COOPERATIVE INVESTMENT GAMES
or
POPULATION GAMES

YARON AZRIELI AND EHUD LEHRER

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Abstract. The model of a cooperative fuzzy game is interpreted as both a population game and a cooperative investment game. Three types of core-like solutions induced by these interpretations are introduced and investigated. The interpretation of a game as a population game allows us to define sub-games. We show that, unlike the well-known Shapley-Shubik theorem on market games [13], there might be a population game such that each of its sub-games has a non-empty core and, nevertheless, it is not a market game. It turns out that, in order to be a market game, a population game needs to be also homogeneous. We also discuss some special classes of population games such as convex games, exact games, homogeneous games and additive games.

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel.
e-mails: azrieliy@post.tau.ac.il and lehrer@post.tau.ac.il
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1. Introduction

The theory of transferable utility cooperative games usually deals with the problem of finding solutions to scenarios where every subset (coalition) of a finite set of players has a given value. Thus, if a coalition is identified with its characteristic vector, a (TU) cooperative game is a real-valued function defined on the vertices of the unit cube. Aubin [1], [2] suggested to expand the set of possible coalitions to the entire cube and thereby enable the players to choose any level of participation in a coalition, not just whether to participate or not. As a consequence, the domain of the game is extended to the entire unit cube. These kind of games are called *cooperative fuzzy games.*

As in the theory of classic cooperative games, solution concepts for cooperative fuzzy games took two main directions. Aubin [1], [2] defined the *core* of a fuzzy game to be the set of all vectors $x = (x^1, \ldots, x^n) \in I\!\!R^n$ that satisfy $\sum_{i=1}^n x^i = v(1, \ldots, 1)$ and $\sum_{i=1}^n x^i c^i \geq v(c^1, \ldots, c^n)$ for all fuzzy coalition $(c^1, \ldots, c^n)$. The logic behind the definition is that, if x^i is understood as the per-unit reward for agent i, then a core element x is resistant to the claims of any (fuzzy) coalition. Other papers discussing cores of fuzzy games include Butnariu [7] where a different definition of the core appears, and Branzei et al. [6] where cores of convex fuzzy games are studied.

Another direction taken in the analysis of fuzzy games is that of a value. Several attempts have been made to axiomatically derive a value function for fuzzy games, similar to the Shapley value in discrete cooperative games [10]. Papers in this direction include Aubin [1], [2], Butnariu [7], Billera and Heath [4] and Tsurumi et al. [14].

In this paper we consider a model similar to that of fuzzy games. Its interpretation, however, is rather different. Let $N = \{1, 2, \ldots, n\}$ be a finite set representing the types of agents in a large population. There is a continuum of agents of each type and $Q^i \geq 0$ is the size of type i ($i = 1, \ldots, n$) agents. The entire population is, therefore, represented by a positive vector $Q = (Q^1, \ldots, Q^n)$, and possible coalitions are identified with the vectors that are (coordinate-wise) smaller than Q: $F(Q) = \{c = (c^1, \ldots, c^n) \in I\!\!R^n;\ 0 \leq c^i \leq Q^i,\ i = 1, \ldots, n\}$. If c^i represents the amount of agents of type i ($i = 1, \ldots, n$) that participate in a coalition, then the total worth of $c = (c^1, \ldots, c^n)$ is given by the real number $v(c)$.

A population game is, therefore, a pair (v, Q), where Q is a vector all of whose coordinates are non-negative, and v is a real valued function defined on the cube $F(Q)$. Such a model may also be interpreted differently. Assume that there are n interacting agents, and that, for every $i = 1, \ldots, n$, the amount of resources available for agent i is $Q^i \geq 0$ (this can be time, money, etc.). Each agent can choose to invest any fraction of his resources in a joint project. If c^i $(0 \leq c^i \leq Q^i)$ is the amount of resources that agent i invests, then the total value of the project is $v(c^1, \ldots, c^n)$.

While the definition of the core proposed by Aubin [1], [2] is plausible in the context of population games, it is not when the model is understood as an investment game. Indeed, assume that in an investment game (v, Q), the allocation x is blocked by the investment $c = (c^1, \ldots, c^n)$. That is, $\sum_{i=1}^n x^i c^i < v(c)$. In this case, each agent i is left with $Q^i - c^i$ uninvested dollars. It might be that, no matter how the agents invest this remainder, the total worth of their investment is smaller than that guaranteed by x. Thus, from a comprehensive perspective on the entire investment x is a stable allocation.

The above argument implies that, in the context of investment games, a core allocation should take into account the total value of investing the entire resources, and not only a fraction of it, possibly in several projects. For that reason we introduce the *comprehensive core* of an investment game. An allocation x is in the comprehensive core of the game (v, Q) if no subset of agents can do better than what is obtained by x, no matter how they choose to invest their resources.

According to the traditional definition, a core element x should be feasible in the sense that $\sum_{i=1}^n x^i Q^i$ equals the worth of the grand coalition $v(Q)$. When interpreting the model as a population game, the grand coalition is comprised of a large number of agents. It is therefore natural to consider situations where splitting the entire population into a few smaller groups is more efficient than forming the grand coalition. This leads us to define another variant of the core, the *split core*. The split core is the set of stable allocations which are feasible when the population splits optimally into several sub-populations.

As mentioned above, the majority of the existing literature interprets the coordinates of a 'fuzzy coalition' as the participation levels of the players in the coalition. Therefore, it was usually assumed that Q is the vector which equals 1 in all of its coordinates. However, under the interpretation of a population game, it is natural to consider other vectors Q. This allows us to define the *subgames* of a given population game in an obvious way: The subgame induced by a sub-population d

is defined by the same v restricted to the subset $F(d)$ that consists of all coalitions that are smaller than or equal to d.

We introduce market games in the context of a large population. Suppose that each agent in the economy has an initial endowment of production factors and a production function. Furthermore, suppose that there are finitely many types in the economy, where a type is characterized by the endowment and the production function the agent is endowed with. This situation naturally induces a population game, called a market game: the worth of a coalition is the maximal product achievable by reallocating the initial resources among the agents of the coalition. It turns out the the core of any subgame of a market game is non-empty. However, unlike Shapley and Shubik [13], the inverse is generally incorrect. There are population games such that each of their subgames has a non-empty core, while they are not market games.

A population game is homogeneous if multiplying the size of a coalition by a positive factor would multiply its worth by the same factor. We show that a population game is a market game if and only if the core of each of its subgames is not empty and moreover, it is homogeneous.

Some special classes of population games are also discussed. An interesting class of cooperative games is that of convex games (Shapley, [12]). The definition of convex games was extended to fuzzy games by Branzey et al. [6]. They proved that, as in the discrete case, convex fuzzy games have a non-empty core. We explore more properties of convex population games, notably that every bounded convex population game is continuous in the interior of the domain (convexity of the game does not imply convexity of the function v, so this is not a trivial result).

Another interesting class is that of exact games (Schmeidler, [9]). The definition of exactness is adapted to population games, and a characterization of exact population games is provided. It is also shown that every subgame of a population game is exact if and only if the game is linear. This is in contrast with discrete cooperative games where exactness of every subgame is equivalent to convexity of the original game.

After formally presenting the model and introducing some basic tools in Section 2, we discuss the various core concepts in Section 3. The (standard) core, the split core and the comprehensive core are formally defined, and we characterize non-emptiness of each one of them. The characterization makes use of several operators defined on population games. These operators can be seen as analogous to the *totally balanced cover* operator in (classical) cooperative games (see Azrieli and Lehrer, [3]). Market games are discussed in Section 4.

In Sections 5 we deal with convex games. In Section 6 we discuss exact games. Section 7 refers to the simple case where there are no externalities. Thus, the worth of a coalition is the sum of the contributions of its maximally homogeneous (i.e., consisting of agents of one type) sub-coalitions. The simple structure of additive games allows us to simplify some of the results of the previous sections. Final remarks and open problems are given in Section 8.

2. The model and preliminary results

2.1. **Basic notation and definitions.**

Let $N = \{1, 2, \ldots, n\}$ be a finite set. In the context of population games, N will be the set of different *types* in the population. When we discuss investment games, N will be interpreted as the set of *agents* participating in the game. For any vector with non-negative coordinates[1] $Q = (Q^1, \ldots, Q^n) \in I\!R^n$, let $F(Q) = \{c \in I\!R^n;\ 0 \leq c \leq Q\}$.

When the model is interpreted as a cooperative investment game, Q^i will be the amount of resources available for player i. Each player can choose to invest any part of his resources. Thus, $F(Q)$ is the set of all possible *joint investments* (or the *investment set*). On the other hand, if N represents the set of types in a given population, then Q^i is some measurement of the number of players of type i. A *coalition* is, therefore, a vector $c = (c^1, \ldots, c^n)$ in $F(Q)$ where c^i is the number of agents of type i that participate in the coalition. Notice that $F(Q)$ is a compact and convex polyhedral set.

For every subset $S \subseteq N$, we denote by Q_S the vector which coincides with Q on the members of S and equals 0 out of S. That is $Q^i_S = Q^i$ if $i \in S$ and $Q^i_S = 0$ otherwise.

Definition 1. *A* cooperative investment game/population game *is a pair* (v, Q) *such that:*

1. $Q \in \mathbb{R}^n$ *and* $Q \geq 0$*;*
2. $v : F(Q) \to \mathbb{R}$ *is a real valued function defined on* $F(Q)$ *such that* $v(0) = 0$*.*

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[1] For two vectors s, t in $I\!R^n$, $s \leq t$ (resp. $s << t$) means that $s^i \leq t^i$ (resp. $s^i < t^i$) for $i = 1, \ldots, n$.

We will use both terms, population games and investment games, to refer to such a pair (v, Q).

For any vector $c \in I\!R^n$, we denote by $|c|$ the l_1 norm of c, that is $|c| = \sum_{i=1}^n |c^i|$. Since we only deal with vectors all of whose coordinates are non-negative, we have that $|c + d| = |c| + |d|$ for any $c, d \in F(Q)$. The $n-1$-dimensional simplex in $I\!R^n$ is denoted by Δ. That is, $\Delta = \{(q^1, \ldots, q^n) \in I\!R^n;\ \sum_{i=1}^n q^i = 1,\ q^i \geq 0,\ i = 1, \ldots, n\}$.

