



**TURUN YLIOPISTO  
UNIVERSITY OF TURKU**

*TALOUSTIEDE  
DEPARTMENT OF ECONOMICS*

***RESEARCH REPORTS***

**On the Existence of Undominated  
Elements of Acyclic Relations**

**Hannu Salonen and  
Hannu Vartiainen**

**No 109**

**October 2004**



**TURUN YLIOPISTO  
UNIVERSITY OF TURKU**

*TALOUSTIEDE  
DEPARTMENT OF ECONOMICS*

***RESEARCH REPORTS***

ISSN 0786-6526

ISBN 951-29-2594-X

**On the Existence of Undominated  
Elements of Acyclic Relations**

**Hannu Salonen and  
Hannu Vartiainen**

**No 109**

**October 2004**

Department of Economics  
University of Turku  
FIN-20014 Turku  
FINLAND

# On the Existence of Undominated Elements of Acyclic Relations

Hannu Salonen  
Department of Economics  
University of Turku  
20014 Turku  
Finland  
e-mail: [hansal@utu.fi](mailto:hansal@utu.fi)

Hannu Vartiainen  
The Yrjö Jahnsson Foundation  
Ludviginkatu 3-5  
FIN – 00130 Helsinki  
e-mail: [hannu.vartiainen@yjs.fi](mailto:hannu.vartiainen@yjs.fi)

JEL Classification Number: D11.

Abstract: We study the existence of undominated elements of acyclic and irreflexive relations. A sufficient condition for the existence is given in the general case without any topological assumptions. Sufficient conditions are also given when the relation in question is defined on a compact Hausdorff space. We study the existence of fixed points of acyclic correspondences, the existence of stable sets, and the possibility of representing the relation by a real valued function.

## 1. Introduction

Existence of undominated (or *maximal*) members of acyclic relations has interested economists for long. These kind of relations appear e.g. in preference theory (for references, see Alcantud [1]; Bergstrom [2]; Campbell and Walker [3]; Walker [4]). The interpretation of being undominated in such applications means that there exists a choice for which there exist no strictly better choices. Since acyclicity seems quite natural in consumer choice theory for example, it is clear why such relations interest economists.

Acyclic relations have applications also in equilibrium theory and in the analysis of dynamic systems. Equilibrium existence results are always some kind of fixed point theorems. Existence of fixed points and existence of undominated members of a relation are closely related problems, so it is not surprising that results in these areas would be potentially useful to economists.

A condition for the existence of undominated members is given in the general case without any topological assumptions (Proposition 1). Such conditions are also given when the relation in question is defined on a compact Hausdorff space. We show e.g. (Proposition 3) that if a closed, acyclic and irreflexive relation on a compact Hausdorff space has in every uncountable closed subset  $Z$  members that are undominated in  $Z$ , then this holds actually for every nonempty subset  $A$ , not just for uncountable closed subsets. We have a result about the possibility of representing the relation by a real valued function, when each closed subset  $Z$  has a member that is undominated in  $Z$  (Proposition 4). Existence of fixed points of acyclic correspondences and existence of Nash equilibrium are also analyzed

(Propositions 7– 10). Finally, we have a result about the existence of stable sets (Proposition 11).

The assumptions about relations on topological spaces are formulated in such a way, that they seem fit to game theoretical applications. For example, given an element  $x$  we declare the set of elements that are dominated by  $x$  closed. In applications in preference theory, it may often be more natural to declare these sets open (see Alcantud [1]). Anyway, it is fruitful to analyze the same problem (existence of undominated members) from different perspectives and with different topological assumptions.

The paper is organized in the following way. In Section 2, we give examples of situations when undominated members do not exist. Examples are simple but reveal something essential about what goes wrong when undominated members do not exist. Notation is introduced in Section 3, and there we give an example of a theorem from the existing literature, to facilitate comparison with our results. The main results are presented in Section 4 and 5. Section 5 is more application oriented, including the fixed point theorems and the result about stable sets.

## 2. Examples

Let us give next two examples of acyclic relations in which undominated members do not exist.

**Example 1.** Let  $X$  be the boundary of the closed unit ball in the two-dimensional plane with center at the origin of the plane. Define a relation  $R$  on  $X$  such that  $xRy$ , if the distance along  $X$  from  $x$  to  $y$  is 1, when we

move from  $x$  to  $y$  clockwise. Since for all  $x$  there is an  $y$  such that  $xRy$ , no undominated members exist. Further  $R$  is acyclic: there are no points  $x_1, \dots, x_n$  such that  $x_i R x_{i+1}$ ,  $i = 1, \dots, n-1$ , and  $x_1 R x_n$ . [You can prove this easily by using the fact that  $X$  has length  $2\pi$ .] Clearly, the relation  $R$  is also irreflexive and closed. In fact, there is a homeomorphism  $f: X \rightarrow X$  such that  $R$  is the graph of  $f$ :  $y = f(x)$  iff  $xRy$ .

So acyclicity and closedness alone do not guarantee the existence of undominated members, even when the relation is “single valued”. But maybe the reason is that  $X$  in Example 1 was not convex? Consider the following example.

**Example 2.** Let  $X = [0, 1]$ . Let  $a \in (1/2, 1)$  be irrational. Let  $R$  be a relation on  $X$  such that  $xRy$ , if  $y = 1 - a + x$ , when  $x \leq a$ , and  $xRy$ , if  $y = x - a$ , when  $a \leq x$ . Graphically,  $R$  consists of two disjoint line segments, one above the diagonal and the other below. Then  $R$  is closed and irreflexive. Since for all  $x$  there is an  $y$  such  $xRy$ , no undominated members exist.  $R$  is also acyclic. To see this, suppose there is a cycle  $\{x_0, \dots, x_n\}$ :  $x_i R x_{i+1}$ ,  $i = 0, \dots, n-1$ , and  $x_0 = x_n$ . We may assume w.l.o.g. that the cycle is minimal in the sense that  $x_i \neq x_j$  when  $i, j < n$ . Clearly  $x_0 + \dots + x_{n-1} = x_1 + \dots + x_n =_{df} S$ . Let  $B$  be the subset of the indices  $i < n$  such that  $x_i < a$ ,  $M$  the subset of the indices  $i < n$  such that  $x_i = a$ , and  $T$  the subset of the indices  $i < n$  such that  $a < x_i$ . Both  $B$  and  $T$  must be nonempty, and denote their cardinalities by  $b$  and  $t$ .  $M$  is either empty or contains only one member. Suppose first  $M$  is empty. Then  $S = \sum_{i \in B} x_i + \sum_{i \in T} x_i = \sum_{i \in B} (x_i + 1 - a) + \sum_{i \in T} (x_i - a) = \sum_{i \in B} x_i + b(1 - a) + \sum_{i \in T} x_i - ta$ . Hence  $b = a(b + t)$ , which is impossible since  $t, b > 0$  and  $a$  is irrational. If  $M$

contains one element, say  $j$ , that is,  $x_j = a$ , then either  $x_{j+1} = 0$  or  $x_{j+1} = 1$ . We get that  $S = \sum_{i \in B} x_i + \sum_{i \in T} x_i + a = \sum_{i \in B} (x_i + 1 - a) + x_{j+1} + \sum_{i \in T} (x_i - a) = \sum_{i \in B} x_i + b(1 - a) + \sum_{i \in T} x_i - ta + x_{j+1}$ . Hence  $a(b + t) = b + x_{j+1}$ , a contradiction since  $x_{j+1} = 0$  or  $x_{j+1} = 1$ .

