A Stronger Case for Transitive Preferences

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Abstract

The assumption that preferences are transitive, or, equivalently, that choice behavior satisfies the Weak Axiom of Revealed Preference, is at the core of most economic theory. While this is a natural assumption, one could ask the degree to which it is restrictive: are there objectives that could not be attained by such behavior that could be attained by choices violating the assumption? It is argued that the answer to this question is no in one setting of choice under random budget sets.

Preliminary and Incomplete; literature survey not included; please do not cite without author's permission. Comments and suggestions welcomed.

1 Introduction

The assumption that agents have transitive preferences, or, approximately equivalently, that agents' choices satisfy the Weak Axiom of Revealed Preference, is at the root of economic theory. The assumption seems very natural and intuitive; in fact, this assumption is just about the definition of what is means for a decision-maker to be "rational."

The normative argument for transitivity is often given by contradiction, using the example of a "Dutch book" or "money pump." An agent who truly adhered to preferences with a violation of the transitivity assumption could be systematically exploited, indefinitely, at no cost to the exploiter. But, in order to begin such exploitation, it would have to be observed that the agent had such preferences. If one considers the decision-making apparatus of a biological being as a product of evolution, one can imagine that adhering to such "inconsistent" choice procedures would rarely be exploited in nature. Could it then be the case that perhaps some benefit could come of using such choice procedures, some of the time? Could some outcome be obtained by a decision-maker who occasionally violates transitivity that would not be accessible via transitive means? If so, intransitivities could well have an evolutionary value.

For many environments, the answer to such a question is, of course, an immediate "no." In any static choice setting, or any setting where the budget set is known in advance, any feasible choice can be easily rationalized by a utility function, and thus is the product of some set of transitive preferences. For random choice models, for any given probability distribution over a (finite) set of choices one can construct an underlying utility function for a logit specification that explains the choices.

This paper explores a model of choice in a setting where the budget set in each period is chosen randomly, and asks whether there are feasible long-run bundles of consumption that can be obtained only by sometimes adopting behavior that could not be rationalized by transitive preferences. This can be thought of as a lower-level model of how consumption actually occurs: at any point in time, a set of opportunities presents itself, and only one can be selected. At the next point in time, the opportunity set will change. The agent is interested in achieving some long-run consumption pattern. Can he behave in a (statistically) consistent way and still achieve any such feasible pattern?

It is shown that, indeed, the intuition that choice behavior that would violate the Weak Axiom of Revealed Preference does not make possible outcomes that could not be obtained by choice behavior satisfying the axiom. This result thus extends this stronger defense of transitivity -- that behavior violating transitivity cannot be of gain in achieving a goal -- to this case.

2 Model

Suppose there are a finite number N of goods. At each point in time $t = 1, 2, 3, \ldots$, the agent is presented with a budget set of two goods drawn from this set, from which the agent may choose exactly one to consume. This subset is chosen randomly according to some probability distribution known to the agent. (Probabilities of presentation of a budget set will be denoted by suitably subscripted q's.) The process is stationary; the presentation probabilities do not change with time, and do not depend on previous choices. The agent's objective is to attain some distribution of consumption $w = (w_1, \ldots, w_N)$, where $w \geq 0$ and $\sum_{i=1}^{N} w_i = 1$.[1] This objective is analogous to the use of time-average payoffs in some repeated game theory.

It will turn out that it will be necessary to consider only three goods, so the notation can be specialized. Consider three budget sets: $B_1 = \{2, 3\}$, $B_2 = \{1, 3\}$, and $B_3 = \{1, 2\}$, and let q_1, q_2, and q_3 be their probabilities of presentation, respectively. Note that budget sets are numbered by the good which is absent. Then, there are eight possible choice functions, given in Table 1. Functions f_3 and f_6 violate the Weak Axiom of Revealed Preference.[2]

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
B_1	2	3	2	3	2	3	2	3
B_2	1	1	3	3	1	1	3	3
B_3	1	1	1	1	2	2	2	2

Table 1. The eight choice functions. Note that f_3 and f_6 exhibit a cycle.

1. Note that the results will all go through if this is relaxed to an assumption that the agent requires some minimum amount w_i of good i, such that $\sum_{i=1}^{N} w_i \leq 1$.

2. To be precise, these functions technically do not violate the axiom, since the condition in the axiom does not hold for any pair of the sets. However, it would not be possible to specify a choice for the budget set $\{1, 2, 3\}$ that would not violate WARP. This slightly inaccurate phrasing will be used for ease of presentation in the body of the text.

In this environment, in order to obtain a particular long-run w, it will in general be necessary to choose randomly for some budget sets. Let p_k, $k = 1, \ldots, 8$ be the probability that function f_k is adopted at any given time. Again, p_k is assumed to be stationary over time.