2.2. **Some operators over population games.**
Let $Q \in I\!R^n$, $Q \geq 0$, be fixed throughout this subsection. We will make use of the following definition:

Definition 2. *(i) A function $g : F(Q) \to I\!R$ is* Super Additive (SA) *if, for every $c, d \in F(Q)$ such that $c + d \in F(Q)$, $v(c) + v(d) \leq v(c + d)$.*
(ii) A function $g : F(Q) \to I\!R$ is called Strongly Super Additive (SSA) *if, for every $c \in F(Q)$ the equation $\sum_{j=1}^L \lambda_j c_j = c$ where $L \in \mathbb{N}$, $\lambda_j \geq 0$ and $c_j \in F(c)$, $j = 1, \ldots, L$ implies that $g(c) \geq \sum_{j=1}^L \lambda_j g(c_j)$.*

We next define several operators over the class of bounded population games that will turn out to be useful in the upcoming sections.

Definition 3. *Let (v, Q) be a population game and let $d \in F(Q)$. Define*

$$\begin{aligned} \mathbf{Sa}v(d) &= \sup\Big\{\sum_{j=1}^L v(c_j);\ L \in \mathbb{N},\ \sum_{j=1}^L c_j = d,\ c_j \in F(d),\ j = 1, ..., L\Big\}. \\ \boldsymbol{SSa}v(d) &= \sup\Big\{\sum_{j=1}^L \lambda_j v(c_j);\ L \in \mathbb{N},\ \sum_{j=1}^L \lambda_j c_j = d,\ \lambda_j \geq 0,\ c_j \in F(d),\ j = 1, ..., L\Big\}. \\ \mathbf{Da}v(d) &= \sup\Big\{\sum_{S \subseteq N} \lambda_S v(d_S);\ \sum_{S \subseteq N} \lambda_S d_S = d,\ \lambda_S \geq 0,\ S \subseteq N\Big\}. \end{aligned}$$

$\mathbf{Sa}v$ is the *super additive cover* of v. $\mathbf{SSa}v$ is the *strong super additive cover* of v. $\mathbf{Da}v$ is the *discrete (strong) super additive cover* of v (recall that d_S is the vector with $d_S^i = d^i$ if $i \in S$ and $d_S^i = 0$ otherwise). Some properties of these operators appear in Lemmas 8 and 9 in the Appendix.
Any population game induces a function on Δ in the following way.

Definition 4. *Let (v, Q) be a bounded population game. Define $u_{v,Q} : \Delta \to I\!R \cup \{+\infty\}$ by $u_{v,Q}(q) = \sup\{\frac{v(c)}{|c|};\ c \in F(Q),\ q = \frac{c}{|c|}\}$.*

Example 1.

Let $n = 2$, $Q = (2, 2)$, and for any $c \in F(Q)$ let $v(c) = (c^1 + c^2)^2$. Fix $q = (q^1, 1 - q^1) \in \Delta$. In order to compute $u_{v,Q}(q)$, we need to maximize $\frac{(c^1+c^2)^2}{c^1+c^2} = c^1 + c^2$ with the constrains that $c \in F(Q)$ and $\frac{c^1}{c^1+c^2} = q^1$. A simple computation yields $u_{v,Q}(q) = \frac{2}{1-\min\{q^1, 1-q^1\}}$. ■

Definition 5. *Assume that $u_{v,Q}$ is bounded. The* concavification *of $u_{v,Q}$, denoted $\mathbf{Cav}u_{v,Q}$, is defined as the minimum of all concave functions $g : \Delta \to \mathbb{R}$ such that $g(q) \geq u_{v,Q}(q)$ for every $q \in \Delta$.*

$\mathbf{Cav}u_{v,Q}$ is a concave function as a minimum of concave functions. Moreover, if $u_{v,Q}$ is continuous, then since Δ is a convex polygon, $\mathbf{Cav}u_{v,Q}$ is also continuous (see Laraki, [8]).

Lemma 1. *Assume that $u_{v,Q}$ is bounded. Then for every $q \in \Delta$,*

$$\mathbf{Cav}u_{v,Q}(q) = \max\Bigg\{ \sum_{i=1}^{n+1} \alpha_i u_{v,Q}(q_i); \quad (i) \quad q = \sum_{i=1}^{n+1} \alpha_i q_i; \quad (ii) \quad \alpha_i \geq 0,\ i = 1, ..., n+1; \quad (iii) \quad \sum_{i=1}^{n+1} \alpha_i = 1; \text{ and } \quad (iv) \quad q_i \in \Delta,\ i = 1, ..., n+1 \Bigg\}.$$

The proof is postponed to the Appendix.

3. Three types of core-like solutions

In this section we give various definitions for the stability of an allocation, all of them inspired by the core concept in classic cooperative games. The first and the second definitions seem more natural when the model is interpreted as a population game, while the third can be justified when the model is understood as a cooperative investment game.

The set of allocations satisfying the first definition is simply called *the core*. This is the standard definition that was used in the theory of cooperative fuzzy games, and it is discussed in subsection 3.1. However, we claim that in the context of population games the set of stable allocations might be larger than the core. The reason is that, when the population is large, it seems natural to consider what can be achieved by splitting into several smaller 'communities' rather than the worth of the entire population. We therefore define the *split core* of a population game to be the set of stable allocations of the worth of some partition of Q and not necessarily of the worth of Q itself. The split core is discussed in subsection 3.2.

When dealing with cooperative investment games, the two previous definitions are hard to justify. In subsection 3.3 we define the *comprehensive core* of a game to be the set of allocations of $v(Q)$ which are immune to deviations of any subset of players $S \subseteq N$, no matter how the players in S invest their resources Q_S.

3.1. **The core.**

Definition 6. *The* core *of the game* (v, Q)*, denoted* $\text{core}(v, Q)$*, is the set of vectors* $x = (x^1, ..., x^n)$ *such that*[2]

1. $xQ = v(Q)$*; and*
2. $xd \geq v(d)$ *for any coalition* $d \in F(Q)$.

Theorem 1. *The following statements are equivalent:*

1. $\text{core}(v, Q)$ *is non-empty.*
2. $u_{v,Q}$ *is bounded from above and* $\boldsymbol{SSa}v(Q) = v(Q)$
3. $u_{v,Q}$ *is bounded from above and* $\textbf{Cav}u_{v,Q}(\frac{Q}{|Q|}) = \frac{v(Q)}{|Q|}$.

Proof. (1) $\Rightarrow$ (2) Let $x \in \text{core}(v, Q)$ and assume that the equation $\sum_{j=1}^{L} \lambda_j c_j = Q$ holds, where $c_j \in F(Q)$, and $\lambda_j \geq 0$, $j = 1, ..., L$. Then $v(Q) = xQ = \sum_{j=1}^{L} \lambda_j x c_j \geq \sum_{j=1}^{L} \lambda_j v(c_j)$, and therefore $v(Q) = \textbf{SSa}v(Q)$. Also, since for every $c \in F(Q)$ $\frac{v(c)}{|c|} \leq \frac{xc}{|c|}$, it follows that $u_{v,Q}$ is bounded from above.

(2) $\Rightarrow$ (3) This is a consequence of Lemma 1. Indeed, since $v(Q) = \textbf{SSa}v(Q)$, it follows that if $\sum_{j=1}^{L} \alpha_j \frac{c_j}{|c_j|} = \frac{Q}{|Q|}$, where $c_j \in F(Q)$, $\alpha_j \geq 0$, $j = 1, ..., L$ and $\sum_{j=1}^{L} \alpha_j = 1$, then $\sum_{j=1}^{L} \alpha_j \frac{v(c_j)}{|c_j|} \leq \frac{v(Q)}{|Q|}$. This implies that $\sum_{i=1}^{n+1} \alpha_i u_{v,Q}(q_i) \leq \frac{v(Q)}{|Q|}$ whenever $\sum_{i=1}^{n+1} \alpha_i q_i = \frac{Q}{|Q|}$ and $\sum_{i=1}^{n+1} \alpha_i = 1$, $\alpha_i \geq 0$, $q_i \in \Delta$, $i = 1, ..., n+1$. By Lemma 1 we get that $\textbf{Cav}u_{v,Q}(\frac{Q}{|Q|}) = \frac{v(Q)}{|Q|}$.

(3) $\Rightarrow$ (1) $\textbf{Cav}u_{v,Q}$ is concave over Δ. Let $x \in I\!R^n$ be a supporting hyperplane for $\textbf{Cav}u_{v,Q}$ at the point $\frac{Q}{|Q|}$. Then $x\frac{Q}{|Q|} = \textbf{Cav}u_{v,Q}(\frac{Q}{|Q|}) =$

[2]for two vectors $z, w \in I\!R^n$, zw denotes the inner product $zw = \sum_{i=1}^{n} z^i w^i$.

$\frac{v(Q)}{|Q|}$, so $xQ = v(Q)$. Also, for every coalition d,

$$xd = |d|x\frac{d}{|d|} \geq |d|\mathbf{Cav}u_{v,Q}\left(\frac{d}{|d|}\right) \geq |d|u_{v,Q}\left(\frac{d}{|d|}\right) \geq |d|\frac{v(d)}{|d|} = v(d)$$

Therefore, $x \in \text{core}(v, Q)$. ∎

The definition of the core is due to Aubin [1], [2]. However, his characterization of non-emptiness of the core is only for the case where the function v is homogeneous on $F(Q)$. This is the case where $v(q) = |q|u_{v,Q}(\frac{q}{|q|})$ for every coalition $q \neq 0$.

3.2. **The split core.**

A core allocation x needs to satisfy two conditions. The first is that x should be feasible: $\sum_{i=1}^n x^i Q^i$ is equal to the worth of the grand coalition $v(Q)$. In a population game the grand coalition consists of a large number of small agents. It might occur that splitting the entire population into a few smaller groups is more efficient than forming the grand coalition. In the split core, we are about to define, feasibility of an allocation is meant in a broader sense than in the core. An allocation is feasible if there is a partition of the grand coalition optimally into several sub-coalitions $c_1, \ldots, c_L$, such that $\sum_{i=1}^n x^i Q^i$ equals the sum of values of these sub-coalitions, $\sum_{i=1}^L v(c_i)$. Formally,

Definition 7. *The* split core *of a population game* (v, Q)*, denoted* $\text{S}-\text{core}(v, Q)$*, consists of all vectors* $x \in I\!R^n$*, such that there exist* $L \in \mathbb{N}$ *and coalitions* $c_1, \ldots, c_L \subseteq F(Q)$ *that satisfy*

1. $\sum_{i=1}^L c_i = Q$;
2. $xQ = \sum_{i=1}^L v(c_i)$; *and*
3. *For any coalition* $d \in F(Q)$, $xd \geq v(d)$.