So convexity of  $X$  does not guarantee the existence of undominated members of  $R$ , although  $R$  is acyclic, irreflexive and closed. In fact, we show in Proposition 1, that if  $X$  is countable and Hausdorff compact, then there exist undominated members if  $R$  is closed, acyclic and irreflexive. As a countable set,  $X$  cannot be convex. It cannot even be connected since it is Hausdorff and countable.

Examples 1 and 2 have one common feature. Starting from any point  $x$  in  $X$ , we can construct infinite sequences  $\{x_n\}$ ,  $x_0 = x$ , such that  $x_n R x_{n+1}$  but  $x_i R x_{n+1}$  does *not* hold for any  $i < n$ . We call these kind of sequences *irreducible*. The existence of such irreducible sequences from any initial value  $x$  in fact precludes the possibility that there are undominated members. We show a partial converse in Proposition 1. Suppose there is some initial value  $x$  such that none of the sequences described above is irreducible. Then there are undominated members, if  $R$  is acyclic and irreflexive, and if every member is in relation to at most finitely many other members. In this result, no topological assumptions are needed.

### 3. Preliminaries

Let  $X$  be a nonempty set, and  $R$  a binary relation on  $X$ , so  $R$  is a subset of  $X \times X$ . The set  $X$  is called the *field* of  $R$ . We may denote  $(x, y) \in R$  by  $xRy$  as usual. A finite subset  $\{x_0, \dots, x_n\} \subset X$  such that  $x_i R x_{i+1}$  for  $i = 0, \dots, n-1$ , is called a *path in  $R$* . If it is clear what relation  $R$  is in question, we may simply say that  $\{x_0, \dots, x_n\}$  is a path.  $R$  is *acyclic*, if  $x_0 \neq x_n$  for every path  $\{x_0, \dots, x_n\} \subset X$  containing at least two different members.  $R$  is *irreflexive*, if  $xRx$  does not hold for any  $x$ .  $R$  is transitive, if  $xRy$  and  $yRz$  imply  $xRz$ , for all  $x, y, z \in X$ . The *transitive closure of a relation  $R$* , denoted by  $R^{Tr}$ , is defined by  $xR^{Tr}y$  iff there is path  $\{x_0, \dots, x_n\}$  such that  $x = x_0$  and  $y = x_n$ .

Given a relation on  $X$  and a nonempty subset  $Y$  of  $X$ , a member  $y \in Y$  is *undominated in  $Y$* , if there is no  $y' \in Y$  such that  $yRy'$  and  $y \neq y'$ . If  $y$  is undominated in  $X$ , we say simply that  $y$  is *undominated*. We may also say in this case that  $R$  *has* (resp. *has not*) undominated members.

An infinite subset  $\{x_n\} = \{x_0, \dots, x_n, \dots\} \subset X$  such that  $x_i R x_{i+1}$ ,  $x_i \neq x_{i+1}$ , for  $i = 0, 1, \dots$ , is called a *dominance sequence*. Note that if  $R$  is irreflexive, the requirements  $x_i \neq x_{i+1}$  are automatically satisfied. If  $R$  is irreflexive and acyclic, also  $x_i \neq x_n$  holds for all  $i$  and  $n$ ,  $i \neq n$ , for any dominance sequence  $\{x_n\}$ .

If  $R$  is a relation on a nonempty set  $X$ , and  $Y$  is a nonempty subset of  $X$ , define *the restriction of  $R$  to  $Y$*  by  $R_{|Y} = R \cap Y \times Y$ . The field of the relation  $R_{|Y}$  is  $Y$ . Then  $R_{|Y}$  has no undominated members, if and only if  $R$  has no undominated members in  $Y$ .

Given a nonempty  $Y \subset X$ , let  $RY = \{x \in X \mid xRy \text{ for some } y \in Y\}$ , and  $YR = \{x \in X \mid yRx \text{ for some } y \in Y\}$ .  $RY$  is called the *inverse image of  $Y$* , and

$YR$  is called *the image of  $Y$* .  $RX$  is the *domain* of  $R$ , and  $XR$  is the *range* of  $R$ . If  $R$  is irreflexive, it has undominated members if and only if the domain is not the whole  $X$ , and in this case  $X \setminus RX$  is the set of undominated members. So undominated members of an irreflexive  $R$  exist precisely when the domain of  $R$  is a proper subset of the field of  $R$ .

If  $X$  is a topological space,  $R$  is *closed* if  $R$  is a closed subset of the product space  $X \times X$  which is equipped with the product topology. The following is part of Theorem 4 by Alcantud (see Alcantud [1] for the whole theorem, and related earlier results).

**Theorem.** *Let  $R$  be an irreflexive and acyclic relation on  $X$ . There are undominated members, iff  $X$  has topology such that  $X$  is compact and  $R\{x\}$  is open for each  $x \in X$ .*

It is well-known that if  $X$  is a compact Hausdorff space and  $R$  is closed, then  $YR$  and  $RY$  are closed for any nonempty closed subset  $Y$  of  $X$ . In particular, this holds for singletons  $\{x\}$ . Further, the *correspondence*  $x \rightarrow \{x\}R$  defined on the domain of  $R$  is *upper semicontinuous* in this case. That is, the subset  $\{x \mid \{x\}R \subset O\}$  is open for any open  $O \subset X$ . Since the function  $f: X \times X \rightarrow X \times X$ ,  $f((x, y)) = (y, x)$  is a homeomorphism, the *inverse of  $R$* ,  $R^{-1} = \{(y, x) \mid (x, y) \in R\}$  is a closed if  $R$  is closed. Then the *correspondence*  $y \rightarrow \{y\}R^{-1} = R\{y\}$  is also upper semicontinuous.

#### 4. Existence of Undominated Members

In this section a relation  $R$  on  $X$  will always be acyclic and irreflexive, unless otherwise explicitly stated.

We will first give conditions for the existence of undominated members without topological assumptions.

Given a relation  $R$  on a set  $X$ , a dominance sequence  $\{x_n\}$  is called *reducible*, if there is  $m > 0$  such that  $x_i R x_{m+1}$  holds for some  $i < m$ . If a dominance sequence is not reducible, then it is *irreducible*. Since  $R$  is acyclic, irreducibility means that  $x_k R x_n$  if and only if  $n = k + 1$ .

**Proposition 1.** *Let  $X$  be a nonempty set and  $R$  a nonempty relation on  $X$  such that  $\{x\}R$  is finite for every  $x \in X$ . There exists undominated members in  $X$ , iff there exists  $x_0 \in X$  such that every dominance sequence starting from  $x_0$  is reducible.*

*Proof. Necessity.* Suppose  $x_0$  is undominated. Then the set of dominance sequences starting from  $x_0$  is empty, and therefore every dominance sequence starting from  $x_0$  is reducible.

*Sufficiency.* Suppose no undominated members exists, and let  $x_0$  be such that every dominance sequence starting from  $x_0$  is reducible. Let  $T$  be the tree with root  $x_0$  whose nodes are all paths  $\{x_0, \dots, x_n\}$  such that  $x_i R x_k$  if and only if  $k = i + 1$ , where  $i = 0, \dots, n - 1$ . If  $\{x_n\}$  is any dominance sequence starting from  $x_0$ , then by reducibility there exists a least index  $k > 0$  such that  $x_i R x_{k+1}$  for some  $i < k$ . Hence all initial segments  $\{x_0, \dots, x_m\}$  of  $\{x_n\}$  with

at most  $k + 1$  members are nodes of  $T$ . So if there exists a node with  $m$  members, then there exist nodes with  $k = 1, \dots, m - 1$  members.

