since $F > F'$, $F(p) \leq F'(p)$ for all $p \in P$, so $h(r)$ is nonpositive and decreasing. Finally,

$$|h(r)| = \left| \int_{\underline{c}}^{r} x(p)[F(p) - F'(p)]\, dp \right| \leq \int_{\underline{c}}^{r} |x(p)[F(p) - F'(p)]|\, dp$$

$$\leq \int_{\underline{c}}^{r} x(p)\, dp \leq V, \tag{A.1}$$

where the first equality uses the fact that $F - F' = 0$ on $[0, \underline{c})$. ∎

Lemma A.2. *Under the hypotheses of Proposition 5,* $D(p, F_t)$ *is differentiable with respect to* $t \in (0,1)$ *with derivative*

$$x(p) \left[\int_{p}^{\overline{p}} q'(r,t) h(r) x(r)\, dr - q(\overline{p}, t) h(\overline{p}) \right]. \tag{A.2}$$

Proof. We first consider the term $Q(\Gamma(\overline{p}, F_t))$ in (15). Since

$$Q(y) = \int_{0}^{y} q(s)\, ds, \tag{A.3}$$

Q is differentiable on $(0, \overline{s})$, where q is continuous. From (23), it follows that Q is differentiable at $\Gamma(\overline{p}, F_t)$ and the derivative of $Q(\Gamma(\overline{p}, F_t))$ with respect to $t \in (0,1)$

is $q(\Gamma(\overline{p}, F_t))h(\overline{p})$. We now consider the integral term in (15). We have already noted that the integrand is indeed integrable on P for all $F \in \mathcal{D}$. For all $t \in (0,1)$, $F_t = 0 \iff F' = 0$ so $z(F_t) = z(F')$. Hence,

$$\int_{p}^{\overline{p}} q(r,t)x(r)\,dr = \int_{\max\{p,z(F')\}}^{\overline{p}} q(r,t)x(r)\,dr, \tag{A.4}$$

because $q(0) = 0$ and $\Gamma(r, F_t) = 0$ on $[0, z(F')]$. Note that the limits of integration in the second integral do not depend on t, so we do not need Leibniz's Rule to differentiate (A.4), which requires extra assumptions. We need to show that $q(\Gamma(r, F_t))x(r)$ is differentiable with respect to $t \in (0,1)$ for almost all $r \in [\max\{p, z(F')\}, \overline{p}]$. The only potential problem is where q is non-differentiable: when $\Gamma(r, F_t) = 0$ and $\Gamma(r, F_t) = \overline{s}$. This is a set of measure zero since (23) rules out the latter and the former occurs only at $r = z(F')$. Hence, the derivative is $q'(r,t)h(r)x(r)$ a.e. Finally, $|q'(r,t)h(r)x(r)| \leq KVx(0)$, so the latter constant provides the required dominating integrable function. We now apply theorem 20.4 in Aliprantis and Burkinshaw (1990, p. 151) to conclude that

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{p}^{\overline{p}} q(r,t)x(r)\,dr &= \frac{\partial}{\partial t}\int_{\max\{p,z(F')\}}^{\overline{p}} q(r,t)x(r)\,dr \\
&= \int_{\max\{p,z(F')\}}^{\overline{p}} q'(r,t)h(r)x(r)\,dr \\
&= \int_{p}^{\overline{p}} q'(r,t)h(r)x(r)\,dr,
\end{aligned} \tag{A.5}$$

where the last equality follows because the theorem permits us to define $q'(r,t)$ however we like at $r = z(F')$, so we let $q'(0) = 0$. Note that this definition is only valid for integration purposes, as in (A.5). ∎

Proof of Proposition 5. (Continued) There are 3 cases to consider.

Case 1. If $z(F') \geq p_M(c)$, then the integral in (24) is constant at

$$\int_{z(F')}^{\overline{p}} q'(r,t)h(r)x(r)\,dr \tag{A.6}$$

for all $p \in (c, p_M(c))$. Hence, (24) becomes

$$\pi^M \left[\int_{z(F')}^{\overline{p}} q'(r,t)h(r)x(r)\,dr - q(\overline{p},t)h(\overline{p}) \right]. \tag{A.7}$$

Differentiating with respect to $p \in (c, p_M(c))$,

$$\pi_p^M \left[\int_{z(F')}^{\overline{p}} q'(r,t)h(r)x(r)\,dr - q(\overline{p},t)h(\overline{p}) \right]. \tag{A.8}$$

Since $\pi_p^M > 0$, the term in square brackets must be nonnegative. Since $\underline{c} \leq c \leq z(F')$ and $q'(0) = 0$ for integration purposes, this is equivalent to

$$\int_{\underline{c}}^{\overline{p}} q'(r,t)h(r)x(r)\,dr \geq q(\overline{p},t)h(\overline{p}). \tag{A.9}$$

The proof continues as in the text.