Here, the split core allocation x is resistent to any blocking coalition, as in the core. However, it might be that $xQ > v(Q)$, meaning that x is not available if the grand coalition is formed. When the entire population splits into the coalitions $c_1, \ldots, c_L$, the total worth is increased and the allocation x becomes available. Notice that if $x \in \text{S}-\text{core}(v, Q)$ and $xQ = \sum_{i=1}^L v(c_i)$ with $\sum_{i=1}^L c_i = Q$, then for any $i = 1, \ldots, L$, $xc_i = v(c_i)$. This means that x is available for each of the coalitions c_i separately.

By definition, we have that $\text{core}(v, Q) \subseteq \text{S}-\text{core}(v, Q)$. Therefore, Theorem 1 provides sufficient conditions for non-emptiness of the split core. The following theorem characterizes games with non-empty split core.

Theorem 2. *Let* (v, Q) *be a population game. Then* $\mathrm{S}-\mathrm{core}(v, Q)$ *is non-empty if and only if* $\boldsymbol{SSa}v(Q) = \boldsymbol{Sa}v(Q)$.

Proof. Consider the auxiliary population game (v', Q) defined by $v'(c) = v(c)$ if $c \neq Q$ and $v'(Q) = \mathbf{Sa}v(Q)$. By the definition of the split core we have that $\mathrm{S}-\mathrm{core}(v, Q) = \mathrm{core}(v', Q)$. By Theorem 1, $\mathrm{core}(v', Q)$ is not empty if and only if $v'(Q) = \mathbf{SSa}v'(Q)$. However, $v'(Q) = \mathbf{Sa}v(Q)$ by definition, and since v and v' coincide on $F(Q) \setminus \{Q\}$ it is obvious that $\mathbf{SSa}v(Q) = \mathbf{SSa}v'(Q)$. Therefore, $v'(Q) = \mathbf{SSa}v'(Q)$ is equivalent to $\mathbf{Sa}v(Q) = \mathbf{SSa}v(Q)$ hence the theorem. ∎

In the following example the game (v, Q) has an empty core but a non-empty split core.

Example 2.

Let $n = 2$, $Q = (1, 1)$ and $v(c) = v(c^1, c^2) = \frac{(c^1)^2}{c^1+c^2}$. Notice that $v(q) = |q| u_{v,Q}(\frac{q}{|q|})$ for every $q \neq 0$. Therefore, $u_{v,Q}$ is just the restriction of v to Δ, that is $u_{v,Q}(q) = (q^1)^2$ for any $q = (q^1, q^2) \in \Delta$. Since $u_{v,Q}$ is convex on Δ it follows that $\mathbf{Cav}u_{v,Q}((1/2, 1/2)) > u_{v,Q}((1/2, 1/2)) = \frac{v(Q)}{|Q|}$ and therefore, by Theorem 1, (v, Q) has an empty core. On the other hand, $\mathbf{SSa}v(Q) = \mathbf{Sa}v(Q)$, so by Theorem 2 the split core of (v, Q) is not empty. ∎

3.3. **The comprehensive core.**

When interpreted as an investment game, the cores discussed in the previous subsections are difficult to justify. An allocation x might not be in the core if there is a blocking investment d such that $v(d) > xd$. This means that a group of players can form an investment d that does better than the share guaranteed by x. However, there is no reference to what remains after investing d. It might be that the yield of the remainder is so low that the total yield is less than the share guaranteed by x.

Recall that, for every $S \subseteq N$, Q_S is the n-dimensional vector which coincides with Q on the coordinates that belong to S, and is equal to zero otherwise. If a coalition S is not satisfied with the allocation x, it means that it has a comprehensive investment (rather than a partial one, as suggested by the core) of its entire resources, Q_S, that yields a higher payoff than xQ_S. By a comprehensive investment we mean investments $c_i \in F(Q_S)$, $i = 1, ..., L$, that satisfy $\sum_{i=1}^{L} c_i = Q_S$. The coalition S of players has a justified claim against x using such a comprehensive investment if $xQ_S < \sum_{i=1}^{L} v(c_i)$. The comprehensive core consists of all those allocations against which there is no justified claim using a comprehensive investment. Formally,

Definition 8. *The* comprehensive core *of a cooperative investment game (v, Q), denoted* $\mathrm{C} - \mathrm{core}(v, Q)$, *is the set of vectors $x = (x^1, ..., x^n)$ such that*

(1) $xQ = v(Q)$; *and*
(2) $xQ_S \geq \sum_{i=1}^{L} v(c_i)$ *for any coalition $S \subseteq N$ and for any investment $c_i \in F(Q_S)$, $i = 1, ..., L$, that satisfy $\sum_{i=1}^{L} c_i = Q_S$.*

It is not hard to see that $\mathrm{core}(v, Q) \subseteq \mathrm{C} - \mathrm{core}(v, Q)$. The following example shows that $\mathrm{core}(v, Q)$ might be empty while $\mathrm{C} - \mathrm{core}(v, Q)$ not.

Example 3.

Let $n = 2,\ Q = (1, 1),\ v(Q) = 2,\ v((1, 1/2)) = 3$, $v((0, t)) = -2t$ for $0 \leq t \leq 1/2$ and $v(c) = 0$ otherwise. We have that $Q = (1, 1/2) + \frac{1}{2}(0, 1)$ but $v(Q) = 2 < 3 = v((1, 1/2)) + \frac{1}{2}v((0, 1))$, so by Theorem 1 the core of (v, Q) is empty. On the other hand, $x = (1, 1)$ is in $\mathrm{C} - \mathrm{core}(v, Q)$. When players 1 and 2 consider forming the mixed coalition $(1, 1/2)$, player 2 is left with an excess amount of $1/2$. Any way he might cuts this amount into pieces yields -1. Thus, the net value is 2, which is what is given to this coalition by x.■

Theorem 3. *The comprehensive core of an investment game (v, Q) is not empty if and only if $v(Q) = \mathbf{DaSa}\mathrm{v}(Q)$.*

Proof. Assume first that $\mathrm{C} - \mathrm{core}(v, Q) \neq \emptyset$ and let $x \in \mathrm{C} - \mathrm{core}(v, Q)$. Assume that $\sum_{S \subseteq N} \lambda_S Q_S = Q$ where $\lambda_S \geq 0$ $S \subseteq N$, and for every $S \subseteq N$, let $c_S^i \in F(Q_S),\ i = 1, \ldots, L_S$, be such that $\sum_{i=1}^{L_S} c_S^i = Q_S$. Then $v(Q) = xQ = \sum_{S \subseteq N} \lambda_S x Q_S \geq \sum_{S \subseteq N} \lambda_S \sum_{i=1}^{L_S} v(c_S^i)$. This implies that $v(Q) = \mathbf{DaSa}v(Q)$.

Conversely, assume that $v(Q) = \mathbf{DaSa}v(Q)$ holds. We define the auxiliary (classic) cooperative game (N, v_N) by $v_N(S) = \mathbf{Sa}v(Q_S)$ for any $S \subseteq N$. We will show that (N, v_N) has a non-empty core. Indeed, by the Shapley-Bondareva theorem (Bondareva [5] or Shapley [11]) it is sufficient to check that if $\sum_{S \subseteq N} \lambda_S \mathbb{I}_S = \mathbb{I}_N$ where $\lambda_S \geq 0$ $S \subseteq N$, then

$\sum_{S\subseteq N}\lambda_S v_N(S)\le v_N(N)$. Notice that $\sum_{S\subseteq N}\lambda_S\mathbb{I}_S=\mathbb{I}_N$ is equivalent to $\sum_{S\subseteq N}\lambda_S Q_S=Q$. By the assumption, if $\sum_{S\subseteq N}\lambda_S Q_S=Q$ then $\sum_{S\subseteq N}\lambda_S v_N(S)=\sum_{S\subseteq N}\lambda_S\mathbf{Sa}v(Q_S)\le v(Q)\le\mathbf{Sa}v(Q)=v_N(N)$. Therefore, the core of (N,v_N) is not empty. Let x be an element in the core of (N,v_N), and for every $i=1,\ldots,n$, define $y^i=\frac{x^i}{Q^i}$. Then,

$$\begin{aligned} yQ &= \sum_{i=1}^{n}\frac{x^i}{Q^i}Q^i=\sum_{i=1}^{n}x^i=v_N(N)=\mathbf{Sa}v(Q)=v(Q);\ \textit{and}\\ yQ_S &= \sum_{i\in S}x^i\ge v_N(S)=\mathbf{Sa}v(Q_S),\ \forall S\subseteq N. \end{aligned}$$

It follows that $y\in \mathrm{C}-\mathrm{core}(v,Q)$. ■

One may wonder weather the condition $\mathbf{DaSa}v(Q)=v(Q)$ in the above Theorem 3 can be replaced by the condition $\mathbf{Da}v(Q)=\mathbf{Sa}v(Q)=v(Q)$. The following example shows that this is not the case.

Example 4.

Let $n=3$, $Q=(1,1,1)$, and define $v(Q)=3$, $v((1,1,0))=v((1,0,1))=v((0,1,1))=2$, $v((1/2,1/2,0))=1+\varepsilon$ and $v(c)=0$ otherwise. Then it is easy to check that $\mathbf{Da}v(Q)=\mathbf{Sa}v(Q)=3=v(Q)$. However, $\mathbf{DaSa}v(Q)=3+\varepsilon$. By Theorem 3, $\mathrm{C}-\mathrm{core}(v,Q)=\phi$.■

4. Market games

4.1. Subgames.

Suppose that the coalition $d\in F(Q)$ is formed. Within d the worth of each coalition is still determined by the same v. Formally, d induces a new population game, called a subgame.

Definition 9. *Let (v,Q) be a population game and fix some $d\in F(Q)$. The* subgame *of (v,Q) with respect to d is (v_d,d), where for every $c\in F(d)$, $v_d(c)=v(c)$.*

Proposition 1. *Let (v,Q) be a population game. Then,*

1. *For any coalition $d\in F(Q)$, the core of the subgame (v_d,d) is not empty iff $\boldsymbol{SSa}v(d)=v(d)$.*
2. *The core of every subgame of (v,Q) is not empty iff v is SSA on $F(Q)$.*

The proof is postponed to the Appendix.

Proposition 2. *Let (v,Q) be a population game. Then,*

(1) *For any coalition $d \in F(Q)$, the split core of the subgame (v_d, d) is not empty iff $\boldsymbol{SSa}v(d) = \boldsymbol{Sa}v(d)$.*
(2) *The split core of every subgame of (v, Q) is not empty iff $\boldsymbol{Sa}v$ is SSA on $F(Q)$.*

The proof is postponed to the Appendix.