We claim first that there exists a natural number  $M$  such that all nodes of  $T$  have at most  $M$  members. Suppose to the contrary that for every  $M$ , there exists a node with more than  $M$  members. Then for every  $M > 1$ , there exists a node with exactly  $M$  members. Now  $T$  is a finite splitting tree: every node has at most a finite number of successor nodes. This follows because  $\{x\}R$  is finite for every  $x \in X$ . Then it follows by the König's Lemma, that  $T$  has an infinite branch. That means that there is a dominance sequence  $\{x_n\}$  starting from  $x_0$  such that every initial segment  $\{x_0, \dots, x_m\}$  is a node of  $T$ . But this means  $\{x_n\}$  is irreducible, a contradiction.

So every node of  $T$  has at most  $M$  members. The root  $x_0$  has finitely many successors because  $\{x_0\}R$  is finite. The subtree  $T(x)$  of  $T$  whose root is  $x \in \{x_0\}R$ , is such that all its nodes have at most  $M - 1$  members. Then it follows by induction that  $T$  has only a finitely many nodes. Since all nodes have only finitely many members, it follows that the subset  $Y = \{x \in X \mid x \text{ is a member of some node of } T\}$  is finite. We show next that if  $\{x_0, \dots, x_m\}$  is any path starting from  $x_0$ , then every member of this path is a member of some node of  $T$  as well.

So let  $\{x_0, \dots, x_m\}$  be a path starting from  $x_0$ . Let  $\{x_0, \dots, x_k\}$  be the greatest initial segment of this path that is a node of  $T$ . That is,  $x_i R x_n$  if and only if  $n = i + 1$ , where  $i = 0, \dots, k - 1$ . Such a greatest initial segment clearly exists. Suppose  $k < m$ , so in particular, the path  $\{x_0, \dots, x_k, x_{k+1}\}$  is not a node of  $T$ . Let  $i$  be the least index such that  $x_i R x_{k+1}$ . Then the member  $x_{k+1}$  belongs to the node  $\{x_0, \dots, x_i, x_{k+1}\}$ . If the path  $\{x_0, \dots, x_i, x_{k+1}, x_{k+2}\}$  is not a node of  $T$ , then this happens only because there exists  $j \leq i$  such that  $x_j R x_{k+2}$ . So let  $j$  be the least

such index, and note that then  $x_{k+2}$  belongs to the node  $\{x_0, \dots, x_j, x_{k+2}\}$ . Continuing in this fashion we get that every member  $x$  of the path  $\{x_0, \dots, x_m\}$  belongs to some node of  $T$ . Since  $\{x_0, \dots, x_m\}$  was chosen arbitrarily, it follows that the subset  $Y = \{x \in X \mid x \text{ is a member of some node of } T\}$  is the set of all members of  $X$  that belong to some path  $\{x_0, \dots, x_m\}$  starting from  $x_0$ .

If  $y \in Y$  is dominated, then there exists  $y' \in Y$  such that  $yRy'$ . To see this, note that if  $yRx$ , then there exists a path starting from  $x_0$  that contains  $y$  and ends to  $x$ . Since  $Y$  is finite and  $R$  is acyclic, not all members of  $Y$  can be dominated, a contradiction with the initial assumption that there are no undominated members. Q.E.D.

Now we turn to the case where  $X$  is a topological space.

**Lemma 1.** *If  $R$  is a nonempty and closed relation on a compact Hausdorff space  $X$  with no undominated members, then there is a minimal nonempty closed  $Y \subset X$  such that every  $y \in Y$  is dominated in  $Y$ .*

*Proof.* Partially order by set inclusion the set  $\mathcal{C}$  of all nonempty closed subsets  $Z$  of  $X$ , such that all members of  $Z$  are dominated in  $Z$ .  $\mathcal{C}$  is nonempty, since by assumption  $X \in \mathcal{C}$ . Let  $\mathcal{T}$  be a maximal totally ordered subset of  $\mathcal{C}$ . Then  $\mathcal{T}$  exists by the Hausdorff Maximality Principle. Let  $Y$  be the intersection of the members of  $\mathcal{T}$ . Then  $Y$  is nonempty and closed, since every  $Z \in \mathcal{T}$  is nonempty and closed and  $X$  is a compact Hausdorff space. Since every  $z \in Z$  is dominated in  $Z$ , we have  $Z \subset RZ$  for all  $Z \in \mathcal{T}$ . Choose  $y \in Y$ . Since  $y$  is dominated in  $Z$ ,  $\{y\}R \cap Z$  is nonempty and closed for any

$Z \in T$ . Therefore  $\{y\}R \cap Y$  is nonempty and closed, and  $y$  is dominated in  $Y$ . Since  $y$  was chosen arbitrarily, we are done. Q.E.D.

Recall that a nonempty closed subset  $Y$  of a metric space is called *perfect*, if  $Y$  contains no points that are isolated in  $Y$ , *i.e.* there is no  $y \in Y$  such that for some open neighbourhood  $V(y)$  of  $y$ ,  $Y \cap V(y) = \{y\}$ . Perfect subsets are uncountable.

**Lemma 2.** *Let  $R$  be a nonempty and closed relation on a compact Hausdorff space  $X$  with no undominated members, and let  $Y \subset X$  be as in Lemma 1. Then  $Y$  is homeomorphic to a compact perfect metric space, and  $Y = R_{|Y}Y = YR_{|Y}$ .*

*Proof.* By Lemma 1,  $R_{|Y}$  has no undominated members, and therefore  $Y = R_{|Y}Y$ . Since  $R_{|Y}$  is viewed as a relation on  $Y$ , we have  $YR_{|Y} \subset Y$ . Let  $Z = YR_{|Y}$ , and note that  $Z$  consists of all those members of  $Y$  that dominate some member of  $Y$ . Since every  $z \in Z$  is dominated by some  $y \in Y$ , we must have  $y \in Z$ . Since  $Y$  is minimal and  $Z$  is closed, we have  $Z = YR_{|Y} = Y$  by Lemma 1.

Let  $\{x_n\} \subset Y$  be any dominance sequence, *i.e.* an infinite sequence such that  $x_i R x_{i+1}$ ,  $i = 0, 1, \dots$ . Such a sequence exists, since  $Y$  contains no undominated members. Since  $Y$  is closed,  $cl\{x_n\} \subset Y$ , where  $clZ$  means the closure of a set  $Z$ . Now  $\{x_n\} = cl\{x_n\}$  is impossible. To see this, suppose  $\{x_n\} = cl\{x_n\}$ , that is,  $\{x_n\}$  is closed. Then  $\{x_n\}_{n \geq 1} = \{x_n\} \cap \{x_n\}R$  would be closed as well, since  $AR$  is closed for any nonempty closed  $A \subset Y$ . By induction, for all natural numbers  $k$ , the dominance sequence  $\{x_n\}_{n \geq k}$  starting from  $x_k$  would be closed. Since  $\{x_n\}_{n \geq m} \subset \{x_n\}_{n \geq k}$  when  $m > k$ , the intersection  $\bigcap_k \{x_n\}_{n \geq k}$  must

be nonempty, because  $Y$  is a compact Hausdorff space. Then for some  $m$ ,  $x_m \in \{x_n\}_{n \geq k}$  for all  $k$ , a contradiction with acyclicity. Therefore  $\{x_n\}$  is a proper subset of  $cl\{x_n\}$ .

