Lost in Translation? Basis Utility and Proportionality in Games*

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abstract

Cooperative and noncooperative games have no representation of players's basis utilities. Basis utility is the natural reference point on a player's utility scale that enables the determination the marginal utility of any payoff or allocation. A player's basis utility can be determined by an observer and other players under standard rationality assumptions. Basis utility allows interpersonal comparison of proportional utility gains relative to the disagreement outcome. Proportional pure bargaining is the unique solution satisfying efficiency, symmetry, affine transformation invariance and monotonicity in pure bargaining games with basis utility. Characterization of the Nash (1950) bargaining solution requires the assumption of the irrelevance of basis utility in games with basis utility. All existing cooperative solution functions become translation invariant once proper account is taken of basis utility. The noncooperative rationality of these results is demonstrated with a proportional bargaining based on Gul (1988). Further noncooperative application is demonstrated by showing that quantal response equilibria with multiplicative error structures (Goeree, Holt and Palfrey (2004)) become translation invariant with specification of basis utility.

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1 Introduction

A cooperative or noncooperative game is a concise mathematical representation of a bargaining problem or strategic interaction. Players in games are generally assumed to be completely rational. As stated by Nash (1950), this rationality implies that players have "full knowledge of the tastes and preferences" of the other players. Therefore, a game must include all information about the value of payoffs or allocations to players that could be apprehended by a rational player (or observer) that might be of value to a rational player in the game.

This paper shows that important information about players's preferences is omitted from cooperative and noncooperative games, as currently defined. This information can be apprehended by rational players in the game and by rational observers. The missing information is the *basis utility* of each player. The basis utility of a player is the reference point on its utility scale that enables another player or an observer to determine the marginal utility of a payoff or allocation to that player.

Some might be surprised by the claim that the marginal utilities of payoffs to players are not determined by the information provided by a game as currently defined. Utilities are sometimes treated as if they were marginal utilities. Further, in experiments the utility of monetary payoffs is often assumed to be equal the cash value. In this case, payoffs happen to correspond to marginal utilities. However, in general, when players are assumed to have expected utility functions and no explicit normalization is provided, it must concluded that marginal utilities are unknown.

In the example of the experiment, the assumption that utility is equal to the cash value provides the normalization: the basis utility must be zero. If the experimenter had added 10 to each monetary payoff in the representation of a player's utility, this would still be a valid representation of the player's preferences. However, we would not be able to properly infer marginal utilities without knowledge of the translation, which directly implies that the player's basis utility is now 10.

The example focuses attention on an important property of expected utility functions: *translation invariance*. Players in games are generally assumed to have expected utility functions. Adding a constant to an expected utility function does not change its representation of a player's preferences and does not change any of the function's decision or game theoretic properties. Basis utility is a point on a player's utility scale. If the scale is translated by adding a constant, basis utility must obviously be translated as well.

If player utility functions are translation invariant, than the solution concepts that determine the outcome of a game should be translation invariant as well. A solution concept is translation invariant if translation of a player's utility function leads to an identical translation of its payoff or allocation. Adding 10 to a player's utility function should lead to its payoff or allocation increasing by 10. This is necessary if the choice of a particular representation of a player's preferences is to have no effect on the real outcome of the game.

The Nash (1950) bargaining solution was the first of many solution concepts that was derived by including the requirement that the solution should be translation invariant. A solution function that fails to be translation invariant will be called *translation dependent*. Myerson (1991: 18) advises that "we should be suspicious any theory of economic behavior" that is translation dependent. This paper shows that what appears to be translation dependence in cooperative games is actually basis utility dependence, or more simply, *basis dependence*. Many cooperative solution concepts defined by axioms, including the Nash bargaining solution, are not properly identified in games extended to allow identification of players's basis utilities. These solutions now require the additional axiom they are not basis dependent.

The principal result of this paper is to show that a large class of solution concepts, (endogenous) proportional solutions, are translation invariant in games with basis utility. To illustrate, consider a cooperative game where two players are bargain over the division of $90. In the event of disagreement, player 1 will receive $10 and player 2 will receive $20. Utility is assumed to be linear in dollars. The proportional solution gives $30 to player 1 and $60 to player 2. Now add 10 to both players's disagreement payoffs and 20 to the cooperative outcome (so that, for example, if players split equally in both cases, their outcomes both increase by 10). Utility is still linear in money but is no longer equal to the number of dollars. Without recognition of basis utility, the proportional outcome appears to be 44 for player 1 and 66 for player 2, as the players will now share on a 20/30 basis. The increase in player 1's payoff is 14 and the increase in player 2's payoff is 6 and translation invariance fails for both players. The proportional pure bargaining solution with basis utility (see section 3.3) gives 40 to player 1 and 70 to player 2, the translation invariant solution that corresponds to the original real outcome.

A related more basic result is that players can make *proportional* interpersonal comparisons of utility in games with basis utility. It has been generally understood that meaningful interpersonal comparisons of utility cannot be made when individuals have expected utility functions.[1] Recognition of basis utility thus requires a revision of the understanding of the potential role of interpersonal comparison of utility in rational bargaining processes.

In the literature on distributive justice going back more than 2000 years, equal and proportional approaches to bargaining have gone side by side (see, e.g., Young (1994)). It is odd that one and only one of these approaches should be considered outside the realm of rational behavior. Proportional allocation has been well-studied by social choice theory (e.g., O'Neill (1980), Young (1988) and Moulin (1987)), but has been largely ignored by game theory. The few exceptions include the proportional nucleolus of Lemaire (1991) and the proportional value of Ortmann (2000) and Feldman

[1] For example, Myerson (1991: 381) writes that translation (and scale) invariance are "... based on an assumption that only the individual decision-theoretic properties of utility scales should matter, and [that] interpersonal comparisons of utility have no decision-theoretic significance [...]."

(1999, 2001) and Kali's (1977) and Roth's (1979) work on (exogenous) proportional pure bargaining according to pre-specified ratios. Other mention of proportional allocation is rare in the economics and game theoretic literature.[2] Recognition of basis utility "rehabilitates" proportional solutions as a legitimate subject of study by those working in the rational behavior framework.

Basis utility is formally defined in section 2 of this paper. Standard rationality assumptions are shown to be sufficient to imply knowledge of a players's basis utilities and the comparability of proportional gains is established.

Cooperative solutions are first taken up in section 3, which develops results for three specific cases: The cooperative equal split bargaining solution, Nash bargaining and proportional pure bargaining. Proportional pure bargaining is shown to be uniquely characterized by efficiency, symmetry, transformation invariance and monotonicity in games with basis utility. Conditions are then developed under which all solutions for the standard characteristic function game are translation invariant.