Proposition 3. *Let (v, Q) be an investment game. Then,*

(1) *For any investment $d \in F(Q)$, the comprehensive core of the subgame (v_d, d) is not empty iff $\boldsymbol{DaSa}v(d) = v(d)$.*
(2) *The comprehensive core of every subgame of (v, Q) is not empty iff $\boldsymbol{DaSa}v = v$ on $F(Q)$.*

Proof. This is a consequence of Theorem 3. ∎

4.2. **Market games.**
Consider a situation where there are many firms in the economy. A firm is characterized by an initial endowment and a production function. There are n types of firms. The initial endowment of firms of type i $(i = 1, \ldots, n)$ is a bundle of production factors $w_i \in I\!R^{\ell}_{+}$. Firms can produce one type of good from the ℓ production factors. Firms of type i produce by a production function $u_i : I\!R^{\ell}_{+} \to I\!R$ that satisfies: (i) $u_i(0) = 0$; and (ii) there is a constant M_i such that for every c either $u_i(c) \leq M_i$ or $u_i(c) \leq |c|M_i$. Nothing like monotonicity or concavity is required.

Suppose that there is a continuum of firms of each type, and that $Q^i > 0$ is some measurement of the number of firms of type i $(i = 1, \ldots, n)$. A coalition that consists of c^i type-i firms is denoted by $c = (c^1, \ldots, c^n)$. The entire endowment of c is $w_c = \sum_{i=1}^{n} c^i w_i$.

In order to maximize production this endowment is split among the firms. That is, w_c is split into bundles $y^i_k \in I\!R^l_+$ $(1 \leq i \leq n,\ 1 \leq k \leq K_i)$, where $\sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k y^i_k = w_c$ and $\sum_{k=1}^{K_i} \gamma^i_k = c^i$ for every $i = 1, \ldots, n$. This means that type-i firms are split into K_i groups, such that group k is of size γ^i_k, and each firm in group k receives the bundle y^i_k. All together, the firms may produce $\sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma^i_k u_i(y^i_k)$.

We define a game in which the value of a coalition c is the maximum achievable production.

Definition 10. *The* market game *induced by $Q = (Q^1, \ldots, Q^n)$ and $\{w_i, u_i\}_{i=1}^{n}$ is the game (v, Q) where for every $c \in F(Q)$,*

$$v(c) = \sup \Big\{ \sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma_k^i u_i(y_k^i);\ \sum_{i=1}^{n} \sum_{k=1}^{K_i} \gamma_k^i y_k^i \leq w_c,\ \sum_{k=1}^{K_i} \gamma_k^i = c^i,\ i = 1, \ldots, n \Big\}.$$

(The obvious constraints $\gamma_k^i \geq 0$ and $y_k^i \in I\!R_+^\ell$ are omitted.)

Remark 1. *(1) For every $c \in F(Q)$, $v(c)$ is finite. Indeed, if $\sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i y_k^i \leq w_c$ and $\sum_{k=1}^{K_i} \gamma_k^i = c^i$, $i = 1, \ldots, n$, then $\sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i u_i(y_k^i) \leq \sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i (M_i + |y_k^i| M_i) \leq (|c| + |w_c|) M_i$. Thus, $v(c) \leq (|c| + |w_c|) M_i$.*
(2) If u_i is continuous and concave, there is a constant M_i such that for every c either $u_i(c) \leq M_i$ or $u_i(c) \leq |c| M_i$. Indeed, when u_i is continuous, there is M_i^1 such that $u_i(c) \leq M_i^1$ when $|c| \leq 1$. Let M_i^2 be the maximum of u_i over Δ. Then, by concavity, if $|c| \geq 1$, $u_i(c) \leq |c| M_i^2 - u_i(0)$. Set, $M_i = \max(M_i^1, M_i^2 + |u_i(0)|)$.

Note that, in the definition of a market game (see Definition 10), $\sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i y_k^i$ is required to be less than or equal to, but not necessarily equal to w_c. Thus, we allow for free disposal of excessive quantity. It is therefore clear that $v(0) = 0$ and that v is monotonically non-decreasing.

Lemma 2. *Every market game has a non-empty core.*

Proof. Let (v, Q) be the market game induced by Q and $\{w_i, u_i\}_{i=1}^n$. By Theorem 1 it is sufficient to show that $\mathbf{SSa}v(Q) = v(Q)$. Fix $\varepsilon > 0$ and suppose that $\mathbf{SSa}v(Q) \leq \sum_{j=1}^L \lambda_j v(c_j) + \varepsilon$ for some integer L, $\sum_{j=1}^L \lambda_j c_j = Q$, $\lambda_j > 0$, $c_j \in F(Q)$, $j = 1, \ldots, L$. Furthermore, suppose that, for every $j = 1, \ldots, L$, the bundles $(y_{j,k}^i)_{i=1}^n$ in $I\!R_+^l$ satisfy $\sum_{i=1}^n \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i y_{j,k}^i \leq w_{c_j}$, $\sum_{k=1}^{K_{ij}} \gamma_{j,k}^i = c_j^i$, $\gamma_{j,k}^i \geq 0$ and $v(c_j) \leq \sum_{i=1}^n \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i u_i(y_{j,k}^i) + \frac{\varepsilon}{L\lambda_j}$. Then,

$$\begin{aligned} \mathbf{SSa}v(Q) &\leq \sum_{j=1}^{L} \lambda_j \left(\sum_{i=1}^{n} \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i u_i(y_{j,k}^i) + \frac{\varepsilon}{L\lambda_j} \right) + \varepsilon \\ &= \sum_{i=1}^{n} \sum_{j=1}^{L} \sum_{k=1}^{K_{ij}} \lambda_j \gamma_{j,k}^i u_i(y_{j,k}^i) + 2\varepsilon \leq v(Q) + 2\varepsilon. \end{aligned}$$

The last inequality is due to $Q^i = \sum_{j=1}^L \lambda_j c_j^i = \sum_{j=1}^L \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i$ $(i = 1, \ldots, n)$ and to $w_Q = \sum_{j=1}^L \lambda_j w_{c_j} \geq \sum_{j=1}^L \lambda_j \sum_{i=1}^n \sum_{k=1}^{K_{ij}} \gamma_{j,k}^i y_{j,k}^i = \sum_{j=1}^L \sum_{i=1}^n \sum_{k=1}^{K_{ij}} \lambda_j \gamma_{j,k}^i y_{j,k}^i$.

It follows that $\mathbf{SSa}v(Q) \leq v(Q) + 2\varepsilon$ for any $\varepsilon > 0$, and therefore, $\mathbf{SSa}v(Q) = v(Q)$. This shows that the core of (v, Q) is not empty. ∎

It is clear that any sub-game of a market game is also a market game. We conclude that if (v, Q) is a market game then each one of its sub-games has a non-empty core. The following example shows that there are games such that each of their subgames has a non-empty core, while they are not market games.

Example 5.

Let $Q = 1$ and $v(t) = t^2$ for every $0 \leq t \leq Q$. Clearly, v is SSA. Therefore, the core of any subgame of v is not empty. If v is a market game, the coalition Q can reallocate the initial endowments of its members so as to produce the quantity 1. The coalition $Q/2$, which has half of the resources that Q has, can reallocate these resources in precisely the same proportions as Q did, and produce in a similar way. By so doing, coalition $Q/2$ may produce half of the production of Q, which is $1/2 > v(Q/2) = 1/4$. Thus, v cannot be a market game. ∎

Definition 11. *The game (v, Q) is* homogeneous *if v is a homogeneous function on $F(Q)$. That is $v(\lambda c) = \lambda v(c)$ whenever $c, \lambda c \in F(Q)$.*

When the game is homogeneous, the worth of any coalition c is $|c|v(\frac{c}{|c|})$, which is the worth of a coalition whose size is 1 and its internal distribution is $\frac{c}{|c|}$ multiplied by the size of c.

Remark 2. *When (v, Q) is homogeneous, the function $u_{v,Q}$ coincides with v on Δ. Moreover, if (v, Q) is homogeneous then $u_{v,Q}$ uniquely determines v on $F(Q)$ via the equation $v(c) = |c|u_{v,Q}(\frac{c}{|c|})$.*

Remark 3. *If v is homogeneous, then $\boldsymbol{Sa}v = \boldsymbol{SSa}v$*

Example 6.

For any $q = (q^1, \ldots, q^n) \in \Delta$ let $e(q) = -\sum_{i=1}^{n} q^i \log(q^i)$ be the entropy of the distribution q. Fix some positive $Q \in I\!R^n$, and consider the homogeneous game (v, Q) defined by $v(c) = |c|e(\frac{c}{|c|})$ for $c \in F(Q)$. (v, Q) reflects a situation where heterogenous coalitions do better than homogenous ones. Since v is concave on Δ and is monotonically non-decreasing on $F(Q)$, by Theorem 4 below, (v, Q) is a market game. ∎

Lemma 3. *Every market game is homogeneous.*

Proof. Since (v, Q) is a market game, by Lemma 2 and by Proposition 1, v is SSA on $F(Q)$. Therefore, if $c \in F(Q)$ and $0 \leq \lambda \leq 1$, then $v(\lambda c) \leq \lambda v(c)$. To obtain the inverse inequality, fix $\varepsilon > 0$ and let $\{\gamma_k^i, y_k^i\}_{1 \leq i \leq n,\ 1 \leq k \leq K_i}$ be such that $\sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i y_k^i \leq w_c$, $\sum_{k=1}^{K_i} \gamma_k^i = c^i$, $i = 1, \dots, n$ and $\sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i u_i(y_k^i) \geq v(c) - \frac{\varepsilon}{\lambda}$. The collection $\{\lambda\gamma_k^i, y_k^i\}_{1 \leq i \leq n,\ 1 \leq k \leq K_i}$ is possible in the definition of $v(\lambda c)$, and therefore $v(\lambda c) \geq \sum_{i=1}^n \sum_{k=1}^{K_i} \lambda \gamma_k^i u_i(y_k^i) \geq \lambda v(c) - \varepsilon$. Since this is true for every $\varepsilon > 0$, we get that $v(\lambda c) \geq \lambda v(c)$ and therefore v is homogeneous. ■

As illustrated by Example 5, the fact that any subgame has a non-empty core does not guarantee that the game is a market game. It turns out that the characterization of market games also requires homogeneity. This is formally stated in the following analogy of Shapley and Shubik's theorem [13]:

Theorem 4. (v, Q) *is a market game if and only if:*
(i) v *is monotonically non-decreasing on* $F(Q)$*;*
(ii) (v, Q) *is homogeneous; and*
(iii) Every subgame of (v, Q) *has a non-empty core.*

Proof. The claim, that if (v, Q) is a market game any of its subgames has a non-empty core is due to Lemma 2 and to the fact that any subgame of a market game is itself a market game. The homogeneity of v is due to Lemma 3. It is also clear by the definition of a market game that v is non-decreasing on $F(Q)$.