The set  $cl\{x_n\}$  is a compact Hausdorff space having a countable dense subset  $\{x_n\}$ . Urysohn's metrization theorem says that the topology of any normal topological space with a countable dense subset is metrizable. Compact Hausdorff spaces are normal (every pair of disjoint closed subsets have disjoint open neighbourhoods), and hence we may view  $cl\{x_n\}$  as a compact metric space.

Take any  $y \in cl\{x_n\} \setminus \{x_n\}$ . There must be a subsequence  $\{x_{n(k)}\}$  converging to  $y$ , since we may view  $cl\{x_n\}$  as a compact metric space. For the same reason the sequence  $\{x_{n(k)+1}\}$  has a subsequence converging to  $z \in cl\{x_n\}$ . Assume w.l.o.g. that  $\{x_{n(k)+1}\}$  converges. Since  $R$  is a closed relation, and  $x_{n(k)} R x_{n(k)+1}$ , it follows that  $y R z$ . Hence every member of  $cl\{x_n\}$  is dominated by some member of  $cl\{x_n\}$ . Since  $cl\{x_n\}$  is a subset of  $Y$ , we must have  $Y = cl\{x_n\}$  by minimality of  $Y$ . Recall that  $\{x_n\} \subset Y$  was an arbitrarily chosen dominance sequence.

If  $y \in Y$  were isolated, we could take a dominance sequence  $\{x_n\} \subset Y$  such that  $y R x_0$ . Then by acyclicity,  $y$  couldn't be a member of  $\{x_n\}$ . Hence  $y$  couldn't be a member of  $cl\{x_n\}$ , since  $y$  is isolated. But then  $y \notin Y$  since  $Y = cl\{x_n\}$ , a contradiction. Hence  $Y$  is perfect. Q.E.D.

**Proposition 2.** *Let  $X$  be a countable, compact Hausdorff space and  $R$  a nonempty and closed relation on  $X$ . Then every nonempty closed  $Z \subset X$  contains members that are undominated in  $Z$ .*

*Proof.* If there were no undominated members in  $X$ , then by Lemma 2 there would exist a nonempty subset  $Y \subset X$  such that  $Y$  is homeomorphic to a compact perfect metric space. But the cardinality of every nonempty perfect set is that of the continuum, a contradiction.

If  $Z \subset X$  is nonempty and closed, then  $R_{|Z}$  is a nonempty and closed relation on a countable, compact Hausdorff space  $Z$ , and the result follows from the first part of this proof. Q.E.D.

Next we drop the assumption that  $X$  is countable, and give some characterizations for the case that each nonempty closed subset  $Z$  of  $X$  has members that are undominated in  $Z$ .

**Proposition 3.** *Suppose  $R$  is a nonempty and closed relation on a compact Hausdorff space  $X$ . Each uncountable closed  $Z \subset X$  contains members that are undominated in  $Z$ , iff each nonempty  $A \subset X$  contains members that are undominated in  $A$ .*

*Proof. Sufficiency.* If each nonempty  $A \subset X$  contains members that are undominated in  $A$ , then this holds for closed nonempty subsets as well.

*Necessity.* The mapping  $Z \rightarrow RZ$  (take  $Z$  and see what members of  $X$  the members of  $Z$  dominate) is monotone on nonempty subsets of  $X$ . Namely, if  $\emptyset \neq Z \subset Z'$ , and  $xRz$  for  $z \in Z$ , then  $x \in RZ$ . But since  $z \in Z'$ , also  $x \in RZ'$ , and therefore  $RZ \subset RZ'$ .

Let  $X(0) = X$ , and define  $X(n+1) = RX(n)$ ,  $n \geq 0$ . Then  $X(n+1) \subset X(n)$  by monotonicity. Since each  $X(n)$  is a closed subset of a compact space  $X$ ,

the intersection  $Z = \bigcap_n X(n)$  is closed as well, and it is nonempty iff each  $X(n)$  is nonempty.

Suppose indeed that  $Z$  is nonempty. If  $Z$  is uncountable, there is by assumption  $y \in Z$  such that  $yRy'$  does not hold for any  $y' \in Z$ . If  $Z$  is countable, then also  $Z$  has a member  $y$  that is undominated in  $Z$ :  $R|_Z$  is a closed (irreflexive and acyclic) relation on  $Z$ , and therefore has undominated members by Proposition 2.

Since  $y \in X(1)$ ,  $y$  cannot be undominated in  $X$ . Hence  $\{y\}R$ , the set of members that dominate  $y$ , is nonempty and closed by closedness of  $R$ . Since  $y \in X(n+1)$  for every  $n$ , it follows that  $\{y\}R \cap X(n)$  is nonempty and closed for every  $n$ . But then  $\{y\}R \cap Z = \bigcap_n (\{y\}R \cap X(n))$  is a nonempty and closed subset of  $Z$ , a contradiction.

It follows that there exists a natural number  $n^*$  such that  $X(n^*+1)$  is empty but  $X(n)$  is nonempty for all  $n < n^*+1$ . That is,  $RX(n^*)$  is empty although  $X(n^*)$  is nonempty and closed, so members of  $X(n^*)$  do not dominate anything. In particular, no member of  $X(n^*)$  dominates another member of  $X(n^*)$ , and hence each  $x$  in  $X(n^*)$  is undominated in  $X(n^*)$ .

Let  $S(n) = X(n) \setminus X(n+1)$ ,  $n < n^*$ , and  $S(n^*) = X(n^*)$ . Then  $S(n)$  contains all members of  $X(n)$  that are undominated in  $X(n)$ . Subsets  $S(n)$  and  $S(m)$  are disjoint when  $n \neq m$ . The union of all  $S(n)$ 's is  $X$ , so  $\{S(n)\}$  is a partition of  $X$ .

Let  $A \subset X$  be nonempty. Let  $n$  be the least index such that  $A \cap S(n)$  is nonempty. Members of this nonempty intersection are undominated in  $A$ .  
Q.E.D.

We can give an analogous characterization in terms of functions  $u : X \rightarrow \mathbf{R}$  ( $\mathbf{R}$  is the set of real numbers) that preserve  $R$  “in one direction”:  $xRy$  implies  $u(x) < u(y)$ .

**Proposition 4.** *Suppose  $R$  is a nonempty and closed relation on a compact Hausdorff space  $X$ . Each uncountable closed  $Z \subset X$  contains members that are undominated in  $Z$ , iff there is a lower semicontinuous function  $u : X \rightarrow \mathbf{R}$  with finite range such that  $xRz$  implies  $u(x) < u(z)$  for all  $x, z \in X$ .*

*Proof. Necessity.* Suppose each uncountable closed  $Z \subset X$  contains members that are undominated in  $Z$ . Then this holds for every nonempty closed  $Z$  by Proposition 2. Let  $X(0) = X$ , and define  $X(n+1) = RX(n)$ ,  $n \geq 0$ . Then  $X(n+1) \subset X(n)$ , each  $X(n)$  is closed, and  $X(n)$  is nonempty iff  $n \leq n^*$  for some  $n^*$  (see the proof of Prop. 3).