Then, the condition that a given vector of probabilities $p = (p_k)_{k=1}^{8}$ achieves the goal of a long-run consumption pattern w is

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} q_2+q_3 & q_2+q_3 & q_3 & q_3 & q_2 & q_2 & 0 & 0 \\ q_1 & 0 & q_1 & 0 & q_1+q_3 & q_3 & q_1+q_3 & q_3 \\ 0 & q_1 & q_2 & q_1+q_2 & 0 & q_1 & q_2 & q_1+q_2 \end{pmatrix} p^T. \tag{1}$$

It is assumed that equation (1) has a solution; that is, that there exists some feasible way to obtain w. Since the matrix has more rows than columns, if a solution exists, then infinitely many solutions exist.

3 Results

Proposition 1. *If $N = 3$, and if the vector w is feasible, there exist distributions p over the choice functions that give probability zero to the functions that violate WARP. In addition, any feasible vector w can be obtained by randomizing over a basis of at most three choice functions.*

Proof. The argument is easiest seen graphically. The space of possible consumption bundles w is presented as the simplex in Figure 1. The eight points on the simplex correspond to one of the eight choice functions f_k. The six functions which do not violate WARP are located on the boundaries of the simplex, and form a hexagonal shape. The two functions which violate WARP, f_3 and f_6, are located interior to the hexagon. Thus, if there is a vector p solving equation (1) such that $p_3 > 0$ or $p_6 > 0$, then there exists another solution such that $p_3 = p_6 = 0$. In other words, whatever choices are being made when using f_3 and f_6 can be "synthesized" using only transitive orders.

The hexagon formed by the points corresponding to the six choice functions satisfying WARP are exactly the vectors w which are feasible. All such points lie within at least one simplex generated by exactly three of the extreme points. Thus, any w can be obtained by using at most three choice functions. □

Corollary 2. *For all finite N, if the vector w is feasible, there exist distributions p over the choice functions that give probability zero to the functions that violate WARP.*

Proof. Observe first that if a cycle exists over four or more goods and budget sets, then there must be a cycle within those goods consisting of exactly three of the goods. Given that, apply the argument of Proposition 1 to those three goods, where the eight choice functions to use are the eight that disagree on those three budget sets, but agree on all other budget sets. □

In practice, what is observed is not the underlying choice function being used, but the aggregate choice behavior when presented with budget sets. Let $P(x;\, B)$ be the probability some good $x \in B$ is chosen from a budget set B. Then, a natural extension of the self-consistency ideas of WARP to this stochastic environment is:

Definition 3. *Observed choice behavior satisfies* ***stochastic WARP*** *if, for any three goods A, B, C, and any three choice sets* $\alpha \supseteq \{A, B\}$*,* $\beta \supseteq \{B, C\}$*, and* $\gamma \supseteq \{A, C\}$*, the conditions*

$$P(A;\alpha) > P(B;\alpha) \text{ and } P(B;\beta) > P(C;\beta) \text{ and } P(C;\gamma) > P(A;\gamma)$$

are not jointly satisfied.

Proposition 4. *There exist distributions p over the choice functions that satisfy stochastic WARP. Furthermore, stochastic WARP is strictly stronger than just requiring* $p_k = 0$ *for all choice functions* f_k *that do not satisfy WARP.*

Proof. Because of symmetry, there are three types of basis for p to consider:

1. The extreme points are all adjacent on the hexagon;
2. Two extreme points are adjacent;
3. Noone of the extreme points are adjacent.

These are investigated in turn.

1. *All adjacent:* Suppose $w = \alpha f_4 + \beta f_7 + \gamma f_8$, for some $\alpha + \beta + \gamma = 1$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$.[3] Then, $P(1|B_3) = \alpha$; $P(3|B_2) = 1$, and $P(2|B_1) = \beta$. Stochastic WARP fails iff all three probabilities are greater than one-half or less than one-half. They cannot all be less (since $1 > 1/2$), and they cannot be all greater, since $\alpha > 1/2$ and $\beta > 1/2$ cannot both hold.
2. *Two adjacent:* Suppose $w = \alpha f_4 + \beta f_5 + \gamma f_8$. Then $P(1|B_3) = \alpha$; $P(3|B_2) = \alpha + \gamma = 1 - \beta$; $P(2|B_1) = \beta$. Since both β and $1 - \beta$ cannot simultaneously be less than or greater than $1/2$, stochastic WARP holds.
3. *None adjacent:* Suppose $w = \alpha f_2 + \beta f_5 + \gamma f_8$. Then $P(1|B_3) = \alpha$; $P(3|B_2) = \gamma$; $P(2|B_1) = \beta$. If, for example, $\alpha = \beta = \gamma = 1/3$, stochastic WARP would fail. □

The intuition for why stochastic WARP is not necessarily satisfied when using choice functions that correspond to extreme points that are not adjacent can be obtained by examining those functions. There are two such groups: $\{f_1,\ f_4,\ f_7\}$ and $\{f_2,\ f_5,\ f_8\}$. Taken together, each group forms a "cycle" in that there is a one-to-one mapping between a function in the group and a good which is never selected by that function. So randomizing over those bases is itself a form of inconsistency. Even in a static environment, randomizing over choice functions that satisfy WARP can still result in overall choice behavior that does not satisfy stochastic WARP.