Now suppose that every subgame of (v, Q) has a non-empty core and that v is a homogeneous non-decreasing function on $F(Q)$. We show that (v, Q) is a market game. Let w_i be the i-th standard basis vector of $\mathbb{R}^n$, and $u_i = v$, $i = 1, \dots, n$ (notice that we take $\ell = n$).

Denote by (r, Q) the market game induced by Q and $\{w_i, v\}_{i=1}^n$. It remains to prove that $r = v$ on $F(Q)$. By the monotonicity of v and since $w_c = c$, we have,

$$r(c) = \sup \Big\{ \sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i v(y_k^i);\ \sum_{i=1}^n \sum_{k=1}^{K_i} \gamma_k^i y_k^i = c,\ \sum_{k=1}^{K_i} \gamma_k^i = c^i,\ i = 1, \dots, n \Big\}$$

(notice that the monotonicity of v enables us to replace the inequality in the definition of a market game with equality). Since every such partition of c is possible also in the definition of $\mathbf{SSa}v(c)$, we have that $r \leq \mathbf{SSa}v$. However, by Proposition 1, and since every subgame of

(v,Q) has a non-empty core, v is SSA. Therefore, by Lemma 9 (7) in the Appendix, $\mathbf{SSa}v = v$, and so $r \leq \mathbf{SSa}v = v$.

To show that $r \geq v$, recall that r is homogeneous because (r,Q) is a market game and that v is homogeneous by assumption. Therefore we can assume w.l.o.g. that $\Delta \subseteq F(Q)$. Fix $c \in \Delta$ and for $i = 1, \ldots, n$ take $K_i = 1$, $\gamma^i = c^i$ and $y^i = c$. Then by the definition of r, $r(c) \geq \sum_{i=1}^{n} c^i v(c) = v(c)$. It follows that $r = v$ on Δ and by homogeneity $r = v$ on $F(Q)$. Therefore, $(v,Q) = (r,Q)$ is a market game. ■

4.3. **More about homogeneous population games.**

Proposition 4. *Let (v,Q) be a homogeneous game. The following are equivalent:*

(1) *Every subgame of (v,Q) has a non-empty core.*
(2) *v is SA on $F(Q)$.*
(3) *v is concave on $F(Q)$.*

Proof. (1) $\Rightarrow$ (2) By proposition 1, if every subgame of (v,Q) has a non-empty core then v is SSA. In particular, v is SA.

(2) $\Rightarrow$ (3) If $c,d \in F(Q)$ and $\alpha \in (0,1)$ then $v(\alpha c + (1-\alpha)d) \geq v(\alpha c) + v((1-\alpha)d) = \alpha v(c) + (1-\alpha)v(d)$ so v is concave.

(3) $\Rightarrow$ (1) For $c,d \in F(Q)$ with $c+d \in F(Q)$ we have, by the concavity and homogeneity of v, $\frac{1}{2}v(c+d) = v(\frac{c+d}{2}) \geq \frac{1}{2}v(c) + \frac{1}{2}v(d)$ so v is SA. By Remark 3, v is SSA and by Proposition 1 every subgame of (v,Q) has a non-empty core. ■

Remark 4. *When (v,Q) satisfies conditions (i), (ii) and (iii) of Theorem 4 we can say more than that (v,Q) is a market game. Indeed, by the proof of Theorem 4, it follows that (v,Q) is a market game where the production functions of all the firms in the economy are homogeneous. Moreover, by Proposition 4 all the production functions are concave.*

Proposition 5. *Every homogeneous game has a non-empty split-core.*

Proof. This is a consequence of Theorem 2 and Remark 3. ■

Recall Example 2 that shows a homogeneous game whose core is empty.

5. Convex population games

Definition 12. *The population game (v,Q) is convex if, whenever s,t,d and $t+d$ are coalitions such that $s \leq t$, it follows that $v(s+d) - v(s) \leq v(t+d) - v(t)$.*

In classic cooperative games, the game (N, v) is convex if $v(S \cup T) - v(S) \geq v(T) - v(S \cap T)$ (this is due to Shapley, 1971). This means that the marginal contribution of any subset of players is monotone increasing with respect to the containment order on the coalitions. Definition 12 above is the analogues in the case of population games: When a population game is convex, the contribution of an additional coalition d is larger when it is being added to a larger existing coalition.

Recently, Branzei et al. [6] defined convexity of a fuzzy game in a slightly different way. However, our definition and theirs are equivalent. In their paper, it is shown that the core of a convex population game (v, Q) is not empty. Moreover, when (v, Q) is convex, the core of (v, Q) coincides with the core of the (convex) cooperative game, which is the restriction of v to the vertices of the cube $F(Q)$ (here it is assumed that all the coordinates of Q equal 1).

In this section, we discuss other properties of convex population games that, to the best of our knowledge, have not appeared elsewhere. The first question arises is whether convexity of the population game (v, Q) is equivalent to convexity of v as a function on $F(Q)$. The following lemma has the answer.

Lemma 4. *(i) If $n = 1$ then convexity of v on $[0, Q]$ is equivalent to convexity of the game (v, Q).*
(ii) For $n \geq 2$, convexity of v on $F(Q)$ does not imply and is not implied by convexity of the game (v, Q).

Proof. (i) This is a consequence of the equivalence of our definition and that of Branzei et al. [6].

(ii) We start with an example where the game (v, Q) is convex but v is not convex on $F(Q)$. Let $Q = (1, 1)$ and define $v(c^1, c^2) = -c^1(1-c^2)$ for $c = (c^1, c^2) \in F(Q)$. v is not convex on $F(Q)$ since, for example, $v((\frac{1}{2}, \frac{1}{2})) = -\frac{1}{4} > -\frac{1}{2} = \frac{1}{2}v((0, 1)) + \frac{1}{2}v((1, 0))$ (actually, v is concave on Δ). However, the game (v, Q) is convex. Indeed, a little bit of algebra gives $v(t + d) - v(t) = t^1d^2 + d^1t^2 - d^1 + d^1d^2$. Similarly, $v(s + d) - v(s) = s^1d^2 + d^1s^2 - d^1 + d^1d^2$. Therefore, if $t \geq s$ then $v(t + d) - v(t) \geq v(s + d) - v(s)$.

On the other hand, let $Q = (2, 2)$ and define $v(c^1, c^2) = (c^1 - c^2)^2$ for any $c \in F(Q)$. Then obviously v is a convex function. Define $s = (0, 1)$, $t = (0, 2)$, $d = (2, 0)$. Then $s \leq t$ but $v(s + d) - v(s) = 0 > -4 = v(t + d) - v(t)$. Therefore, (v, Q) is not convex. ∎

Example 7.

Let f be a convex real function defined on $\mathbb{R}$ and let Q be a positive vector in $\mathbb{R}^n$. Define $v(c) = f(|c|)$ for every $c \in F(Q)$. Then (v, Q) is a convex game. This is a simple application of the previous Lemma 4 (i). ■

We move on to discuss continuity of convex population games. By the previous lemma, convexity of (v, Q) does not imply that v is a convex function. Therefore, a matter of interest is whether convexity of (v, Q) guarantees continuity of v on $F(Q)$. The following example shows that, in general, convex games need not be continuous.

Example 8.

Let $Q = (1, \ldots, 1)$.

(1) Define $v(c) = -1$ for every $c \in F(Q) \setminus L$, where L is the line connecting 0 with the point $(0, \ldots, 0, 1)$ and $v(c) = 0$ for every $c \in L$. (v, Q) is convex and not continuous.
(2) Define $v(c) = 0$ for every $c \in F(Q) \setminus L$, where L is the line connecting $(1, \ldots, 1, \frac{1}{2})$ with Q and $v(c) = 1$ for every $c \in L$. (v, Q) is convex and not continuous. ■

Lemma 5. *Let (v, Q) be a bounded convex game.*

(1) *Suppose that c is in $F(Q)$. For every $\varepsilon > 0$ there is $\delta > 0$ such that if $c \leq d$ and $|d - c| < \delta$ then $v(d) \leq v(c) + \varepsilon$.*
(2) *Suppose that d is in $F(Q)$. For every $\varepsilon > 0$ there is $\delta > 0$ such that if $c \leq d$ and $|d - c| < \delta$ then $v(d) \geq v(c) - \varepsilon$.*
(3) *If c is in the interior of $F(Q)$, then there is a neighborhood $U \subseteq F(Q)$ of c such that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $c' \in U$, $c' \leq d$ (resp. $c' \geq d$) and $|d - c'| < \delta$, then $v(d) \leq v(c') + \varepsilon$ (resp. $v(d) \geq v(c') - \varepsilon$).*

Proof. We prove (1). Let c be in $F(Q)$ and let $\varepsilon > 0$. Suppose in contradiction to the lemma, that there is a sequence d_n such that $d_n \searrow c$ and $v(d_n) > v(c) + \varepsilon$ for every $n = 1, 2, \ldots$. Define $e_n = d_n - c$. Because of convexity, it can be shown by induction that for every integer k, $v(c + ke_n) - v(c) \geq k(v(c + e_n) - v(c)) \geq k\varepsilon$, provided that $c + ke_n \in F(Q)$. However, since $c + e_n$ is in $F(Q)$ and $e_n \to 0$, for every k there is n such that $c + ke_n \in F(Q)$. This means that $v(c + ke_n) \geq k\varepsilon + v(c)$, which contradicts the boundness of v. This proves (1). A similar idea proves (2).

As for (3), if c is the interior of $F(Q)$, there is a neighborhood U of c which is bounded away from the boundary of $F(Q)$. Thus, for every integer k, there is $\delta > 0$ such that when $e \in F(Q)$ satisfies $|e| < \delta$, then for every $c' \in U$, $c' + ke$ and $c' - ke$ both are in $F(Q)$. This implies that $k(v(c' + e) - v(c')) \leq v(c' + ke) - v(c')$. Therefore, $v(c' + e) - v(c') \leq \frac{v(c'+ke)-v(c')}{k} \leq \frac{2M}{k}$, where $-M \leq v \leq M$. Since $\frac{2M}{k}$ goes to zero as k goes to infinity, (3) is proven. ■

Proposition 6.

(1) *A bounded convex game is continuous in the interior of* $F(Q)$.
(2) *If a bounded convex game is continuous in* 0 *and* Q*, then it is continuous in* $F(Q)$.