Let  $S(n) = X(n) \setminus X(n+1)$ ,  $n < n^*$ , and  $S(n^*) = X(n^*)$ . Then  $S(n)$  contains all members of  $X(n)$  that are undominated in  $X(n)$ . Note that since each  $X(n+1)$  is closed,  $n < n^*$ , the union  $S(0) \cup \dots \cup S(n) = X \setminus X(n+1)$  is open, and that subsets  $S(n)$  and  $S(m)$  are disjoint when  $n \neq m$ . The union of all  $S(n)$ 's is  $X$ , so  $\{S(n)\}$  is a partition of  $X$ .

Define a function  $u$  on  $X$  by  $u(x) = n^* - n$ , where  $n$  is the unique number such that  $x \in S(n)$ . Now  $u$  represents  $R$  in the sense that  $xRz$  implies  $u(x) < u(z)$ . Namely,  $u(x) = k$  iff  $x \in S(n^* - k)$ . Since  $x$  is undominated in  $X(n^* - k)$ ,  $z \in S(n)$  implies  $n < n^* - k$ . Hence  $u(z) = n^* - n > k = u(x)$ . Fix  $a \in \mathbf{R}$ , and note that  $\{x \in X \mid a < u(x)\}$  is open, and so  $u$  is lower semicontinuous.

*Sufficiency.* Suppose there is a lower semicontinuous function  $u : X \rightarrow \mathbf{R}$  with finite range such that  $xRz$  implies  $u(x) < u(z)$  for all  $x, z \in X$ . We may assume w.l.o.g. that  $u[X] = \{0, \dots, n^*\}$ , the set of first  $n^* + 1$  natural numbers. Let  $T(n) = \{x \in X \mid n - 1 < u(x) \leq n\}$ , and note that unions  $T(k) \cup T(k + 1) \cup \dots \cup T(n^*)$  are open,  $k \leq n^*$ , and that  $\{T(n) \mid 0 \leq n \leq n^*\}$  is a partition of  $X$ .

Let  $Z$  be a nonempty closed subset of  $X$ , and let  $k$  be the largest number such that  $Z \cap T(k)$  is nonempty. For each  $x \in Z \cap T(k)$ ,  $u(x) = k$ , and if  $xRy$  then  $u(x) < u(y)$ . But then  $y \notin Z$ . So members of  $Z \cap T(k)$  are undominated in  $Z$ . Q.E.D.

We will now formulate some sufficient conditions for the existence of undominated elements. In the first, we use the reducibility concept.

Let  $R$  a relation on  $X$ . We say that dominance sequences starting from  $x \in X$  are *uniformly reducible*, if there is  $M > 0$  such that to each dominance sequence  $\{x_n\}$ ,  $x_0 = x$ , there is  $n \leq M$  such that  $x_i R x_{n+i}$  for some  $i < n$ .

**Proposition 5.** *Suppose  $R$  is a nonempty and closed relation on a compact Hausdorff space  $X$ . If every uncountable closed subset  $Z \subset X$  contains a member  $z$  such that dominance sequences starting from  $z$  are uniformly reducible, then  $R$  has undominated members.*

*Proof.* Suppose  $R$  has no undominated members. Then there is a minimal closed subset  $Y$  by Lemma 1 such that every  $y \in Y$  is dominated in  $Y$ . By Lemma 2,  $Y$  is homeomorphic to a compact perfect metric space, and so  $Y$  is uncountable.

Define  $\{z\}R^n$  by  $(\{z\}R^{n-1})R$ , when  $n > 0$  and  $\{z\}R^0 = \{z\}$ ,  $z \in X$ . Then each  $\{z\}R^n$  is closed, and so are the finite unions of these sets. By assumption, there is  $y_0 \in Y$  such that dominance sequences starting from  $y_0$  are uniformly reducible.

*Claim.*  $\bigcup_n \{y_0\}R^n$  is closed.

*Proof.* Since the dominance sequences starting from  $y_0$  are uniformly reducible, there is  $M > 0$  such that for every dominance sequence  $\{y_n\}$  that starts from  $y_0$ , there is  $n \leq M$  such that  $y_i R x_{n+1}$  and  $i < n$ .

Let  $\{x_i\} \subset \bigcup_n \{y_0\}R^n$  be a sequence converging to  $x$ , not necessarily a dominance sequence. Since  $x_i \in \bigcup_n \{y_0\}R^n$ , there exists a shortest path  $\{y_0, \dots, y_k, x_i\}$ . Since  $X$  contains no undominated members, this path extends to a dominance sequence  $\{y_n\}$  starting from  $y_0$  such that  $y_{k+1} = x_i$ . Since  $\{y_0, \dots, y_k, x_i\}$  is a shortest path from  $y_0$  to  $x_i = y_{k+1}$ , we have that  $y_i R y_j$  if and only if  $j = i + 1$ , when  $j \leq k + 1$ . By uniform reducibility,  $k + 1 \leq M$ .

Therefore  $\{x_i\} \subset \{y_0\}R \cup \dots \cup \{y_0\}R^M$ , which is a closed set, and therefore the limit  $x$  of  $\{x_i\}$  is in this set also, and so  $x \in \bigcup_n \{y_0\}R^n$ . *End.*

Now  $y_0$  is such that  $y_0$  does *not* dominate any  $y \in \bigcup_n \{y_0\}R^n$ . By Lemma 2 there is  $y' \in Y$  such that  $y_0$  dominates  $y'$ , so  $\bigcup_n \{y_0\}R^n \cap Y$  is a proper closed subset of  $Y$ . By Lemma 1, there is  $z \in \bigcup_n \{y_0\}R^n \cap Y$  such that  $z$  is undominated in  $\bigcup_n \{y_0\}R^n \cap Y$ . Since  $z \in Y$ ,  $z R y$  holds for some  $y \in Y$ , and then necessarily  $y \notin \bigcup_n \{y_0\}R^n$ . But  $y \in \{z\}R \subset \bigcup_n \{y_0\}R^n$  by construction, a contradiction. Q.E.D.

If  $X$  is a topological space, we say that  $A \subset X$  is *sequentially closed*, if for any sequence  $\{a_n\} \subset A$  converging to  $a$ , also  $a \in A$ . Every closed subset is sequentially closed. In metric spaces, sequentially closed sets are closed.

Recall the definition of the transitive closure  $R^{Tr}$  of a relation  $R$  on  $X$  (see Section 3). Since  $R$  is acyclic and irreflexive,  $\{x\}R^{Tr} = \bigcup_n \{x\}R^n \setminus \{x\}$  for all  $x \in X$ , where  $\{x\}R^0 = \{x\}$  and  $\{x\}R^{n+1} = (\{x\}R^n)R$  for  $n \geq 0$ .

**Proposition 6.** *Suppose  $R$  is a nonempty and closed relation on a compact Hausdorff space  $X$ . If every uncountable closed subset  $Z$  of  $X$  contains some  $z$  such that  $\{z\}R^{Tr}$  is sequentially closed, then  $R$  has undominated members.*

*Proof.* Suppose there are no undominated members. Then there is a minimal closed set  $Y$  by Lemma 1 such that every  $y \in Y$  is dominated in  $Y$ . By Lemma 2,  $Y$  is homeomorphic to a compact perfect metric space, and so  $Y$  is uncountable.