3. The notation for choice functions is abused slightly to also represent the points on the simplex.

4 Conclusion

The standard normative justification for the use of transitive preferences is that intransitivities in preferences make an agent subject to systematic exploitation. However, the conditions for that systematic exploitation are, in some sense, relatively likely to occur. It is unlikely that simply by chance the conditions would occur in nature. A strategic individual could create those conditions, but only after having found out an intransitivity exists; the informational requirements for that are perhaps implausibly stringent.

This paper proposes to augment this justification further, by additionally noting that behaving in such a way that is inconsistent with the assumption does not allow the agent to attain results that could not be obtained via means consistent with the assumption. Thus, in addition to the potential cost of exploitability, at least in the settings discussed here, there is no benefit to violating transitivity. Thus, the justification for the assumption can be phrased in cost-benefit terms -- that is to say, it can itself be formulated in terms of economic principles.

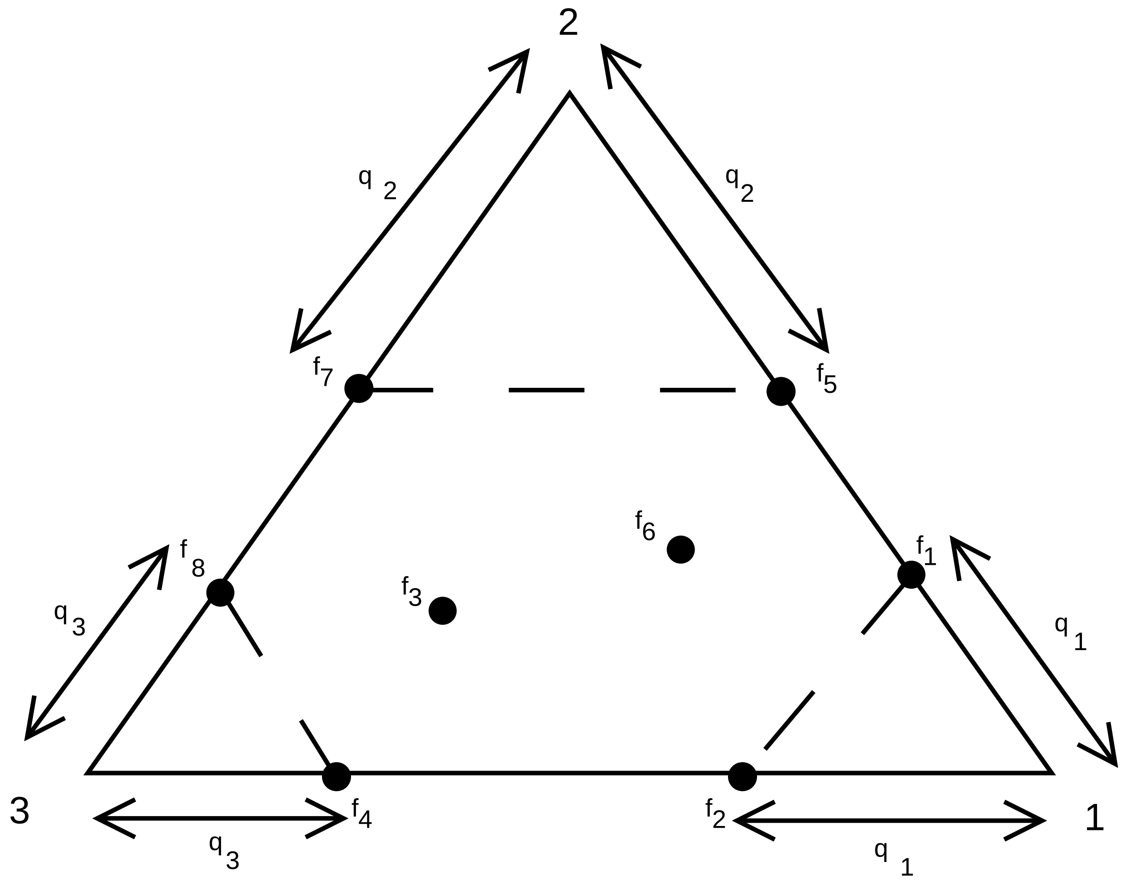

Figure 1. Feasible consumption bundles as a function of the budget set probabilities.