Section 4 examines noncooperative games. An implementation of proportional bargaining based on Gul (1988) affirms the noncooperative rationality of basis utility dependent cooperative solutions. In the limiting case, a player's probability of selection to propose is proportional to her expected payoff. The equilibrium is translation invariant when selection probabilities are based on marginal utilities. Basis dependence shows up in a different noncooperative context in Goeree, Holt and Palfrey (2004), who find that quantal response equilibrium based on response functions with multiplicative error structures better model experimental data than additive error models. They consider multiplicative error models to be translation dependent. These models are shown instead to be basis utility dependent.

Section 5 returns to cooperative pure bargaining. Equal split is characterized in general pure bargaining games. The similarity of equal split, Kalai-Smorodinsky (1975) and proportional bargaining and their relationship to the proportional solutions of Kali (1977) and Roth (1979) are considered. Several approaches are offered to structure the expanded universe of pure bargaining solutions.

2 Basis utility

2.1 Definition

The standard representations of games abstracts from the *real outcomes* that generate utility for players. These outcomes are the true inputs to players's utility functions. Except in the case of market games, utility functions are generally defined on strategy spaces in noncooperative games and coalitions in cooperative games. A rudimentary

[2] See, however, the comments of Aumann and Maschler (1985: 210), fn. 26 and Hart and Mas-Colell (1996: 595), fn. 10 on the apparent translation dependence of proportional solutions.

framework for the description of real outcomes is developed here in order to provide a clear definition of basis utility.

Let $\mathcal{U}$ be the universe of real outcomes. $\mathcal{U}$ is constructed from a set of *objects* $\mathcal{O}$. An individual object can be a physical object such as a car or a chair or an abstract good such as the right to receive a certain percentage of the profits of an undertaking. Objects may have a quantity dimension, suppressed here for notational simplicity.

Approximately, a real outcome X is a subset of $\mathcal{O}$ and $\mathcal{U}$ is the set of all such X: $\mathcal{U} = \{X : X \subseteq \mathcal{O}\}$. Note that $\varnothing \subset \mathcal{U}$, where $\varnothing$ is the empty set. The definition of real outcomes is imprecise because of the potential for a quantity dimension for an object. For example, in a bargaining problem involving a certain volume x of water, water could be the only object in $\mathcal{O}$. In this case, $\mathcal{U}$ could be understood to be the set of all volumes of water from zero to the entire volume. $\mathcal{U}$ could thus comprise an infinite number of outcomes.

The utility of a player is defined on $\mathcal{U}$, for any player i, $u_i : \mathcal{U} \to \mathbb{R}$. The function u_i is considered to be player i's utility as defined by an observer of the game. Each player j constructs utility scale $u_i^j : \mathcal{U} \to \mathbb{R}$ for player i.

In a noncooperative game with n players, any pure strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ determines a set of real outcomes $X = \{X_i\}_{i=1}^n$. These real outcomes determine the utility to players of strategy profile s^*, so that $u_i(s^*) = u_i(X_i(s^*))$. The utility of mixed strategies can then be defined in the conventional way.

In cooperative games based on the characteristic function a simple approach that is sufficient for many situations is to consider that there is a subset $\mathcal{O}(S)$ of $\mathcal{O}$ available to every coalition $S \subseteq N$, where N is the set of players. For each player i and any allocation $X_i(S)$ to i by coalition S, $X_i(S)$ must be feasible, so that $X_i(S) \subseteq \mathcal{O}(S)$. Further, the set of all allocations by S must also be feasible: $\bigcup_{i \in S} X_i(S) \subseteq \mathcal{O}(S)$. The traditional feasible set of allocations open to coalition S, $W(S) \subset \mathbb{R}^S$ ("W" for the *worth* of coalition S in a cooperative game non-transferrable utility) can then be generated from the set of feasible real allocations and probability distributions over this set.[3]

The basis utility of a player is the utility of receiving no real outcome:

Definition 2.1 *The basis utility of a player i with utility function u_i defined on a universe of real outcomes $\mathcal{U}$ is $u_i(\varnothing)$.*

[3]These definitions do not cover every possible game structure. For example, a pure strategy payoff could be a mixture of real outcomes in a noncooperative game. In cooperative games, there could differing costs associated with providing real outcomes to different players (e.g., different transportation costs) or other reasons why real outcomes might not be completely transferrable. These examples, however can be handled with simple elaborations of the framework described here.

2.2 Identifiability

Standard rationality assumptions imply an observer or players in a game can identify a player's basis utility. Formally, since a player's utility function u_i is defined on $\mathcal{U}$ and $\varnothing \subset \mathcal{U}$ it must be that $u_i(\varnothing)$ is defined. This, however, does not settle the question of whether identifying basis utility somehow might require a "higher level" of rationality than the assessment of the utility to a player of other outcomes in $\mathcal{U}$. As an intuitive matter, it is hard to avoid the conclusion that determining a player's basis utility could not be any harder than determining the utility of all the other outcomes in $\mathcal{U}$. Still, since players in existing games are not assumed to make this assessment, this argument might not be entirely persuasive.

A further argument can be developed around probabilistic outcomes. The utility of any lottery involving two non-null outcomes is clearly within the conventionally assumed scope of knowledge of rational players. It is again then intuitive that the assessing the utility of receiving a non-null outcome with a probability p and nothing otherwise, a *probabilistic outcome*, should also be with this scope of knowledge. Basis utility is then directly inferred.[4] However, it could instead be argued that this is begging the question and that probabilistic outcomes cannot be determined without knowledge of basis utility.

An apparently bulletproof approach to demonstrating that identifying basis utility does not require a higher level of rationality on the part of a player or observer is to consider the utility of a divisible object in a limit process as it "volume" goes to zero. The object could be a volume of water or wine or the size of a royalty to be paid. The utility of any individual outcome in this limit process is clearly identifiable. Let X_μ represent an outcome where $\mu \in (0, 1)$ is the fraction of X received. Then, making only the conventional technical assumption of the continuity of u_i, $\lim_{k\to\infty} u_i(X_{1/k})$ is player i's basis utility. Note that we can assume that there is an infinitely divisible object X in $\mathcal{O}$ even if X is not part of a potential payoff or allocation for any player.

Proposition 2.1 *The basis utility of a player is known by all players and observers in a standard game.*

2.3 Interpersonal comparisons of utility

A player's basis utility is a natural common reference point that, in a natural manner, enables the determination of the marginal utilities to a player of all possible outcomes to a player. These marginal utilities enable all players and observers to determine the proportional utility gains of any outcome relative to the player's individually rational

[4] Assume player (or observer) j assigns $u_i^j(A|p_1)$ as the utility to player i of receiving outcome A with probability p_1 and $u_i^j(A|p_2)$ to receiving A with probability p_2. Then, since $u_i^j(A|p_k) = p_k u_i^j(A) + (1 - p_k)U(\varnothing)$, j can clearly infer that i's basis utility, according to j's scale, must be $u_i^j(\varnothing) = 1/(p_1 - p_2)\{p_1 u_i^j(A|p_2) - p_2 u_i^j(A|p_1)\}$.

outcome. While players and observers may not construct the same utility scales for a player, they will compute the same proportional gains. Basis utility thus enables proportional interpersonal comparisons of utility gains.