By assumption,  $\{y\}R^{Tr}$  is sequentially closed for some  $y \in Y$ . Since  $Y$  can be viewed as a metric space,  $Y \cap \{y\}R^{Tr}$  is closed. Since every member of  $Y$  is dominated in  $Y$ ,  $Y \cap \{y\}R^{Tr}$  is nonempty. Since  $y$  is not a member of  $\{y\}R^{Tr}$ ,  $Y \cap \{y\}R^{Tr}$  is a proper closed subset of  $Y$ . By Lemma 1, there is  $z \in Y \cap \{y\}R^{Tr}$  such that  $z$  is undominated in  $Y \cap \{y\}R^{Tr}$ . Since  $z \in Y$ ,  $zRx$  holds for some  $x \in Y$ , and then necessarily  $x \notin Y \cap \{y\}R^{Tr}$ . But  $x \in \{z\}R \subset \{y\}R^{Tr}$  by construction, a contradiction. Q.E.D.

**Corollary 1.** *Suppose  $R$  is a nonempty and closed relation on a compact Hausdorff space  $X$ .  $R$  satisfies the conditions of Prop. 5 or Prop. 6, iff for each uncountable closed  $Z \subset X$ , there is  $z \in Z$  which is undominated in  $Z$ .*

*Proof. Necessity.* If  $Z$  is countable, then by Prop. 1  $Z$  contains a member  $z$  such that  $z$  is undominated in  $Z$ . If  $Z$  is uncountable, then apply Prop. 5 and 6 to the relation  $R_{\downarrow Z}$  on  $Z$ .

*Sufficiency.* By Prop. 4, there is a function  $u : X \rightarrow \mathbf{R}$  such that  $xRz$  implies  $u(x) < u(z)$  for all  $x, z$ . If  $\{x_n\} \subset X$  is a dominance sequence, then the image  $u[\{x_n\}]$  is infinite. But  $u[X]$  is finite by Prop. 4, so there are no dominance sequences. So the sufficient conditions of Prop. 5 and 6 are automatically satisfied. Q.E.D.

## 5. Fixed Points Theorems and Applications

Let  $F: X \rightrightarrows X$  be a correspondence: for each  $x \in X$ ,  $F(x) \subset X$ . A correspondence  $F$  is nonempty valued (finite valued), if  $F(x)$  is nonempty (finite) for all  $x$ . A correspondence  $F$  is acyclic (irreflexive), if its graph  $grF = \{(x, y) \mid y \in F(x), x \in X\}$  is acyclic (irreflexive). We call a correspondence *reducible*, if for some  $x_0$ , all dominance sequences in  $grF$  starting from  $x_0$  are reducible. In other words, if there exists an infinite sequence  $\{x_n\}$  with first member  $x_0$  such that  $x_{i+1} \in F(x_i)$ ,  $i = 0, 1, \dots$ , then for some  $n > 0$ , there exists  $i < n$  such that  $x_{n+1} \in F(x_i)$ . If  $X$  is a topological space, then a correspondence  $F$  is closed, if its graph  $grF$  is closed in the product topology of  $X \times X$ .

We say that  $x \in X$  is a fixed point of  $F: X \rightrightarrows X$ , if  $x \in F(x)$ .

**Proposition 7.** *If  $X$  is a nonempty set, and  $F: X \rightrightarrows X$  is a nonempty and finite valued, acyclic and reducible correspondence, then  $F$  has a fixed point.*

*Proof.* If  $F$  has no fixed points, then  $F$  is irreflexive. By Proposition 1, the relation  $grF$  has then undominated members. That is, for some  $x \in X$ , there is no  $y \in X$  such that  $(x, y) \in grF$ . Since  $F$  is nonempty valued, this is impossible. Q.E.D.

**Proposition 8.** *Suppose  $X$  is countable, compact Hausdorff space. If the correspondence  $F : X \Rightarrow X$  is closed, nonempty valued and acyclic correspondence, then  $F$  has a fixed point.*

*Proof.* Apply Proposition 2 and the proof of Proposition 7. Q.E.D.

Given a correspondence  $F : X \Rightarrow X$ , let  $F^0(x) = \{x\}$ , and  $F^n(x) = F[F^{n-1}(x)]$  for  $n > 0$ , for every  $x \in X$ .

**Proposition 9.** *Suppose  $X$  is a compact Hausdorff space, and the correspondence  $F : X \Rightarrow X$  is closed, nonempty valued and acyclic correspondence. If every uncountable closed subset  $Z$  of  $X$  contains a member  $z$  such that  $\bigcup_{n>0} F^n(z)$  is sequentially closed, then  $F$  has a fixed point.*

*Proof.* Note that  $z \notin \bigcup_{n>0} F^n(z)$  since  $F$  is acyclic. Apply Proposition 6 and the proof of Proposition 7. Q.E.D.

An  $n$ -person normal form game is  $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$ , where  $S_i$  is a nonempty set of strategies of player  $i = 1, \dots, n$ , and  $u_i$  is a real valued utility function of player  $i = 1, \dots, n$  defined on  $S = S_1 \times \dots \times S_n$  the set of strategy

profiles. Given a strategy profile  $s$  and player  $i$ , denote  $s = (s_{-i}, s_i)$ , where  $s_{-i}$  denotes the strategies chosen by  $i$ 's opponents, and  $s_i$  denotes the strategy chosen by  $i$ . A strategy profile  $s^*$  is a *Nash equilibrium*, if the inequality

$$u_i(s^*) \geq u_i((s_{-i}, s_{-i}^*))$$

holds for all  $i$  and all  $s_i \in S_i$ . Let  $B_i(s) = \{x_i \mid u_i((x_i, s_{-i})) \geq u_i((y_i, s_{-i}))\}$ , for all  $y_i \in S_i$  denote the set of *best replies* of player  $i$  against a strategy profile  $s \in S$ . For all strategy profiles  $s$ , let  $B(s) = B_1(s) \times \dots \times B_n(s)$ , and let  $B : S \Rightarrow S$  denote the correspondence that to each  $s$  assigns the set  $B(s)$ . Then  $s^*$  is a Nash equilibrium, if and only if  $s^* \in B(s^*)$ . We say that  $B$  is the best reply correspondence of the game  $G$ .

**Proposition 10.** *Suppose  $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$  is an  $n$ -person normal form game having a best reply correspondence  $B : S \Rightarrow S$ . If  $B$  is a nonempty and finite valued, acyclic and reducible correspondence, then  $G$  has a Nash equilibrium.*

*Proof.* By Proposition 7.

Q.E.D.

A Nash equilibrium existence result based on Propositions 8 and 9 can be formulated analogously.

In game theoretic applications, dominance sequences are usually called best reply sequences. Acyclicity of a best reply correspondence  $B$  means that there cannot exist cycles in any best reply sequence (or path). Reducibility of  $B$  means that there exists a strategy profile  $s$  such that if  $\{s^k\}$  is a best reply sequence starting from  $s$ , then for some  $k$ ,  $s^{k+1}$  is a best reply not just against

$s^k$  but also against some other  $s^i$ ,  $i < k$ . Finite valuedness of  $B$  means simply that each player has only a finitely many best replies against every strategy profile.

As a last application, we show that stable sets exist, if a relation has undominated members in every nonempty closed subset.

Given a relation  $R$  on  $X$ , a nonempty  $S \subset X$  is *stable*, if (i)  $x, y \in S$  implies that neither  $xRy$  nor  $yRx$  holds; (ii)  $x \in X \setminus S$  implies that  $xRy$  for some  $y \in S$ .