Proof. (1) Let c be in the interior of $F(Q)$. Lemma 5 states that if $q_n \searrow c$ or $q_n \nearrow c$, then $v(q_n) \to v(c)$. Fix $\varepsilon > 0$ and let q_n be a sequence that converges to c, where neither $c \leq q_n$ nor $c \geq q_n$ hold.

Lemma 5 (3) ensures the existence of a neighborhood U around c with the property that there is $\delta > 0$ such that if $c' \in U$, $c' \leq d$ and $|d-c'| < \delta$, then $v(d) \leq v(c')+\varepsilon$. Since $q_n \to c$, for sufficiently large[3] n, $\min(c, q_n) \in U$, $|c - \min(c, q_n)| < \delta$, $|q_n - \min(c, q_n)| < \delta$ and $q_n \in U$. Thus, $v(c) \leq v(\min(c, q_n)) + \varepsilon$ and $v(q_n) \leq v(\min(c, q_n)) + \varepsilon$. Since $q_n \in U$ (now q_n plays the role of c' and $\min(c, q_n)$ plays the role of d in Lemma 5 (3)) $v(c) \geq v(\min(c, q_n)) - \varepsilon$ and $v(q_n) \geq v(\min(c, q_n)) - \varepsilon$. Thus, $|v(c) - v(\min(c, q_n))| \leq \varepsilon$ and $|v(q_n) - v(\min(c, q_n))| \leq \varepsilon$ which implies that $|v(q_n) - v(c)| \leq 2\varepsilon$. This completes the proof of (1).

As for (2), notice first that if (v, Q) is convex then, for any $d \geq c$ in $F(Q)$, the game $(v_{c,d}, d-c)$ defined by $v_{c,d}(e) = v(c+e)$ is also convex. Therefore, (1) ensures that v is continuous in the relative interior of every facet.

The proof of (2) is by induction on the dimension of the game, n. If $n = 1$, then by Lemma 4 (i), convexity of (v, Q) implies that v is convex, and a convex function is continuous if it is continuous in the boundary. Assume that (2) holds for every game of dimension less than n and we prove the assertion for n.

Let F be a facet of $F(Q)$. There are c and d such that $F = \{e;\ c \leq e \leq d\}$. In order to show continuity of v in F it is enough to show continuity of v in c and d. Let e be a coalition such that $c + e \in F$.

[3] $\max(s, t)$ ($\min(s, t)$) is the coalition whose i-th coordinate is the maximum (minimum) of s^i and t^i.

Due to convexity of (v, Q),

$$(1) \qquad v(e) \leq v(c+e) - v(c) \leq v(Q) - v(Q-e).$$

However, if $|e|$ is sufficiently small, then both sides of (1) are close to zero. This proves continuity at c. Continuity at d is shown in a similar way.

Now, let two coalitions c and d be in the boundary of $F(Q)$. If c and d are sufficiently close to each other, then there is a point e which is in the same facet as c and in the same facet (possibly different) as d. Furthermore, e is close to both c and d. Thus, continuity of v in every facet implies continuity within the boundary of $F(Q)$.

It remains to show that if c is in the boundary and d is in the interior and both are close to each other, then $v(c)$ and $v(d)$ are close. Since c is in the boundary, either $\min(c, d)$ or $\max(c, d)$ is in the boundary. If $q = \min(c, d)$ is in the boundary, then

$$(2) \qquad v(d-q) \leq v(d) - v(q) \leq v(Q) - v(Q-d+q).$$

Thus, when d is sufficiently close to c, $d - q$ is close to zero and the two sides of (2) are close to zero. Therefore, $v(d)$ must be close to $v(q)$. Since q and c are in the boundary, $v(q)$ and $v(c)$ are close to each other, which shows that $v(d)$ and $v(c)$ are close to each other.

The proof is similar when $\max(c, d)$ is in the boundary. Thus, v is continuous, as required. ■

It is clear that any sub-game of a convex game is convex and therefore, any sub-game of a convex game has a non-empty core. It implies that, if (v, Q) is convex, then v is SSA. A natural question is whether the inverse, namely whether strong super additivity implies convexity, is correct. The following example shows that the answer is no.

Example 9.

Consider a case where $n = 1$, $Q = 1$ and

$$v(t) = \begin{cases} t/2, & \text{if } 0 \leq t \leq 1/4; \\ 3t/2 - 1/4, & \text{if } 1/4 \leq t \leq 1/2; \\ t, & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

v is SSA (and continuous) but not convex on $[0, 1]$. Therefore, by Lemma 4 (i), (v, Q) is not a convex game. ■

6. Exact population games

The following definition is due to Schmeidler [9].

Definition 13. *The cooperative game* (N, v) *is* exact *if for every coalition* $S \subseteq N$ *there exists* x *in the core of* (N, v) *such that* $\sum_{i \in S} x^i = v(S)$.

This definition has its natural counterpart in the context of population games:

Definition 14. (v, Q) *is an* exact population game *if for every* $c \in F(Q)$ *there exists* $x \in \text{core}(v, Q)$ *such that* $xc = v(c)$.

The following proposition characterizes exact population games.

Proposition 7. (v, Q) *is an exact population game iff the following conditions hold:*

1. v *is concave on* $F(Q)$.
2. v *is homogeneous on* $F(Q)$.
3. $v(\alpha c + (1-\alpha)Q) = \alpha v(c) + (1-\alpha)v(Q)$ *for any* $c \in F(Q)$ *and for any* $\alpha \in (0, 1)$.

Proof. Assume that (1)-(3) hold. By rescaling, if needed, we can assume that $\Delta \subseteq F(Q)$. For any $q \in \Delta$ denote by q^{-n} the $n-1$ dimensional vector consisting of the first $n-1$ coordinates of q. Thus, the n-th coordinate of q, q^n, is equal to $1 - |q^{-n}|$. Consider the set $D = \{(q, t);\ q^{-n} \in \Delta,\ t < v(q)\}$. Fix $c \in \Delta$ and define $L = \{\alpha(c^{-n}, v(c)) + (1-\alpha)((\frac{Q}{|Q|})^{-n}, v(\frac{Q}{|Q|}));\ \alpha \in [0, 1]\}$. L is the line connecting $(c^{-n}, v(c))$ and $((\frac{Q}{|Q|})^{-n}, v(\frac{Q}{|Q|}))$. L is a convex set in $I\!R^n$, and since v is concave, D is also a convex set. Furthermore, the interior of D is not empty.

Fix $\beta \in (0, 1)$ and let $\alpha \in (0, 1)$ be such that $\beta = \frac{\alpha}{|\alpha c + (1-\alpha)Q|}$. Notice that $1 - \beta = \frac{(1-\alpha)|Q|}{|\alpha c + (1-\alpha)Q|}$. By conditions (2) and (3),

$$\begin{aligned} v\left(\beta c + (1-\beta)\frac{Q}{|Q|}\right) &= v\left(\frac{\alpha c + (1-\alpha)Q}{|\alpha c + (1-\alpha)Q|}\right) = \\ &= \frac{v(\alpha c + (1-\alpha)Q)}{|\alpha c + (1-\alpha)Q|} = \beta v(c) + (1-\beta)v\left(\frac{Q}{|Q|}\right). \end{aligned}$$

The above equality implies that the sets D and L are disjoint. Moreover, the line segment L is on the boundary of D. The separation theorem ensures that there is a hyperplane in $\mathbb{R}^n$ that separates L from D, and since L is on the boundary of D, it follows that L is contained in this hyperplane. This implies that there is a vector $x \in I\!R^n$ such that $xq \geq v(q)$ for every $q \in \Delta$ with equality for every $q \in L$. In particular it means that x is in core(v, Q) and that $xc = v(c)$. Therefore, we proved the assertion for any $c \in \Delta$. However, since (v, Q) is homogeneous it is easy to see that it will hold for any $c \in F(Q)$.

As for the converse, assume that (v, Q) is exact. Let $c, d \in F(Q)$ and fix $\alpha \in (0, 1)$. Let $x \in$ core(v, Q) be such that $x(\alpha c + (1-\alpha)d) = v(\alpha c + (1-\alpha)d)$. Then $\alpha v(c) + (1-\alpha)v(d) \leq \alpha xc + (1-\alpha)xd = x(\alpha c + (1-\alpha)d) = v(\alpha c + (1-\alpha)d)$, so v is concave.

Next, we show that v is homogeneous. Fix $c \in F(Q)$ and $\alpha > 0$ such that $\alpha c \in F(Q)$. Let $x, y \in$ core(v, Q) be such that $xc = v(c)$ and $y\alpha c = v(\alpha c)$. Since both x and y are in the core, $xc = v(c) \leq yc$ and $\alpha xc \geq v(\alpha c) = \alpha yc$. It follows that $v(\alpha c) = \alpha v(c)$.

Finally, for some $c \in F(Q)$ and $\alpha \in (0, 1)$, let $x, y \in$ core(v, Q) be such that $xc = v(c)$ and $y(\alpha c + (1-\alpha)Q) = v(\alpha c + (1-\alpha)Q)$. Then $\alpha v(c) + (1-\alpha)v(Q) = \alpha xc + (1-\alpha)xQ \geq v(\alpha c + (1-\alpha)Q) = \alpha yc + (1-\alpha)yQ \geq \alpha v(c) + (1-\alpha)v(Q)$, so we have an equality. ■

Proposition 8. *Suppose that there are n algebraically independent coalitions $c_1, ..., c_n$ such that*
(i) $c_i >> 0$, $i = 1, ..., n$;
(ii) the subgame (v_{c_i}, c_i) is exact, $i = 1, ..., n$; and
(iii) $Q = c_1$.
Then, v is linear: $v(d) = xd$ for some $x \in I\!R^n$.

Proof. Without loss of generality the standard simplex, Δ, is a subset of $F(Q)$. Since (v, Q) is exact, by Proposition 7, v is homogenous. Thus, in order to prove the proposition, it is enough to prove that v is linear in Δ.