Consider a cooperative game in which players have expected utility, with representation w for an observer and w^j for each participant. These representations are based on utility scales constructed for each player. Let u_i be the utility function for player i constructed by an observer and let u_i^j be the utility function for i constructed by player j.

The marginal utility $MU_i(Z)$ of a payoff Z to a player i is the player's utility with the payoff minus the utility without the payoff. The player's utility without the payoff is the player's basis utility.

Definition 2.2 $MU_i(Z) = u_i(Z) - u_i(\varnothing)$.

Assume that $MU_i(Z)$ is the marginal utility to i of Z according to an observer's utility scale. Similarly, define $MU_i^j(Z)$ be this marginal utility according to player j's utility scale for i.

Proposition 2.2 *Consider players i, j and k in a cooperative game w where $w(\bar{i}\,) \neq u_i(\varnothing)$ and consider any outcome $Z \subset \mathcal{U}$. Then*

$$\frac{MU_i(Z)}{MU_i(w(\bar{i}\,))} = \frac{MU_i^j(Z)}{MU_i^j(w(\bar{i}\,))} = \frac{MU_i^k(Z)}{MU_i^k(w(\bar{i}\,))}.$$

The ratio of any player's marginal utility for any real outcome to the marginal utility of the player's individually rational playoff must be the same according to all players and observers.

Proof: For any player j with representation u_i^j of player i's preferences there must be constants a^j and $b^j > 0$ such that $u_i = a^j + b^j u_i^j$ because players have expected utility. Let $MU_i(w(\bar{i}\,))$ be the marginal utility of i's individually rational payoff. Then

$$\begin{aligned}
\frac{MU_i^j(Z)}{MU_i^j(w(\bar{i}\,))} &= \frac{u_i^j(Z) - u_i^j(\varnothing)}{w^j(\bar{i}\,) - u_i^j(\varnothing)} \\
&= \frac{(a^j + b^j u_i(Z)) - (a^j + b^j u_i(\varnothing))}{(a^j + b^j w(\bar{i}\,)) - (a^j + b^j u_i(\varnothing))} \\
&= \frac{u_i(Z) - u_i(\varnothing)}{w(\bar{i}\,) - u_i(\varnothing)} \qquad = \qquad \frac{MU_i(Z)}{MU_i(w(\bar{i}\,))}.
\end{aligned}$$

All ratios are clearly well-defined when $w(\bar{i}) \neq u_i(\varnothing)$. □

This simple result is significant, as the standard understanding is that players in a game and observers of the game cannot make interpersonal comparisons of player's utilities.[5] Most fundamentally, this means that it is impossible to determine if a real outcome Z means more to one player than another. This remains true in the absolute sense: there is no way to tell if i values Z more than does i. However, recognition of basis utility makes it possible to determine which player would experience the greatest proportional gain in utility relative to individually rational payoffs. This knowledge may be relevant to players's evaluation of the fairness of allocations, and, so, enter into the bargaining process. Recent work by Gächter and Riedl (2005) finds that people commonly utilize proportional comparisons in making normative assessments about the fairness of distributional outcomes.

3 Cooperative Games

3.1 Soft bargaining and the equal split solution

In the equal split solution players split the payoff equally, irrespective of disagreement payoffs. This *soft bargaining* – in contrast the commonly expected *hard bargaining* – might be justified by fairness considerations that are rational in a larger context.[6] Soft bargaining is observed in a variety of experiments, most dramatically in the ultimatum game.[7] This example demonstrates the basis dependence of the equal split solution and defines key elements of the formal bargaining framework employed in this paper.

Assume players 1 and 2 can share \$100 if they can agree on its division. Otherwise they receive \$10 and \$30, respectively. Start with players's utilities defined as equal to the dollars received. The bargaining game is then $B = (d, S)$, where $d = (10, 30)$ is the disagreement point and $S = \{(x, y) \in \mathbb{R}^2 | \, x + y \leq 100\}$ is the set of feasible bargaining outcomes. Refer to d and S as *elements* of the bargaining game B.

Definition 3.1 *A bargaining game is* symmetric *if all comparable elements of the game are symmetric. All elements are* comparable *unless specifically excluded from comparison. An element Z of a bargaining game is symmetric if and only if for any players i and j in the game and any vector $x \in Z$, $y \in Z$ as well, where $y_i = x_j$ and $y_j = x_i$ and $y_k = x_k$ for all $k \neq i, j$.*

[5] See footnote 1. Perhaps the maximum degree of comparability established to date is by Myerson (1977), who develops conditions under which there must be a pair of transformations according to which outcomes for players in two-player bargaining games reflect equal gains. Nash bargaining does not satisfy the conditions, but proportional bargaining in games with basis utility does.

[6] See, e.g., Huck and Oechssler (1999) and Lopomo and Ok (2001).

[7] In the ultimatum game one player makes a take-it or leave-it offer to another, and each player receives zero if it is rejected. Equal split does not describe the structure of an ultimatum game, but the choice between equal split and, say, Nash bargaining provides a useful cooperative perspective.

Definition 3.2 *If a cooperative solution is* symmetric *then all players receive equal allocations in a symmetric game.*

Definition 3.3 *A cooperative solution is* efficient *if, for $x = F(d, S)$, there is no $y \in S$ such that $y \geq x$ and $y \neq x$.*

Proposition 3.1 *The TU equal split solution identified by efficiency, symmetry and the noncomparability of disagreement payoffs is basis dependent.*

Proof: Efficiency, symmetry and the noncomparability of disagreement payoffs determine $ES(B) = (50, 50)$. Define B^* by adding 100 to player 1's utility function: $B^* = (d^*, S^*)$, where $d^* = (110, 30)$ and $S^* = \{(x, y) \in \mathbb{R}^2 | \, x + y \leq 200\}$. The axioms again imply a symmetric outcome: $ES(B^*) = (100, 100)$. But this corresponds to $100 - 100 = \$0$ for player 1. Translation invariance implies the outcome $(150, 50) = ES(B) + (100, 0) = (150, 50)$. The basis points of B and B^* are clearly $(0, 0)$ and $(100, 0)$. B^* is no longer symmetric if basis utility is made a comparable element. Translation of B^* to make it symmetric restores the proper allocations. $\square$

Representation of the basis point allows for a natural transformation invariant representation of equal split. This is done now for the case of linear utility functions. Generalization to general pure bargaining games is deferred to theorem 5.1.