**Proposition 11.** *Suppose  $R$  is a nonempty, closed, irreflexive and acyclic relation on a compact Hausdorff space  $X$ , and assume that each nonempty closed  $Y \subset X$  contains a member  $y$  such that  $y$  is undominated in  $Y$ . Then a stable set exists.*

*Proof.* Let the subsets  $S(n), X(n), n = 0, \dots, n^*$ , be defined as in the proof of Prop. 3. So  $X(0) = X$ , and  $X(n+1) = RX(n) \subset X(n)$ , and  $S(n) = X(n) \setminus X(n+1)$ . It was shown in the proof of Prop. 3, that every member of  $X(n^*)$  is undominated in  $X(n^*)$  and  $X(n^*+1)$  is empty.

If  $n^* = 0$ , then  $S(0) = X$  is a stable set. If  $n^* > 0$ , then  $S(1)$  is nonempty. Since members of  $S(1)$  are undominated in  $X(1) = X \setminus S(0)$ , we have that  $S(1) \subset RS(0)$ . Let  $k$  be the least index such that  $S(k)$  is not a subset of  $S(0) \cup RS(0)$ . Let  $T(1) = S(0) \cup (S(k) \setminus RS(0))$ . Then  $T(1)$  and  $RT(1)$  are disjoint. If their union is  $X$ , then  $T(1)$  is a stable set.

Suppose that  $T(k)$  is defined such that  $T(k)$  and  $RT(k)$  are disjoint. If their union is  $X$  then  $T(k)$  a stable set. If their union is not  $X$ , then define  $T(k+1)$  by  $T(k+1) = T(k) \cup (S(m) \setminus RT(k))$ , where  $m$  is the least index such

that  $S(m)$  is not a subset of  $T(k) \cup RT(k)$ . Since there are only  $n^* + 1$  subsets  $S(n)$ , there must exist  $k$  such that  $T(k)$  is a stable set. Q.E.D.

## References

1. Alcantud, J. C. R.: Characterization of the existence of maximal elements of acyclic relations. *Economic Theory* **19**, 407-416 (2002)
2. Bergstrom, T. C.: Maximal elements of acyclic relations on compact sets. *Journal of Economic Theory* **10**, 403-404 (1975)
3. Campbell, D. E., Walker, M.: Maximal elements of weakly continuous relations. *Journal of Economic Theory* **50**, 459-464 (1990)
4. Walker, M.: On the existence of maximal elements. *Journal of Economic Theory* **16**, 470-474 (1977)

**Department of Economics**  
**University of Turku**  
**FIN-20014 TURKU**  
**FINLAND**  
<http://aws.tt.utu.fi>

tel 02-333 5399,  
02-333 5400  
fax 02-333 5893

## **RESEARCH REPORTS**

1. Hannu Salonen: Monopoly power and price expectations. June, 1989
2. Luis Alvarez Jr: The effects of uncertain parameter expectations on the optimal investment of the firm. Part 1: Known timing size. September, 1989
3. Hannu Salonen: Bertrand equilibrium and price competition. September, 1989
4. Luis Alvarez Jr: The effects of uncertain parameter expectations on the optimal investment of the firm. Part 2: Unknown shift size. January, 1990
5. Luis Alvarez Jr: The almost certain asymptotical stability of Walrasian and Marshallian stochastic market - adjustment processes. March, 1990
6. Hannu Salonen: An axiomatization of Nash equilibrium. June, 1990
7. Erkki Koskela and Matti Virén: Capital Market Imperfections, Taxation and Household Saving. May, 1991
8. Matti Virén: Financial Innovations and Currency Demand: Some New Evidence. July, 1991
9. Erkki Koskela, Heikki Loikkanen and Matti Virén: Housing Markets, Taxation and Financial Liberalization in Finland. August, 1991
10. Luis Alvarez Jr: Yrityksen odotukset ja optimaalinen investointi. September, 1991
11. Hannu Koiranen and Jussi Kojamo: Some Experiments with Bayesian Estimators. November, 1991
12. Luis Alvarez Jr: Timing uncertainty and optimal control. November, 1991
13. Christian Starck and Matti Virén: Forecasting the Behaviour of Bankruptcies, December, 1991
14. Petri Lehto: Kilpailun karttaminen ja ylisuuret voitot oligopolistisilla markkinoilla. December, 1991
15. Erkki Koskela and Matti Virén: An economic model of auto thefts in Finland. February, 1992
16. Matti Virén: Some Long-Run Trends in Finnish Financial Variables. March 1992
17. Erkki Koskela and Matti Virén: Household saving and uncertainty: Evidence from Finland. May, 1992
18. Matti Virén: Omaisuusrikollisuus taloudellisena ongelmana. July, 1992
19. Jaakko Pehkonen and Matti Virén: Measuring Changes in Age-Income Profiles over Time: Evidence from Finnish Panel Data. August, 1992
20. Erkki Koskela and Matti Virén: Is There a Laffer Curve between Government Size and Private Output: Some Evidence from a Market Price Approach. August, 1992
21. Hannu Koiranen: Estimation of the Phillips Curve: Finnish Quarterly Data, 1922 - 1991. November, 1992
22. Luis Alvarez Jr: Investment under price and wage uncertainty: A generalization. May, 1993
23. Luis Alvarez Jr: Constant returns to scale, output price uncertainty, and the optimal demand for inputs. June, 1993
24. Camilla Harismaa, Erkki Koskela and Matti Virén: Taxation and Household Saving in the Nordic Countries. July, 1993
25. Matti Virén: A Note on Interest Rate Policy During the Great Depression. August, 1993

26. Erkki Kivistö: Taloustieteen mikroteoria II. September, 1993. Opintomoniste.
27. Hannu Salonen: Continuity Properties of Bargaining Solutions. September, 1993
28. Markku Stenborg: Effort Incentives in Oligopoly. September, 1993
29. Luis Alvarez Jr: Irreversibility, uncertainty, and the optimal investment of a neoclassical firm. September, 1993
30. Jaakko Pehkonen and Matti Virén: Age Profiles of Earnings and Income: Evidence from Finnish Panel Data. October, 1993
31. Matti Virén: A Test of an Economics of Crime Model. October, 1993
32. Luis Alvarez Jr: Irreversibility, uncertainty, and decreasing returns to scale. October, 1993
33. Luis Alvarez Jr: Irreversibility and Myopic Rules of Investment: Uncertainty vs. Certainty. November, 1993
34. Markku Stenborg: Folk Theorem for One-shot Agency Games. November, 1993
35. Mika Karjalainen: Velkanneutraliteettiteoreema - empiirinen tutkimus budjettialijäämien vaikutuksesta kulutukseen. Joulukuu, 1993
36. Markku Stenborg: Untruthful Equilibrium in Multi-Organization Mechanism. December, 1993
37. Petri Lehto: Maintenance Problems of Collusive Behavior: The Case of Fluctuating Demand. December, 1993
38. Erkki Koskela and Matti Virén: Taxation, House Prices and Household Saving in the Nordic Countries. December, 1993
39. Markku Stenborg: Debt and Equity as Optimal Financial Mechanism. February, 1994
40. Kari Takala and Matti Virén: Chaos and Nonlinear Dynamics: Evidence from Finland. February, 1994
41. Matti Virén: A Test of an Economics of Crime Model. Some Further Evidence. February, 1994
42. Markku Stenborg: Contracts and Commitments in Games: A Review Article. May, 1994
43. Matti Virén: Onko yhteiskunta vastuussa rikollisuuden kasvusta? Toukokuu, 1994
44. Kari Takala and Matti Virén: Macroeconomic Effects of Bankruptcies. May, 1994
45. Matti Virén: Verotuksen ja sosiaalietujen vaikutus kotitalouksien käytettävissä oleviin tuloihin. Lokakuu 1994
46. Esko Vuorela: Vuodenaikojen mukaan määräytyvät nopeusrajoitukset. Suomessa tehdyn kokeilun tulosten tarkastelua. Marraskuu 1994
47. Erkki Kivistö: Ekonometrian perusteet. Joulukuu 1994. Opintomoniste
48. Petri Lehto: Collusive Behavior and Multimarket Contact with Differing Demands. December 1994
49. Matti Virén: Omaisuusrikollisuus Suomessa. Tammikuu 1995
50. Luis Alvarez Jr: American options, infinite expiration, and waiting: A non-standard approach. Toukokuu 1995
51. Luis Alvarez Jr, Vesa Kanninen and Jan Södersten: Theory of Tax-Induced Investment Spurts. Toukokuu 1995
52. Luis Alvarez Jr, Vesa Kanninen and Jan Södersten: Tax Reform Uncertainty, Investment Spurts and Capital Markets. Toukokuu 1995
53. Matti Virén: Verotus, tulonsiirrot ja köyhyysloukut 1995. Toukokuu 1995
54. Luis Alvarez Jr: Demand Uncertainty and Production Opportunities: The Effects of Excess Supply on Production. Toukokuu 1995
55. Kari Takala and Matti Virén: Testing Nonlinear Dynamics, Long Memory and Chaotic Behaviour with Macroeconomic Data. June 1995
56. Kari Takala and Matti Virén: Bankruptcies, Indebtedness and the Credit Crunch. June 1995
57. Hannu Salonen: Strategic Complements, Equilibrium and Information. October 1995