Case 2. If $c < z(F') < p_M(c)$, then $q'(r,t)$ may be discontinuous at $z(F')$. We therefore split $[c, p_M(c)]$ into $[c, z(F')]$ and $[z(F'), p_M(c)]$. On $(c, z(F'))$, the derivative is again (A.8), leading to (A.9). On $(z(F'), p_M(c))$, we have the same situation as in the text, because $\Gamma(r, F_t) > 0$.

Case 3. If $z(F') \leq c$, then $\Gamma(r, F_t) > 0$ on $(c, p_M(c))$, so we again have the same situation as in the text. ∎

References

C.D. Aliprantis, O. Burkinshaw, Principles of Real Analysis, Academic Press Ltd., London, England, 1990.

M.R. Baye, J. Morgan, Information gatekeepers on the internet and the competitiveness of homogeneous product markets, Amer. Econ. Rev. 91 (2001), 454-74.

M.R. Baye, J. Morgan, Price dispersion in the lab and on the internet: theory and evidence, RAND J. Econ. 3 (2004), 449-66.

M.R. Baye, J. Morgan, P. Scholten, Information, search, and price dispersion, Handbook on Economics and Information Systems (Elsevier, forthcoming).

R. Bénabou, Search, price setting, and inflation, Rev. Econ. Stud. 55 (1988), 353-373.

R. Bénabou, Inflation and efficiency in search markets, Rev. Econ. Stud. 59 (1992), 299-329.

R. Bénabou, Search market equilibrium, bilateral heterogeneity, and repeat purchases, J. Econ. Theory 60 (1993), 140-158.

R. Bénabou, R. Gertner, Search with learning from prices: does increased uncertainty lead to higher markups? Rev. Econ. Stud. 60 (1993), 69-94.

K. Burdett, K.L. Judd, Equilibrium price dispersion, Econometrica 51 (1983), 955-969.

P.A. Diamond, A model of price adjustment, J. Econ. Theory 3 (1971), 156-68.

C. Fershtman, A. Fishman, The "perverse" effects of wage and price controls in search markets, Europ. Econ. Rev. 38 (1994), 1099-1112.

N.B. Haaser, J.A. Sullivan, Real Analysis, Dover, New York, NY, 1991.

M.A. Khan, Y.N. Sun, Non-cooperative games on hyperfinite Loeb spaces, J. Math. Econ. 31 (1999), 455-492.

M.T. Rauh, A model of temporary search market equilibrium, J. Econ. Theory 77 (1997), 128-153.

M.T. Rauh, Heterogeneous beliefs, price dispersion, and welfare improving price controls, Econ. Theory 18 (2001), 577-603.

M.T. Rauh, Non-cooperative games with a continuum of players whose payoffs depend on summary statistics, Econ. Theory 21 (2003), 901-906.

M.T. Rauh, Wage and price controls in the equilibrium sequential search model, Europ. Econ. Rev. 48 (2004), 1287-1300.

M.T. Rauh, Nonstandard foundations of equilibrium search models, J. Econ. Theory, forthcoming.

[16] J.F. Reinganum, A simple model of equilibrium price dispersion, J. Polit. Econ. 87 (1979), 851-858.

[17] R. Rob, Equilibrium price distributions, Rev. Econ. Stud. 81 (1985), 1283-1308.

J. Robert, D.O. Stahl, Informative price advertising in a sequential search model, Econometrica 61 (1993), 657-686.

W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, Inc., New York, NY, 1976.

D.F. Spulber, Market Microstructure: Intermediaries and the Theory of the Firm, Cambridge University Press, Cambridge, MA, 1999.

D.M. Topkis, Supermodularity and Complementarity, Princeton University Press, Princeton, NJ, 1998.

X. Vives, Oligopoly Pricing, MIT Press, London, England, 1999.

Fig. 1. The marginal benefit Γ of another search.