Definition 3.4 *A* proper n-player pure bargaining game *is represented by the triple $B = (\xi, d, S)$, where $\xi \in \mathbb{R}^N$ is the basis point, $d \in \mathbb{R}^N$ is the disagreement point, $S \subset \mathbb{R}^N$ is the set of feasible alternatives and $\mathbb{R}^N$ is the n-dimensional space indexed by the set N of the n players $i = 1, \ldots, n$.*

Definition 3.5 *Direct or Hadamard multiplication is represented by the symbol $\odot$. If a and b are both n-vectors, then $a \odot b = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$. If $a \in \mathbb{R}^N$ and B is a subset of $\mathbb{R}^n$, then $a \odot B = \{x \in \mathbb{R}^n | \, y \in B$ and $x = a \odot y\}$.*

Definition 3.6 *Direct addition and subtraction are represented by the symbols $\oplus$ and $\ominus$. If $a \in \mathbb{R}^N$ and B is a subset of $\mathbb{R}^n$, then $a \oplus B = \{x \in \mathbb{R}^n | \, y \in B$ and $x = a + y\}$ and direct subtraction is defined analogously. Direct multiplication has precedence.*

Definition 3.7 *A solution F for a proper n-player pure bargaining game $B = (\xi, d, S)$ is* affine transformation invariant *if and only if $F(s \odot \xi + c, s \odot d + c, s \odot S \oplus c) = s \odot F(\xi, d, S) + c$ for any $s > 0^N \in \mathbb{R}^N$ and $c \in \mathbb{R}^N$.*

Proposition 3.2 *Assume, in a two-player bargaining game $B = (\xi, d, S)$, that the efficient surface of S is linear and that the bargaining solution is efficient, symmetric,* *independent of individual payoffs and affine transformation invariant. Let the maximum utility obtainable by player 1 while player 2 obtains at least ξ_2 be M_1 and the similar maximum for player 2 be M_2. The solution is then the equal split solution and the payoffs are*

$$ES(\xi, d, S) = \left(\frac{1}{2}(M_1 + \xi_1), \frac{1}{2}(M_2 + \xi_2)\right).$$

Proof: First translate utilities so that the basis point is zero. Then rescale so that players's utility is one-to-one transferrable by multiplication with constants $x = (1, (M_1 - \xi_1)/(M_2 - \xi_2))$. Symmetry gives the outcome $(1/2(M_1 - \xi_1), 1/2(M_1 - \xi_1))$. Reverse transformation completes the proof. □

3.2 The Nash bargaining solution

Changes in basis utility have no effect on the Nash (1950) bargaining solution. However, in proper pure bargaining games this independence must be assumed since it cannot be inferred from Nash's axioms of efficiency, symmetry, transformation invariance and independence from irrelevant alternatives[8]

Let x^N be the $1 \times n$ vector with $x_i^N = x$ for all $i \in N$. The game normalized so that the disagreement point is 0^N and the Nash solution is 1^N can no longer be made symmetric by IIA in the sense of definition 3.1 because it cannot be guaranteed that basis utility will be symmetric as well.

Proposition 3.3 *Characterization of the Nash bargaining solution in a proper pure bargaining game $B = (\xi, d, S)$ requires that basis utility be declared a noncomparable element of the game.*

Basis utility is readily identified in Nash's (1950) detailed bargaining example in which two players, Bill and Jack, bargain over personal items. Table 1 reports the utility of these items to each player. Nash assumes utilities are additive. This is a strong assumption in as much as it not only requires that there are no external effects, but that utilities are marginal utilities as well. This implies that the basis point in the example is $\xi = (0, 0)$. When the game is normalized the basis point becomes $\xi^* = (-1, -1.2)$. This is illustrated in Figure 1.

If the reported utilities need not be marginal utilities, then translating Bill's utilities adding one to each outcome would have no effect on the real bargain reached. The Nash solution of the original game is $(24, 11)$ for Bill and Jack. In the transformed

[8] Independence of irrelevant alternatives in proper pure bargaining games requires that if $B = (\xi, d, S)$, $B^* = (\xi, d, S)$, $S \subset T$, $x = F(B^*)$ and $x \in S$, then $x = F(B)$.

game the outcome is $(30, 10.33)$. Translation invariance is clearly violated because Jack's utility changes.[9]

3.3 Proportional bargaining

Kalai (1977) and Roth (1979) study a type of proportional solution that is independent of basis utility. Individual worths are normalized to zero so that the feasibility set completely represents the standard pure bargaining game. Kalai's principal result is that a solution that is weakly Pareto optimal, homogeneous (i.e., $c\,F(S) = F(c\,S)$ for $c > 0$), strongly individually rational and monotonic must be proportional. Strikingly, however, the proportions are exogenously determined.

Social choice models of *endogenous proportional* allocation where individual worths determine relative shares are studied by O'Neill (1980), Moulin (1987) and Young (1988). There have been no endogenous proportional pure bargaining solutions similar to these models. Representation of basis utility allows the disagreement point to determine an endogenous vector of proportionality. Since individual worths are the only measure of the strength of players in a pure bargaining game, this result has an obvious natural interpretation.

The set S of feasible bargaining outcomes is required to be convex, compact, comprehensive and nonlevel. Comprehensive means that if $y \in S$ and $x < y$, then $x \in S$ as well. Nonlevel means that if x is a weakly efficient allocation, then it must be (strongly) efficient as well. That is, if $x \in S$ and there is no $y \in S$ such that $y > x$, then there is no $y \in S$ such that $y \geq x$ and $y \neq x$.

Definition 3.8 *The proportional solution for the proper bargaining game $B = (\xi, d, S)$ with $d > \xi$ is the unique point z on the efficient surface of S such that, for some $c > 0$,*

$$c = \frac{z_i - \xi_i}{d_i - \xi_i}, \qquad i = 1, 2, \ldots, n. \tag{1}$$

Definition 3.9 *A bargaining solution F is* monotonic *if and only if given any two proper pure bargaining games $B = (\xi, d, S)$ and $B^* = (\xi, d, T)$ where $S \subset T$, $F(B^*) \geq F(B)$.*

Theorem 3.1 *The proportional bargaining solution is the unique pure bargaining solution that is efficient (def. 3.3), symmetric (def. 3.2), affine transformation invariant (def. 3.7) and monotonic in all proper pure bargaining games with $d > \xi$.*

[9] In the original game Bill gives Jack the book, whip, ball and bat. Jack gives Bill the pen, toy and knife. In the transformed game, with 1/3 probability the original solution obtains and with 2/3 probability the exchange is the same except that Bill keeps his ball.

Proof: Create a normalized game $B^* = (\xi^*, d^*, S^*)$ from $B = (\xi, d, S)$ where $d > \xi$ with $\xi^* = 0^N$ and $d^* = 1^N$ as follows. Let $S^0 = S \ominus \xi$ (def. 3.6). Then define $\delta = d - \xi$ and let $S^* = \delta^{-1} \odot S^0$, where $\delta^{-1} = (1/\delta_1, 1/\delta_2, \ldots, 1/\delta_n)$.