58. Juha Niemelä: Odotusteoria ja korkojen aikasarjaominaisuudet. November 1995
59. Hannu Salonen: Revealing Non-Verifiable Information in a Cournot Duopoly. November 1995
60. Jarkko Heinonen: Cointegration Analysis of Finnish Prices of Beef and Pork. December 1995
61. Erkki Koskela and Matti Virén: An Occupational Choice Model of Crime Switching. December 1995
62. Esko Vuorela: Talvinopeusrajoitukset ja liikenneonnettomuudet. Joulukuu 1995
63. Klaus Kultti and Hannu Salonen: The Role of Anonymity in Quality Control Problems. February 1996
64. Seppo Heikkilä and Hannu Salonen: Iterated dominance, equilibrium, and rationalizability in ordered games. May 1996
65. Matti Virén: Modelling Crime and Punishment. July 1996
66. Seppo Heikkilä and Hannu Salonen: On the Existence of Extremal Stationary Distributions of Markov Processes. September 1996
67. Pekka Tsupari ja Matti Virén: Työllistämiskynnyksen hinta. Marraskuu 1996
68. Matti Virén: Testing the “natural rate of suicide” hypothesis. December 1996
69. Petri Kangas: Ajassa vaihtelevan riskipreemion estimointi korkorakenteesta ARCH-M -mallin avulla. Huhtikuu 1997
70. Juha Niemelä: Analysing the longrun relationship between the nominal interest rate and expected inflation. Toukokuu 1997
71. Sami Niemelä: Kiintopistealgoritmit ja yleisen tasapainon numeerinen ratkaiseminen. Syyskuu 1997
72. Haruo Imai and Hannu Salonen: The Representative Nash Solution for two-sided Bargaining Problems. December 1997
73. Jukka Virtanen: The Criminal Life-Cycle: An Argument for Selective Deterrence. December 1997
74. Hannu Salonen and Hannu Nurmi: A Note on Rough Sets and Common Knowledge Events December 1997
75. Hannu Salonen: Bayesian Predictions and the Correlated Equilibrium. September 1998
76. Jussi Tolvi: Outliers in Time Series: A Review. October 1998
77. Esko Vuorela: Aksiomaattinen nopeustutkimus. Joulukuu 1998
78. Jukka Virtanen: Private and Public Deterrence of Predatory Crime. December 1998
79. Esko Vuorela: Onnettomuusstabiliteetti. Joulukuu 1998
80. Eija Pohjansaari: Understanding Bargaining Games: Experimental Evidence and Future Approaches. December 1998
81. Jussi Tolvi: A nonlinear moving average test as a robust test for ARCH. May 1999
82. Hannu Salonen: Beliefs, Filters and Measurability. August 1999
83. Sami Niemelä: Tuotannon mallintaminen yleisen tasapainon malleissa. Marraskuu 1999
84. Jussi Tolvi: Small sample properties of a joint ARCH-bilinearity test. December 1999
85. Hannu Salonen: Markov Perfect Equilibria in Large Stochastic Games. March 2000
86. Hannu Salonen: Approximations of Equilibria and the Reduction relation. June 2000
87. Kari Friman: Ehdollinen siirtymämatriisi yritysten luottokelpoisuuden muutosten indikaattorina. Marraskuu 2000
88. Hannu Salonen: Take away. January 2001
89. Matti Virén: Hirvikolarit: onnettomuuksia vai kuolemantuottamuksia? Helmikuu 2001

90. Jussi Tolvi: Nonlinear model selection in the presence of outliers. March 2001
91. Jouni Jokinen ja Matti Virén: Eduskunnan menot 1945-2001. Syyskuu 2001
92. Saija Kaukonen ja Matti Virén: Varkausrikosten rangaistukset 1992-1999. Syyskuu 2001
93. Matti Virén: Testing the predictions of the Obstfeld - Rogoff model on spillover effects of consumption shocks. September 2001
94. Hannu Salonen: On completeness of knowledge models. November 2001
95. Jussi Tolvi: Suomalaisten makrotaloudellisten aikasarjojen stationaarisuus ja pitkän muistin ominaisuudet. Tammikuu 2002
96. Hannu Salonen: Common Theories. March 2002
97. Hannu Salonen: Nash Equilibria in Normal Form Games. April 2002
98. Matti Virén: Rahat vanhoille, velat nuorille. Varallisuuskajakauman muutosten tarkastelua. Kesäkuu 2002
99. Hannu Salonen: On Completeness of Contracts. September 2002
100. Sami Niemelä: Yleisen tasapainon numeeristen mallien soveltaminen. Lokakuu 2002
101. Pekka Aarnisalo, Klaus Kultti and Hannu Salonen: Insurance contracts: an experimental study. October 2002
102. Jouni Jokinen and Matti Virén: A Review of the Organisations of the Ministries of Finance. December 2002
103. Jussi Tolvi: Long memory in the Finnish stock market. December 2002
104. Jorma Lehtimäki ja Seppo Reinamo: Toiminnot yliopiston kustannuslaskennassa. Maaliskuu 2003
105. Hannu Salonen and Kari Saukkonen: On Continuity of Arrowian Social Welfare Functions. November 2003
106. Klaus Kultti and Hannu Salonen: Market for votes. March 2004
107. Matti Virén: Kuka arvonlisän todella maksaa? Elokuu 2004
108. Päivi Kankaanranta: Suomalaisten kotitalouksien sijoituskäyttäytyminen. Elokuu 2004
109. Hannu Salonen and Hannu Vartiainen: On the Existence of Undominated Elements of Acyclic Relations. October 2004.