We first claim that for every $i = 1, ..., n$ and for every $d \in \Delta$, v is linear in the interval between $\frac{c_i}{|c_i|}$ and d. Indeed, since $c_i >> 0$, there is $\varepsilon > 0$ such that $c_i >> \varepsilon d$. By assumption, the subgame (v_{c_i}, c_i) is exact and by Proposition 7, for every $\beta \in (0, 1)$, $v\left(\frac{\beta c_i + (1-\beta)\varepsilon d}{|\beta c_i + (1-\beta)\varepsilon d|}\right) = \frac{\beta}{|\beta c_i + (1-\beta)\varepsilon d|} v(c_i) + \frac{1-\beta}{|\beta c_i + (1-\beta)\varepsilon d|} v(\varepsilon d)$. Since v is homogenous this is equal to $\frac{\beta |c_i|}{|\beta c_i + (1-\beta)\varepsilon d|} v(\frac{c_i}{|c_i|}) + \frac{(1-\beta)\varepsilon}{|\beta c_i + (1-\beta)\varepsilon d|} v(d)$. This implies linearity in the interval between $\frac{c_i}{|c_i|}$ and d because $\frac{\beta |c_i|}{|\beta c_i + (1-\beta)\varepsilon d|}$ is onto $(0, 1)$, as a function of β.

Next we show that $v(\sum_{i=1}^n \alpha^i \frac{c_i}{|c_i|}) = \sum_{i=1}^n \alpha^i v(\frac{c_i}{|c_i|})$, when $\sum_{i=1}^n \alpha^i = 1$ and $\alpha^i \geq 0,\ i = 1, ..., n$. This is a simple consequence (by induction) of the previous claim. Thus, v is linear in the convex hull of $\{\frac{c_i}{|c_i|}\}_{i=1}^n$, denoted C.

Now, let $d \in \Delta$. Since $c_1, ..., c_n$ are independent, there are coefficients (not necessarily positive) $\gamma^i,\ i = 1, ..., n$, such that $d = \sum_{i=1}^n \gamma^i \frac{c_i}{|c_i|}$. Furthermore, this representation of d as a linear combination of the $\frac{c_i}{|c_i|}$s is unique. Since all $\frac{c_i}{|c_i|}$ are in Δ, the sum of the coefficients is 1.

¿From here on the proof follows an induction on the number of negative coefficients. If this number is 0, then $d \in C$, a case that has been proven above. Now suppose that if the number of negative coefficients is less than k, then $v(d) = \sum_{i=1}^n \gamma^i v(\frac{c_i}{|c_i|})$. We prove this assertion when the number of negative coefficients is k.

We know that for every $j = 1, ..., n$ and $\beta \in (0, 1)$, $v(\beta \frac{c_j}{|c_j|} + (1 - \beta)d) = \beta v(\frac{c_j}{|c_j|}) + (1 - \beta)v(d)$. However, if $\gamma^j < 0$ and β is sufficiently close to 1, the number of negative coefficients of $\beta \frac{c_j}{|c_j|} + (1 - \beta)d = \sum_{i \neq j}(1 - \beta)\gamma^i \frac{c_i}{|c_i|} + \big(\beta + (1 - \beta)\gamma^j\big)\frac{c_j}{|c_j|}$ is $k - 1$. Thus, when β is sufficiently close to 1, $\beta v(\frac{c_j}{|c_j|}) + (1 - \beta)v(d) = \sum_{i \neq j}(1 - \beta)\gamma^i v(\frac{c_i}{|c_i|}) + \big(\beta + (1 - \beta)\gamma^j\big)v(\frac{c_j}{|c_j|})$. By rearranging the last equality, we obtain that $v(d) = \sum_{i=1}^n \gamma^i v(\frac{c_i}{|c_i|})$, as desired. ■

Corollary 1. *The subgame (v_c, c) is exact for any $c \in F(Q)$ iff v is linear: $v(d) = xd$ for some $x \in I\!R^n$.*

Remark 5. *In classic cooperative games, the game (N, v) is convex if and only if each of his subgames is exact. However, in this new model, the fact that every subgame of the game (v, Q) is exact is stronger than convexity of (v, Q). Indeed, by Corollary 1 every subgame of (v, Q) is exact iff v is linear on $F(Q)$.*

7. Additive population games

When a game is additive, the contribution of any type to the total worth of a coalition is independent of the amount of players of other types in the coalition. This is when there is no external effects of the presence of players of other types. Formally,

Definition 15. *(v, Q) is an* additive population game *if there exist functions $g_i : I\!R \to I\!R,\ i = 1, 2, \ldots, n$, such that $g_i(0) = 0,\ i = 1, \ldots, n$, and for any coalition $c = (c^1, \ldots, c^n) \in F(Q),\ v(c) = \sum_{i=1}^n g_i(c^i)$.*

The special structure enables us to provide a simpler characterization of additive games with non-empty core.

Proposition 9. *Let (v,Q) be an additive game with $g_1,\dots,g_n$ corresponding to it. The following are equivalent:*

(1) *(v,Q) has a non-empty core.*
(2) *$\frac{g_i(t)}{t} \le \frac{g_i(Q^i)}{Q^i}$ for every $0<t<Q^i$ and for every $i=1,\dots,n$.*

Proof. ($2\Rightarrow 1$) for $i=1,\dots,n$ define $x^i = \frac{g_i(Q^i)}{Q^i}$. Then $xQ = v(Q)$ and for every $c\in F(Q)$, $xc = \sum_{i=1}^n \frac{g_i(Q^i)}{Q^i}c^i \ge \sum_{i=1}^n \frac{g_i(c^i)}{c^i}c^i = v(c)$, so $x\in \text{core}(v,Q)$.

($1\Rightarrow 2$) Let $x\in\text{core}(v,Q)$ and assume that (2) doesn't hold. Then there is some $i\in\{1,\dots,n\}$ and $\tilde{t}\in(0,Q^i)$ such that $\frac{g_i(\tilde{t})}{\tilde{t}} > \frac{g_i(Q^i)}{Q^i}$. Consider the coalition d defined by $d^i=\tilde{t}$ and $d^j=Q^j$ for any $j\neq i$. Since x is in the core,

$$x^i\tilde{t} + \sum_{j\neq i} x^jQ^j = xd \ge v(d) = g_i(\tilde{t}) + \sum_{j\neq i} g_j(Q^j) \tag{3}$$

$$\sum_{j=1}^n x^jQ^j = xQ = v(Q) = \sum_{j=1}^n g_j(Q^j) \tag{4}$$

Subtract (3) from (4) to obtain $x^iQ^i - x^i\tilde{t} \le g_i(Q^i) - g_i(\tilde{t})$. Using the assumption we obtain,

$$x^i\left(\frac{Q^i}{\tilde{t}} - 1\right) \le \frac{g_i(Q^i)}{\tilde{t}} - \frac{g_i(\tilde{t})}{\tilde{t}} < \frac{g_i(Q^i)}{\tilde{t}} - \frac{g_i(Q^i)}{Q^i} \tag{5}$$

Rearranging the terms in (5) gives $x^iQ^i < g_i(Q^i)$. However, this is impossible because one can consider the coalition whose ith coordinate equals Q^i and all other coordinates equal 0. Thus, we got the desired contradiction. ■

8. Discussion and final comments

8.1. **Another possible definition of market games.** When (v,Q) is interpreted as an investment game, there are only n individuals interacting as and not a continuum of players. These individuals might be firms that combine forces for a joint production. Suppose that firm i has an initial endowment Q^iw_i (Q^i is a number and $w^i\in\mathbb{R}^\ell_+$) and a production function u_i. When firm i invests c^iw_i ($c^i\le Q^i$) we say that the investment is $c=(c^1,...,c^n)$. In this case the total endowment is $w_c = \sum_{i=1}^n c^i w_i$. By splitting this endowment as $\sum_{i=1}^n d_i = w_c$, the investment c can produce $\sum_{i=1}^n u_i(d_i)$. The value of c is therefore the maximal achievable production. Formally,

Definition 16. *The* investment-market game *induced by* $Q = (Q^1, ..., Q^n)$ *and* $\{w_i, u_i\}_{i=1}^n$ *is the game* (v, Q) *where for every* $c \in F(Q)$,
$v(c) = \sup\left\{ \sum_{i=1}^n u_i(d_i);\ \sum_{i=1}^n d_i \le w_c \right\}$.

It turns out that in order for an investment-market game to have a non-empty core, the production functions should be strongly super-additive (SSA):

Lemma 6. *If all the production functions* u_i *are SSA, then the investment-market game induced by* $Q = (Q^1, ..., Q^n)$ *and* $\{w_i, u_i\}_{i=1}^n$ *has a non-empty core.*

The proof appears in the Appendix. Similar to Theorem 4 we obtain,

Proposition 10. (v, Q) *is an investment-market game with strongly super-additive production functions iff the core of any subgame of* (v, Q) *is non-empty and* v *is non-decreasing.*

The proof is similar to that of Theorem 4 and is therefore omitted.

8.2. **Weakly convex games.** Another possible extension of the definition of cooperative convex games to investment games is the following:

Definition 17. *A game* (v, Q) *is* weakly convex *if for every two coalitions* s *and* t,

$$v(s) + v(t) \le v(\max(s,t)) + v(\min(s,t)).$$

It seems that this family of games deserves a separate investigation.

8.3. **Non-atomic games.** A non-atomic game is a triple $(I, \mathcal{B}, v)$, where I is the unit interval, $\mathcal{B}$ is the Borel sigma-field of subsets of I, and v is a real function defined over $\mathcal{B}$ that satisfies (1) $v(\emptyset) = 0$; and (2) if $v(S) > 0$ then there is $T \subseteq S$, $T \in \mathcal{B}$, such that $0 < v(T) < v(S)$.

Let μ_i, $i = 1, ..., n$, be n measures on $(I, \mathcal{B})$ such that μ_i and μ_j are mutually singular whenever $i \ne j$. Denote $\mu = (\mu_1, \mu_2, ..., \mu_n)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $f(0) = 0$. Then, $v(S) = f \circ \mu(S) = f(\mu_1(S), \mu_2(S), ..., \mu_n(S))$ is a non-atomic game. Furthermore, since any two measures are mutually singular, the range of the n-dimensional measure $(\mu_1, \mu_2, ..., \mu_n)$ is a cube, $F(Q)$. Thus, the game v just defined (with the set I of players and coalitions in $\mathcal{B}$) induces a population game (v, Q).

On the other hand, any population game induces an equivalent non-atomic game similar to the one described above.

The advantage of the model of population games is that it is defined in a finite Euclidian space which allows for a relatively simple analysis.