Define $c = \max\{c \mid c\,1^N \in S^*\}$. Let π^i be an n-vector with $\pi^i_i = c + \epsilon$, $\epsilon > 0$, and $\pi^i_j = 0$ for $j \neq i$. Define T to be the comprehensive set based on the convex hull generated by the points $\{c\,1^N, \pi^1, \pi^2, \ldots, \pi^n\}$. Choose ϵ small enough that $T \subset S^*$ and note that $c\,1^N \in T$ and T is symmetric with respect to players. By symmetry $c^N = F(0^N, 1^N, T)$. Monotonicity now requires that $c^N = F(0^N, 1^N, S^*)$. Any other outcome will require at least one player to receive less than c.

Restoring the utility scales of the original game, we find that, for every player i, $F_i(\xi, d, S) = (d_i - \xi_i)c + \xi_i$, or $c = (F_i(\xi, d, S) - \xi_i)/(d_i - \xi_i)$ for every player i. □

Remark 3.1 *Proportional bargaining is defined here only for games where $d > \xi$. With little work, the solution can be extended to games where $d \geq \xi$ through a limit argument. In these games $d_i = \xi_i$ implies $F_i(\xi, d, S) = \xi_i$ so long as there is at least one player j with $d_j > \xi_j$. The allocation to these players is the same as the bargaining game without the players i that have $d_i = \xi_i$.*

Remark 3.2 *Proportional solutions for the case where there exists a $d_i < \xi_i$ are problematic. If there is also a $d_j > \xi_j$, then the arguments of theorem 3.1 do not carry through because affine transformation cannot create a symmetric bargaining problem and there is no hint of a solution. On the other hand, if there is no $d_j > \xi_j$ and $d < \xi$, the bargaining problem can be made symmetric and the arguments of theorem 3.1 hold. However, the result is unacceptable. The lower the disagreement outcome relative to a player's basis utility, the stronger the player becomes. This is because proportional solutions reflect magnitudes but cannot reflect signs.*

3.4 General cooperative games

The worth of the null coalition currently conveys no information. One way to represent basis utility in coalitional games is to let the null coalition represent the basis point. In continuity with the notation already used, let $w(\varnothing) = \xi$.

Definition 3.10 *A game w in* proper characteristic function form *is a set function from the coalitions $S \subset 2^N$, $S \neq \varnothing$, to $\mathbb{R}^S$, with $w(\varnothing) \equiv \xi$.*

Note that affine transformation of the characteristic function requires that the worth of the null coalition must be transformed as well. This definition simplifies for transferrable utility games. Any game in proper form has a marginal representation.

Definition 3.11 *Let w be a proper game with $w(\varnothing) = \xi \in \mathbb{R}^N$. Let w^* be the* marginal *form of w. For any coalition $S \subset N$, let ξ_S be the restriction of ξ to the players in S, with $\xi_\varnothing \equiv 0$. Then $w^*(S) = w(S) \ominus \xi_S$ (def. 3.6).*

In a marginal form game $w^*(\varnothing) = 0^N$. Proposition 2.1 shows that all player and observers know the basis point in a game. Therefore, it seems reasonable to assume that $\xi = 0^N$ unless explicitly stated otherwise, and, hence, that the standard characteristic function is, effectively, a marginal form game.

Definition 3.12 *A solution function is in* marginal form *if it does not reference the basis point.*

Definition 3.13 *Let ϕ^* be a solution function with marginal form and let w be a cooperative game in proper characteristic function form with $w(\varnothing) = \xi$ and marginal representation w^*. Then define*

$$\phi^*(w) = \phi^*(w^*) + \xi.$$

The result of the application of a solution function in marginal form to a proper form game is defined to be the solution on the marginal form game plus the basis point. The marginal form is directly analogous to the 0-normalized standard characteristic function game. The proper form of the solution function can now be inferred. The following proposition trivially follows.

Proposition 3.4 *All marginal form solutions are translation invariant.*

Definition 3.14 *Let ϕ be a solution function defined on proper characteristic function games. Let $v = w$, except that $w(\varnothing) \neq w(\varnothing)$. Basis utilities change in v. If $\phi(v) = \phi(w)$ for any game w and any value $w(\varnothing)$, then ϕ is* basis independent.

Proposition 3.5 *The property identified as translation invariance in standard form games without representation of basis utility is actually* basis independence.

Proof: If a solution appears translation invariant in standard characteristic function games, it must be independent of basis utility as the implied basis point is always $(0, 0)$, even after translation. If a solution is basis dependent, it must appear translation dependent because the implied basis point is not subjected to translation. $\square$

4 Noncooperative games

The principal task of this section is to demonstrate the noncooperative rationality of basis utility with a translation invariant model of proportional bargaining. Additionally, quantal response equilibrium is shown to provide a pure noncooperative application of basis utility. There is at least one prior point of contact between basis utility and noncooperative games. A player in the Hart and Mas-Colell (1996) model of the consistent NTU value that is 'removed' due to a breakdown in negotiations receives zero terminal payoff. This payoff must function like the player's basis utility if the game is be translation invariant.

4.1 Noncooperative proportional bargaining

Proportional bargaining (th. 3.1) is modeled using the basic setup developed by Gul (1998) to implement the Shapley value. Two players are endowed with productive assets that yield an income stream. In each time period, a player is selected to bid a constant stream of payments for the other's resources. If the bid is accepted, the bidder receives the assets of the acceptor, the acceptor receives the promised payments and bargaining ends. If the bid is rejected, this stage game repeats.

Let $w(\overline{12}\,)$ be the worth of the joint assets, let $w(\overline{1})$ and $w(\overline{2})$ be the individual worths and let $w(\varnothing) = (\xi_1, \xi_2)$. The game must be superadditive: $w(\overline{12}\,) > w(\overline{1}\,) + w(\overline{2}\,)$. The common discount *factor* is δ, with $0 < \delta < 1$. Let c_t^i be the value of the assets owned by player i at time t. The utility provided to player i at time t is defined to be $(1-\delta)\,c_t^i$. The present discounted utility at time t_0 to player i given an prospective asset holding history $\{c_t^i\}_{t=t_0}^{\infty}$ is then

$$U^i(t_0) = \sum_{t=t_0}^{\infty} \delta^{t-t_0}\,(1-\delta)\,c_t^i. \tag{2}$$

The selection of players to propose differs from the Gul model. Both players submit bids before the bidder is selected. The probability of selecting player i to bid is set proportional to player j's bid for i's assets. This selection procedure can be seen as a natural way to reflect the impact of a player's strength on the bargaining process. Let b_t^{ji} be the bid by player j for i's assets at time t. The probability p_t^i of i's selection to bid in period t, conditional on b_t^{ji} and b_t^{ij} is

$$p_t^i = \frac{b_t^{ji} - \xi_i}{(b_t^{ji} - \xi_i) + (b_t^{ij} - \xi_j)}. \tag{3}$$

Computing probabilities based on the marginal utility of bids makes selection probabilities independent of the translation of players's utility scales.

Given that no bid has been accepted, the complete history of the game prior to time t, is $h_{t-1} = (b_k^{ij}, b_k^{ji})_{k=1}^{k=t-1}$. The complete set of all such possible histories prior to time t is H_{t-1}. A strategy for player i at time t given a history $h_{t-1} \in H_{t-1}$ is $\sigma_t^i(w, \delta, h_{t-1}) = (b_t^{ij}, r_t^i)$, where i will accept any bid $b_t^{ji} \geq r_t^i$ if j is selected to bid. Let σ_t^i contain a single strategy for each possible history to time $t-1$. A complete strategy for i is $\Sigma^i = (\sigma_t^i)_{t=1}^{t=\infty}$, the set of all strategy profiles is $\Sigma = \Sigma^1 \times \Sigma^2$ and a complete description of the game is then $\Gamma_1 = (\Sigma, (U^1, U^2), w, \delta)$.

Theorem 4.1 *In the unique stationary subgame perfect equilibrium of Γ_1 i offers j*

$$\bar{b}^{ij} = \delta \frac{w(\bar{j}) - \xi_j}{\sum_{i=1}^{i=2} w(\bar{i}) - \xi_i} \left(w(\overline{12}) - \xi_1 - \xi_2\right) + (1-\delta)(w(\bar{j}) - \xi_j) \; + \; \xi_j,$$

and $\bar{r}^j = \bar{b}^{ij}$. The expected utilities during bargaining and at any time t before a bidder is selected are the allocations determined by proportional pure bargaining (eq. 1)

$$\bar{U}^i = \frac{w(\bar{i}) - \xi_i}{\sum_{j=1}^{j=2} w(\bar{j}) - \xi_j} \left(w(\overline{12}\,) - \xi_1 - \xi_2\right) \; + \; \xi_i, \qquad i = 1, 2.$$

Proof: Considering stationary strategies, history is irrelevant and each player computes optimal strategies under the assumption that if the current bid is rejected that agreement will be reached in the next time period. Expected utility before selection of a bidder is

$$\bar{U}^i = p^i \Big(w(\overline{12}\,) - \bar{b}^{ij}\Big) + p^j \, \bar{b}^{ji}, \qquad i = 1, 2; \; j \neq i.$$

Equilibrium bids are player's continuation values and are the solution of the equations

$$\bar{b}^{ij} = \delta \bar{U}^j + (1-\delta)\, w(\bar{j}), \qquad i = 1, 2; \; j \neq i.$$

□

Proposition 4.1 *The game Γ_1 is translation invariant.*

Proof: Let $w^*(\varnothing) = \xi^* = (\xi_1 + x, \xi_2)$, $w^*(\overline{12}) = w(\overline{12}) + x$, $w^*(\bar{1}) = w(\bar{1}) + x$ and $w^*(\bar{2}) = w(\bar{2})$. Then $\bar{b}^{ji^*} = \bar{b}^{ji} + x$, $\bar{U}^{i^*} = \bar{U}^i + x$, $\bar{b}^{ij^*} = \bar{b}^{ij}$ and $\bar{U}^{j^*} = \bar{U}^j$. □

Remark 4.1 *Theorem 4.1 easily generalizes to n-player pure bargaining games.*

Remark 4.2 *Selection probabilities can be based on the average of both players's proposals,* e.g., $p^i = (b^{ji} + b^{ii})/((b^{ji} + b^{ii}) + (b^{ij} + b^{jj}))$. *The outcome in the limit, as* $\delta \to 1$, *is the same. However, expected utility for* $\delta < 1$ *is no longer exactly the proportional solution.*

Remark 4.3 *Complete translation invariance can easily be shown in (NTU) hyperplane games, and with some work, in general NTU games.*

Remark 4.4 *A TU and NTU implementation of proportional pure bargaining based on the game of Hart and Mas-Colell (1996) is included in Feldman (2002).*

4.2 Quantal response equilibria

Quantal response equilibria (QRE) are a refinement introduced by McKelvey and Palfrey (1995). "Trembles" or misperceptions of payoffs cause deviations from best response and are modeled with a statistical response function. Under general conditions a unique equilibrium is selected as the size of trembles goes to zero. QRE is defined with an additive error structure. This guarantees translation invariance. Goeree, Holt and Palfrey (2004) (GHP) introduce regular QRE. One feature of these equilibria is that they allow a multiplicative error structure, which the authors find provides a better fit to experimental data. GHP consider the multiplicative error model to be translation dependent and thus that "[t]ranslation invariance is not plausible in settings where the magnitudes of perception errors or preference shocks depend on the magnitudes of expected payoffs." (2004: 19.)

A regular n-player QRE for may be defined as follows. Let $S_i = (s_{i1}, s_{i2}, \ldots, s_{iJ_i})$, be i's pure strategy set, where J_i is the number of i's pure strategies. Let $\sigma_i \in \Sigma_i$ be a mixed strategy over S_i, let $\sigma \in \Sigma$ be a complete profile of mixed strategies, and let σ_{-i} represent the strategy profile of all players except i. Player i's expected payoff from a strategy profile σ is $\pi_i(\sigma)$.

Represent undisturbed payoffs as a function of strategy choice to any i given σ_{-i} by the function $\bar{\pi}_i(\sigma) = (\pi_i(s_{i1}, \sigma_{-i}), \pi_i(s_{i2}, \sigma_{-i}), \ldots, \pi_i(s_{iJ_i}, \sigma_{-i}))$. Collect the $\bar{\pi}_i$ into the profile $\bar{\pi}(\sigma) = (\pi_1(\sigma), \pi_2(\sigma), \ldots, \pi_n(\sigma))$. Player i's perceived payoff from strategy j, $\hat{\pi}_{ij}(s_{ij}, \sigma_{-i})$, is affected by a privately observed random disturbance that may be a function of her strategy choice: $\hat{\pi}_{ij}(s_{ij}, \sigma_{-i}) = g(\bar{\pi}_i(s_{ij}, \sigma_{-i}), \epsilon_{ij})$.

Let $P_i : \bar{\pi}_i \to \Sigma_i$ be the regular quantal response function for player i. The regular QRE is a reduced form approach because P_i implies $g(\bar{\pi}_i, \epsilon_i)$ and the distribution of ϵ_i. GHP place restrictions directly on the response functions of regular QRE that ensure representation of boundedly rational choice behavior. A strategy profile σ is a regular QRE if and only if $P_i(\bar{\pi}_i(\sigma_{-i})) = \sigma_i$ for all $i = 1, \ldots, n$.

The canonical quantal response function based on multiplicative error is the power model, under which the probability of i playing strategy j is

$$P_{ij} = \frac{(\pi_{ij})^{\frac{1}{\mu}}}{\sum_{k=1}^{J_i} (\pi_{ik})^{\frac{1}{\mu}}}, \tag{4}$$

where $\mu \geq 0$ is a constant determining players's discrimination ability. As $\mu \to 0$ the probability of all players playing their best response goes to one. McKelvey and Palfrey prove, for logit response functions, that there is generically a unique branch of the equilibrium correspondence based on the discrimination parameter that contains the unique regular QRE under no discrimination and a perfect discrimination QRE that is also a Nash equilibrium. This branch can be thought of as representing a learning process that leads to a unique Nash equilibrium.