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9. Appendix

Proof of Lemma 1:

Denote by $w(q)$ the right-hand side of the equality. First, it is clear that $w \geq u_{v,Q}$ on Δ. In addition, one can show that w is concave. Therefore, $w \geq \mathbf{Cav}u_{v,Q}$. On the other hand, if $q = \sum_{i=1}^{n+1} \alpha_i q_i$ where $\sum_{i=1}^{n+1} \alpha_i = 1$, $q_i \in \Delta$, $\alpha_i \geq 0$, $i = 1, ..., n+1$, then by concavity of $\mathbf{Cav}u_{v,Q}$ we have $\mathbf{Cav}u_{v,Q}(q) \geq \sum_{i=1}^{n+1} \alpha_i \mathbf{Cav}u_{v,Q}(q_i) \geq \sum_{i=1}^{n+1} \alpha_i u_{v,Q}(q_i)$. It follows that $w \leq \mathbf{Cav}u_{v,Q}$. It is enough to consider convex combinations of no more than $n+1$ elements by the Caratheodory theorem. ∎

The following lemma relates properties of the game v with properties of $u_{v,Q}$.

Lemma 7. *(i) If v is Lipschitz with constant K, then $u_{v,Q}$ is continuous and bounded by K.*
(ii) If v is bounded and is Lipschitz in a neighborhood of zero, then $u_{v,Q}$ is bounded.

Proof. (i) Since $v(0) = 0$, we have $|v(c)| = |v(c) - v(0)| \leq K|c - 0|$ for any $c \in F(Q)$, so $u_{v,Q}$ is bounded by K. To show that $u_{v,Q}$ is continuous, fix $\varepsilon > 0$ and let $p, q \in \Delta$ be such that $|p - q| < \frac{\varepsilon}{K}$. By the definition of $u_{v,Q}$ there is $c \in F(Q)$ such that $p = \frac{c}{|c|}$ and $\frac{v(c)}{|c|} \geq u_{v,Q}(p) - \varepsilon$. Define $d = |c|q$ (if d is not in $F(Q)$, take the minimal $\delta > 0$ such that $d = (|c| - \delta)q \in F(Q)$. Since p, q, are close the argument will still hold). We have that $|v(d) - v(c)| \leq K|d - c| = K|c||q - p| \leq \varepsilon|c|$. It follows that $u_{v,Q}(q) \geq \frac{v(d)}{|d|} = \frac{v(d)}{|c|} \geq \frac{v(c) - \varepsilon|c|}{|c|} \geq u_{v,Q}(p) - 2\varepsilon$. By symmetry we get that $u_{v,Q}$ is continuous.

(ii) Let $\delta > 0$ be such that if $|c| < \delta$ then $\frac{|v(c)|}{|c|} \leq K$ where K is the Lipschitz constant. If $|c| \geq \delta$ then $\frac{v(c)}{|c|} \leq \frac{M}{\delta}$ where M is a constant bounding v. It follows that $\frac{v(c)}{|c|} \leq \max\{K, \frac{M}{\delta}\}$ so $u_{v,Q}$ is bounded. ∎

The following lemmas describe some of the properties of the operators **Sa** and **SSa**.

Lemma 8. *Let v, v' be two bounded functions on $F(Q)$ with $v(0) = v'(0) = 0$. Then*
(1) $v \leq \mathbf{Sa}v$.
(2) If $v \leq v'$ then $\mathbf{Sa}v \leq \mathbf{Sa}v'$.

(3) If v is SA, then $v = \mathbf{Sa}v$.
(4) $\mathbf{Sa}v$ is SA on $F(Q)$.
(5) $\mathbf{Sa}v = \mathbf{SaSa}v$.
(6) The infimum of any family of SA functions is SA.
(7) $\mathbf{Sa}v = \inf\{g;\ g \geq v$ and g is SA$\}$.

Lemma 9. *Let v, v' be two bounded functions on $F(Q)$ with $v(0) = v'(0) = 0$. Then*
(1) $v \leq \boldsymbol{SSa}v$.
(2) If $v \leq v'$ then $\mathbf{SSa}v \leq \mathbf{SSa}v'$.
(3) If v is SSA, then $v = \mathbf{SSa}v$.
(4) $\mathbf{SSa}v$ is SSA on $F(Q)$.
(5) $\mathbf{SSa}v = \mathbf{SSaSSa}v$.
(6) The infimum of any family of SSA functions is SSA.
(7) $\mathbf{SSa}v = \inf\{g;\ g \geq v$ and g is SSA$\}$.

The proof of Lemma 8 is similar to that of Lemma 9 and is therefore omitted.

Proof of Lemma 9:

(1) - (3) are clear. As for (4), fix $c \in F(Q)$ and assume that the equation $\sum_{j=1}^{L} \lambda_j c_j = c$ holds where $\lambda_j \geq 0$ and $c_j \in F(c),\ j = 1, \ldots, L$. Let $\varepsilon > 0$. By the definition of $\mathbf{SSa}v$, for any $j = 1, \ldots, L$, there exist investments $c_{j_1}, \ldots, c_{j_{K_j}}$ in $F(c_j)$ and non-negative numbers $\alpha_{j_1}, \ldots, \alpha_{j_{K_j}}$ such that $\sum_{i=1}^{K_j} \alpha_{j_i} c_{j_i} = c_j$ and $\sum_{i=1}^{K_j} \alpha_{j_i} v(c_{j_i}) \geq \mathbf{SSa}v(c_j) - \frac{\varepsilon}{\sum_{j=1}^{L} \lambda_j}$. It follows that,

$$\begin{aligned} \sum_{j=1}^{L} \lambda_j \mathbf{SSa}v(c_j) &\leq \sum_{j=1}^{L} \lambda_j \left(\sum_{i=1}^{K_j} \alpha_{j_i} v(c_{j_i}) + \frac{\varepsilon}{\sum_{j=1}^{L} \lambda_j} \right) = \\ &= \varepsilon + \sum_{j=1}^{L} \sum_{i=1}^{K_j} \lambda_j \alpha_{j_i} v(c_{j_i}) \leq \varepsilon + \mathbf{SSa}v(c). \end{aligned}$$

The last inequality is due to the fact that $\sum_{j=1}^{L} \sum_{i=1}^{K_j} \lambda_j \alpha_{j_i} c_{j_i} = c$. Since $\varepsilon > 0$ is arbitrary we have (4).

(5) follows from (3) and (4). To prove (6) let $\{g_\alpha\}_{\alpha \in I}$ be a family of SSA functions and define $w = \inf_{\alpha \in I} g_\alpha$. Assume that the equation $\sum_{j=1}^{L} \lambda_j c_j = c$ holds where $\lambda_j \geq 0$ and $c_j \in F(c),\ j = 1, \ldots, L$. Then for some fixed $\tilde{\alpha} \in I,\ g_{\tilde{\alpha}}(c) \geq \inf_{\alpha \in I} \sum_{j=1}^{L} \lambda_j g_\alpha(c_j) \geq \sum_{j=1}^{L} \lambda_j w(c_j)$.

Since this is true for every $\tilde{\alpha}$ we get that $w(c) \geq \sum_{j=1}^{L} \lambda_j w(c_j)$, so w is SSA.

(7) follows from the previous claims. Indeed, denote $w = \inf\{g;\ g \geq v \text{ and } g \text{ is SSA}\}$. By (4), $\mathbf{SSa}v$ is SSA on $F(Q)$ and by (1) it is above v. Therefore, $w \leq \mathbf{SSa}v$. On the other hand, if g is above v then by (2) $\mathbf{SSa}g \geq \mathbf{SSa}v$. If g is also SSA, then by (3) $g = \mathbf{SSa}g \geq \mathbf{SSa}v$. Since this is true for every such g, it follows that $w \geq \mathbf{SSa}v$ so $w = \mathbf{SSa}v$. ■

Proof of Proposition 1: (1) is a consequence of Theorem 1 (more precisely, from the equivalence of Theorem 1 (1) and Theorem 1 (2)) when applied to the sub-game (v_d, d).
As for (2), assume that the core of every sub-game is not empty. Then by (1) $v = \mathbf{SSa}v$. By Lemma 9, v is SSA. On the other hand, if v is SSA then $v = \mathbf{SSa}v$, which implies that any sub-game has a non-empty core. ■

Proof of Proposition 2: (1) is a consequence of Theorem 2. To see (2) notice that by (1) the split core of every sub-game is not empty iff $\mathbf{SSa}v = \mathbf{Sa}v$ on $F(Q)$. However, this is equivalent to $\mathbf{Sa}v$ being SSA. Indeed, if $\mathbf{SSa}v = \mathbf{Sa}v$ then by Lemma 9 $\mathbf{Sa}v$ is SSA. Conversely, if $\mathbf{Sa}v$ is SSA then by the same lemma, $\mathbf{SSaSa}v = \mathbf{Sa}v \leq \mathbf{SSa}v \leq \mathbf{SSaSa}v$, so we have $\mathbf{Sa}v \leq \mathbf{SSa}v$. ■

Proof of Lemma 6: Let (v, Q) be the investment-market game induced by Q and $\{w_i, u_i\}_{i=1}^{n}$. By Theorem 1 it is sufficient to show that $\mathbf{SSa}v(Q) = v(Q)$. Fix $\varepsilon > 0$ and suppose that $\mathbf{SSa}v(Q) \leq \sum_{j=1}^{L} \lambda_j v(c_j) + \varepsilon$ for some integer L, $\sum_{j=1}^{L} \lambda_j c_j = Q$, $\lambda_j > 0$, $c_j \in F(Q)$, $j = 1, ..., L$. Furthermore, suppose that for every $j = 1, ..., L$ the vectors $(d_{ij})_{i=1}^{n}$ in $I\!R_{+}^{l}$ satisfy $\sum_{i=1}^{n} d_{ij} \leq \sum_{i=1}^{n} c_j^i w_i$ and $v(c_j) \leq \sum_{i=1}^{n} u_i(d_{ij}) + \frac{\varepsilon}{L\lambda_j}$.
Then,

$$\begin{aligned}\mathbf{SSa}v(Q) &\leq \sum_{j=1}^{L} \lambda_j \left(\sum_{i=1}^{n} u_i(d_{ij}) + \frac{\varepsilon}{L\lambda_j} \right) + \varepsilon = \sum_{j=1}^{L} \sum_{i=1}^{n} \lambda_j u_i(d_{ij}) + 2\varepsilon \leq \\ &\leq \sum_{i=1}^{n} u_i \left(\sum_{j=1}^{L} \lambda_j d_{ij} \right) + 2\varepsilon,\end{aligned}$$

where in the last inequality we used the fact that each u_i is SSA. Notice that $\sum_{i=1}^{n} \left(\sum_{j=1}^{L} \lambda_j d_{ij} \right) \leq \sum_{i=1}^{n} Q^i w_i$. It follows that $\mathbf{SSa}v(Q) \leq$

$v(Q) + 2\varepsilon$ for any $\varepsilon > 0$, and therefore, $\mathbf{SSa}v(Q) = v(Q)$. This shows that the core of (v, Q) is not empty. ■