Figure 2 shows a simple three-player coordination game Γ_2. The strategy profiles (U, L, W) and (D, R, E) are both Nash equilibria. The QRE equilibrium using the power response function is (U, L, W) when $x = 0$. However, increasing all of player 3's payoffs by one by setting $x = 1$ leads to the selection of (D, R, E). This apparent translation dependence disappears if response probabilities in eq. 4 are determined by marginal utilities. Basis and not translation dependence appears in the quantal power response function.

Proposition 4.2 *All quantal response functions using marginal utilities, payoffs relative to basis utilities, are translation invariant.*

5 Focal Points, Monotonicity and Pure Bargaining

This section completes the presentation of pure bargaining results and provides some interpretation. Equal split is first characterized in proper pure bargaining games.

5.1 Equal split in general pure bargaining games

Definition 5.1 *Let $B = (\xi, d, S)$ be an n-player pure bargaining game. Consider a set $x^i \in S$, $i = 1 \ldots, n$ and a $y \in \mathbb{R}^N$. For any i, let $x^i \in S$ maximize x^i_i subject to the further restriction that $x^i_j \geq y_j$ for all $j \neq i$. Then the* maximal aspirations point relative to y *is $M_y = M(y, S) = (x^1, \ldots, x^n)$. Define $M_\xi = M(\xi, S)$ as the* ξ-maximal aspirations point *of B and $M_d = M(d, S)$ as the* d-maximal aspirations point *of B.*

The ξ-maximal aspirations point shows the most a player can receive when all other players receive at least their basis utility. The standard d-maximal aspirations point represents the most a player can receive when all others receive at least their disagreement payoffs.

Definition 5.2 *Let $B = (\xi, d, S)$ and $B^* = (\xi, d, T)$ have a common maximal aspirations reference point y. A solution F is* restricted monotonic *if and only if $M_y = M(y, S) = M(y, T)$ and $S \subset T$ imply that $F(B^*) \geq F(B)$.*

Restricted monotonicity weakens the definition of monotonicity (def. 3.9) by requiring that two feasibility sets share the same maximal aspirations point.

Theorem 5.1 *The equal split solution is the unique solution for the game $B = (\xi, d, S)$ that is efficient (def. 3.3), symmetric (def. 3.2), affine transformation invariant (def. 3.7), restrictedly monotonic (def. 5.2), and shows noncomparability of disagreement payoffs and comparability of $M_\xi = M(\xi, S)$ (see def. 3.1).*

Proof: Normalize B so that $\xi^* = 0^N$ and $M^*_\xi = 1^N$ and define $c = \max\{c \mid c\,1^N \in S^*\}$. Let π^i be an n-vector with $\pi^i_i = 1$, and $\pi^i_j = 0$ for $i \neq j$. Define T to be the comprehensive set based on the convex hull generated by the points $\{c\,1^N, \pi^1, \pi^2, \ldots, \pi^n\}$. By symmetry $c^N = F(0^N, 1^N, T)$. Restricted monotonicity then requires that $c^N = F(0^N, 1^N, S^*)$ as well. $\square$

The equal split solution is the point on the line between ξ and M_ξ that intersects the efficient surface of S. The sense of equality in the general equal split solution is in the nature of a proportionality property. Consider the range from any player's maximal expectations to their basis utility. Each player loses relative to maximal aspirations or gains relative to basis utility in equal proportion.

Proposition 5.1 *Let $x = ES(\xi, d, S)$ be the equal split solution, let b be the ξ-maximal aspirations point and let b be strictly greater than ξ. Then there is a k such that*

$$\frac{b_i - x_i}{b_i - \xi_i} = k \quad \textit{and} \quad \frac{x_i - \xi_i}{b_i - \xi_i} = 1 - k, \qquad\qquad i = 1, 2, \ldots, n.$$

5.2 Monotonic solutions

Equal split is a direct variation on the Kalai and Smorodinsky (1975) solution. Basis utility replaces the disagreement point and ξ-maximal aspirations replace d-maximal aspirations. There is an analog to proposition 5.1 for Kalai-Smorodinsky bargaining. Thus equal split, Kalai-Smorodinsky and proportional bargaining are all monotonic and have proportional qualities. The relationship between monotonicity and proportionality shown by Kalai (1977) also appears in these endogenously proportional solutions. However, Kalai's (1975) solutions are homogeneous and not translation invariant because the exogenous proportionality vector is translation dependent.

There is an essential similarity between equal split, Kalai-Smorodinsky and proportional bargaining. Given efficiency, symmetry, transformation invariance and the appropriate monotonicity axiom, the salience of any two reference points identifies a solution. These solutions are intuitive. Two points determine a line. The solution is the intersection of this line with the efficient bargaining surface. Monotonicity merely identifies this intersection mathematically. This simplicity can seem like a weakness. There is little subtlety and no sense of marginal equilibrium. However, this simplicity is likely a strength. Schelling writes

> [G]ame characteristics that are relevant to sophisticated mathematical solutions ... might not have the power of focusing expectations and influencing the outcome ... except when the same solution can be reached by an alternative less sophisticated route. (1960: 113, edited)

Indeed, the less sophisticated the nonmathematical route, the greater the power of focusing expectations might reasonably be. The salience of two reference points makes a monotonic solution a focal point.

5.3 Pure bargaining choices

Section 5.2 provides a reference-point based approach focusing expectations in bargaining. Given such a focus, bargaining mechanisms consistent with these expectations might then be favored. If the salience of only the disagreement point is thought to guide or allow expectations to move toward Nash bargaining, this approach becomes more complete.

Figure 4 illustrates another approach to solution selection, one based on the characteristics of bargaining outcomes. The primary choice is between equal and proportional gain. There is no 'soft' variant of proportional pure bargaining because the disagreement point is essential to proportional outcomes. The next choice is then between the soft and hard variants of equal gain bargaining, with equal split being the soft bargaining solution. There are two variants of equality-based hard bargaining. Monotonic Kalai-Smorodinsky bargaining provides gains that are in strict equal proportion relative to the disagreement and maximal aspirations points. IIA-based Nash outcomes deviate from this strict equality when and to the extent that doing so will increase the product of player's payoffs relative to the disagreement point.

The nature of noncooperative implementations provides the last approach to comparing pure bargaining models. Nash bargaining results when players have equal participation in the game (e.g., Binmore, Rubinstein and Wolinsky (1986) and Hart and Mas-Colell (1996)). Theorem 4.1 and Feldman (2002) show that proportional bargaining results when players's probability of proposing is proportional to their expected payoff. Moulin (1984) shows that Kalai-Smorodinsky bargaining can be implemented in a game where players first bid *probabilities* for the right to propose, the player with the highest bid proposes first and the second player has the right to make a last counteroffer with the probability of the winning bid. Finally, Huck and

Oechssler (1999) find equal split is the equilibrium outcome in an evolutionary setting and Lopomo and Huck (2001) find equal split in cases of interdependent preferences. None of these models can be considered inherently more rational than the others, but each has aspects that make it more relevant to particular bargaining environments.

6 Conclusion

Recognition of basis utility expands pure bargaining theory with two new endogenous proportional solutions. Equal split provides a model of commonly observed experimental outcomes. Pure proportional bargaining is the pure bargaining version of the TU proportional value of Ortmann (2000) and the NTU proportional value of Feldman (1999, 2002). With the Kalai-Smorodinsky (1975) bargaining solution they form a versatile family of monotonic pure bargaining models. Basis utility also allows moves of nature in noncooperative games, such as the selection of proposers and trembles, to be conditioned on payoffs without creating translation invariant equilibria.

Endogenous proportionality was lost to game theory without basis utility, which was obscured in part by the mechanics of translation invariance. Basis utility expands the range of interpersonal comparisons that can be made in the expected utility framework beyond those of Kalai (1977) and Myerson (1977). Thompson's (1998: 197) negotiation text sees consensus interpersonal comparison and proportionality as "the heart of equity theory." Proportionality, here, should be taken in the sense of the ratios of Kalai (1977) and propositions 2.2 and 5.1 and not simply proportional bargaining. This is not a new idea. Moulin (1999) quotes Aristotle in his survey of social choice allocation rules: "Equals should be treated equally, and unequals, unequally in proportion to relevant similarities and differences."

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Item	Utility to Bill	Utility to Jack
Bill's items:		
book	2	4
whip	2	2
ball	2	1
bat	2	2
box	4	1
Jack's items:		
pen	10	1
toy	4	1
knife	6	2
hat	2	2

Table 1: Bargaining example from Nash (1950).

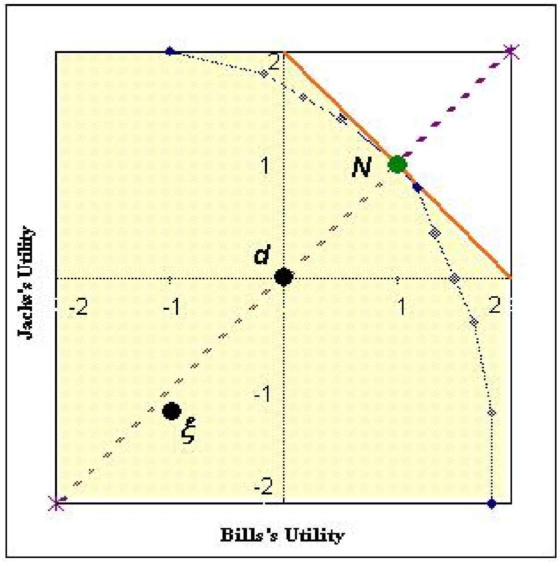

Figure 1: Nash (1950) example normalized and including basis utility point $\xi = (-1, -1.2)$.

W	L	R
U	3, 3, 5 + x	1, 1, 1 + x
D	1, 1, 1 + x	2, 2, 1 + x

E	L	R
U	2, 2, 1 + x	1, 1, 1 + x
D	1, 1, 1 + x	4, 4, 2 + x

Figure 2: Coordination game Γ_2.

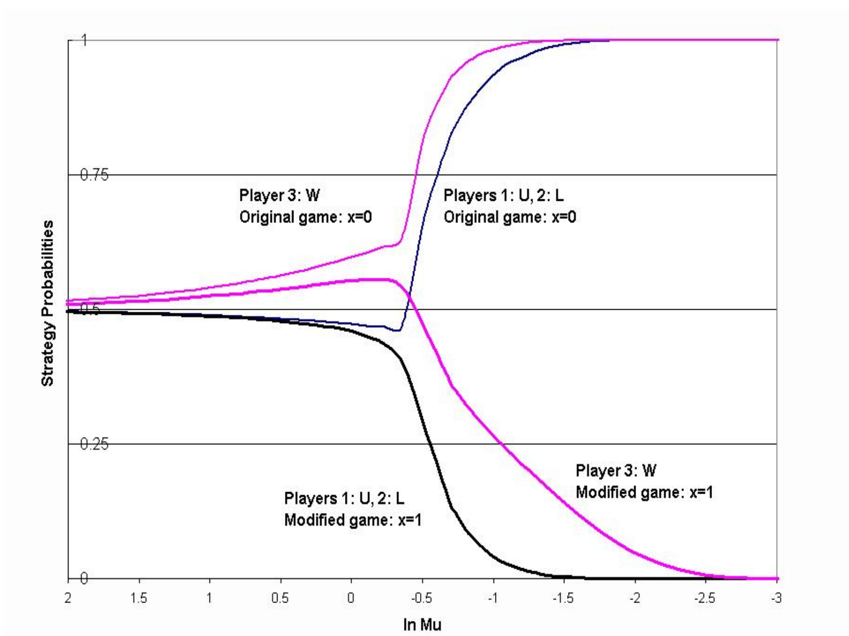

Figure 3: Quantal response graph for Γ_2 showing apparent translation dependence.

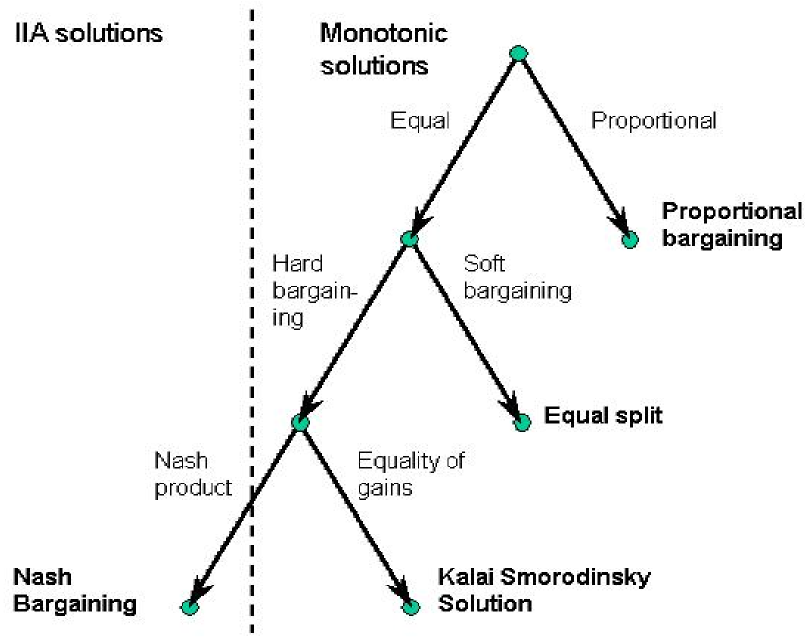

Figure 4: Tree of IIA and monotonic bargaining solutions.